On Approximate Welfare- and Revenue-Maximizing Equilibria for Size-Interchangeable Bidders

(Extended Abstract)

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ABSTRACT

This paper introduces a novel relaxation of *Walrasian equilibrium* (WE) which we call *Restricted Envy-Free Pricing* (REFP), an algorithm to compute this outcome for the case of size-interchangeable bidders (a generalization of single-minded bidders introduced in this paper), and a heuristic for searching among these outcomes for one that maximizes revenue. We provide theoretical bounds for our algorithms where possible, and run extensive experiments to evaluate their performance on both a synthetic distribution, and one obtained from real-world web-usage data. Compared to other benchmarks in the literature, our algorithms perform well on the metrics of revenue and efficiency, without incurring too many violations of the *true* WE conditions.

Keywords

Walrasian Equilibrium, Restricted Envy-Free Pricing

1. GENERAL OVERVIEW

In a **centralized combinatorial matching market (CCMM)** [5, 6], a market maker offers a set U of n heterogeneous **goods** to m consumers (or **bidders**), the latter of which are interested in acquiring combinations (or **bundles**) of goods. In general, there are multiple copies of each good i, but the total supply of each good is finite. Bidder j's preferences are captured by a **valuation** function that describes how j values each bundle. In general, a bidder's valuation function can be an arbitrary function of the set of all bundles. CCMMs are a fundamental market model with many practical applications, such as: estate auctions, transportation networks, wireless spectrum allocation and electronic advertising markets; and thus, these markets have been extensively studied in the literature [1, 2, 8, 11, 12].

Given a CCMM, a market **outcome** is an allocation-pricing pair $(\boldsymbol{X}, \boldsymbol{p})$, where $\boldsymbol{X}$ describes an allocation of goods to bidders, and $\boldsymbol{p}$ ascribes prices to goods. While $\boldsymbol{X}$ is a matrix, in our model we assume that $\boldsymbol{p}$ is a vector, which precludes any form of price discrimination (all copies of the same good must have the same price). Furthermore, we assume **item pricing**, not bundle pricing, so that the price of a bundle is the sum of the prices of all the goods (items) in the bundle. Both of these assumptions—no price discrimination and item pricing—are most natural. Given a market outcome, we assume quasi-linear utilities, meaning bidder j's utility is defined to be the difference between their valuation for the bundle they are allocated, and its price.

In a CCMM, of paramount concern is what properties are desirable in an outcome. In this work, we focus on a fundamental market outcome known as **Walrasian equilibrium** (WE) [15]. An outcome is said to be a WE if two properties hold: (1) all bidders are envy-free (EF), meaning the outcome is utility-maximizing for all, and (2) the market clears (MC), meaning the price of any unallocated good is zero. A WE is a fundamental market outcome that ensures that market participants are maximally happy with the outcome while at the same time supply meets demand. Moreover, by the first welfare theorem of economics, any allocation that is part of a WE is also maximizes (social) welfare (i.e., utility of the bidders and the market maker).

While of great theoretical importance, the WE concept suffers from two important drawbacks. First, a WE might not exist, even for relatively simple forms of bidders' valuation functions. Second, even when one does exist, the revenue of a WE outcome can be low, in particular as low as zero. Revenue in this context is defined as the total income of the market maker, i.e. $\sum_{i,j} x_{ij} p_i$.

A well-known approach to simultaneously address both existence and low revenue is to relax *only* the MC condition, and instead require that the price of unallocated goods is some, possibly non-zero, reserve price. This approach is known as *Envy-Free Pricing* (EFP) [3, 7, 8] and has been extensively studied in the case of single-minded bidders (where bidders are interested in exactly one bundle –including any bundle that contains their preferred bundle–; see, for example [8, 4]). Unlike a WE, an EFP always exists. An outcome in which no goods are allocated, and all are priced at infinity is a trivial, albeit undesirable, example of an EFP. This is not the only approach to relaxing a WE that has been proposed. For example, to address the existence issue, Postlewaite and Schmeidler [13] define an ϵ-WE in which every bidder is envy-free up to a ratio of 1-ϵ, and Huang, Li, and Zhang [9] try to maximize the ratio of envy-free bidders to all bidders. Note that in all of these approaches, one only relaxes *either* the EF or the MC condition.

Appears in: *Proc. of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2017), S. Das, E. Durfee, K. Larson, M. Winikoff (eds.), May 8–12, 2017, São Paulo, Brazil.*

2. OUR APPROACH

In our work, we go one step further and relax *both* the EF and MC conditions. We propose a relaxation of the EF condition where only winners (bidders that are part of the allocation) are EF, and further relax the MC condition such that unallocated goods are priced *at least* at the reserve. We call this new solution concept *Restricted Envy-Free Pricing* (REFP). Similar to an EFP, this solution concept always exists (same trivial example as before). However, whereas for a fixed allocation, an EFP might not exist, a REFP always exists, even if an allocation has been decided upon beforehand. Thus, our solution concept provides a stronger guarantee of existence and paves the way for fast computational methods to find such outcomes. In particular, we investigate what REFP entails for single-minded bidders, and show that for size-interchangeable bidders (a generalization of the single-minded case we introduce in this paper where bidders are interested in a bundle of certain size, but are indifferent among various sources) we can compute REFP in polynomial time, given a fixed allocation. In the case of single-minded bidders, there exist polynomial-time algorithms to find nearly welfare-maximizing allocations [10]. We extend these algorithms to size-interchangeable, and use them to compute REFP outcomes.

