On weakly and strongly popular rankings

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What are popular rankings?

Popularity is all about robustness against majority swings!

We have a set V of voters and a set C of candidates. Each voter submits a ranking π_v (strictly ordered list) of the candidates. Whether a voter prefers one ranking π to another σ is calculated based on the Kendall distance.

$$K(\pi, \sigma) = |\{(a, b)| \ \pi \text{ and } \sigma \text{ disagree on the relative rank of } a \text{ and } b\}|$$

Goal: A ranking π of the candidates avoiding the scenario that a majority of voters prefer another candidate ranking.

A ranking π is **weakly popular** if compared against any other ranking σ, π is preferred to σ by at least half of the voters (Van Zuylen et al. [1]).