As in the case of EFP, we remain interested in computing outcomes with maximal revenue. Drawing inspiration from algorithms proposed for EFP in the case of unit-demand and single-minded bidders, we propose and evaluate algorithms to find revenue-maximizing REFP in the case of size-interchangeable bidders. These algorithms work by exploring a space of reserve prices: for each candidate reserve price, they find an EFP, and then, among all outcomes seen, they choose one with maximal revenue.

Alternatively, given a candidate reserve price, instead of computing an outcome that simultaneously yields an allocation and corresponding prices, one could have instead solved for an allocation that respects the reserve price and then solved for a corresponding set of supporting prices, each one being at least the reserve. This two-step process (first solve for an allocation and then for prices) fails in the case of EFP, because, given an allocation, envy-free prices might not exist; however, it is viable in the case of REFP because *restricted* envy-free prices always exist. This begs the question: given an allocation, are these prices efficiently computable?

In this work, we answer this question in the affirmative for two special cases: single-minded bidders and size- interchangeable bidders. In the case of single-minded bidders, we show that finding a set of revenue-maximizing REFP reduces to the problem of finding a welfare-maximizing allocation. In the more complicated case of size-interchangeable bidders, we derive necessary and sufficient conditions for finding restricted envy-free prices, given a fixed allocation, and propose a greedy heuristic to find approximately welfare-maximizing allocations. Our characterization of restricted envy-free prices is a linear characterization and thus, together with our greedy heuristic, we succeed in finding REFP for size-interchangeable bidders in polynomial-time.

Our linear characterization is agnostic as to the objective function being optimized. Thus, we present a powerful two-step framework where we first solve for an allocation, and then for restricted envy-free prices for *any* linear objective function of the prices. We then apply this methodology to solve, in particular, for revenue-maximizing REFP for a fixed allocation and reserve price, and use this algorithm at the heart of a heuristic (Algorithm 1) to find revenue maximizing REFP among all allocations and reserve prices. We evaluate the performance of our revenue-maximizing heuristic by running extensive experiments, using both synthetic and real-world data and by feeding it allocations obtained with two different objectives: (1) **egalitarian**, which maximizes the number of winners, and (2) **welfare-maximizing**, which maximizes total utility. Compared to other benchmarks in the literature, it performs well on the metrics of revenue and efficiency, without incurring too many EF and MC violations.

ALGORITHM 1: Revenue-maximizing heuristic.

Input: Market M and solution concept S
Output: A pricing $\boldsymbol{p}$ and an allocation $\boldsymbol{X}$
1. Find an initial allocation $\boldsymbol{X}$.
2. For all $x_{ij} > 0$:
   2.1 Set a reserve price r as a function of x_{ij} and M.
   2.2 Find $(\boldsymbol{X}, \boldsymbol{p})$ for concept S using reserve r.

Output a pair $(\boldsymbol{X}, \boldsymbol{p})$ with maximal seller revenue.

Our size-interchangeable model is motivated by the Trading Agent Competition Ad Exchange game (TAC AdX) [14], which in turn models online ad exchanges in which agents face the challenge of bidding for display-ad impressions required to fulfill advertisement contracts, after which they earn the amount the advertiser budgeted. Other settings captured by this model include the problem of how to allocate specialized workers to firms, and how to compensate the workers, where each firm requires a certain number of workers to produce an output that yields a certain revenue.

3. EXPERIMENTAL RESULTS SUMMARY

Experimental results using Algorithm 1 show that, in general, our algorithms performs well across different markets on the metrics of revenue, efficiency, and time, with very few violations of the EF and MC conditions. Although our heuristic searches only REFPs, we nevertheless obtain outcomes that are close to EFPs, even when we seed our heuristic with a welfare-maximizing allocation, rather than an egalitarian one. In other words, our two step approach of first fixing an allocation, and then making only *winners* envy-free, seems to be a reasonable way to find nearly EFP outcomes, in which *losers* are also envy-free.

Also of interest is the fact that it is not always the case that the egalitarian algorithms achieve fewer EF violations than the utilitarian ones. The original intent of the egalitarian objective was to increase the number of winners, so that solving for restricted envy-free prices where only winners are envy-free, would yield fewer EF violations—fewer losers would mean fewer opportunities to violate EF. However, egalitarian allocations end up allocating goods to bidders with low rewards which, together with individual rationality, keeps prices low, which of course yields low revenue, but also yields EF violations of greater magnitude, given that the allocated goods are necessarily cheap.

Acknowledgments

This work was supported by NSF Grant #RI-1217761.

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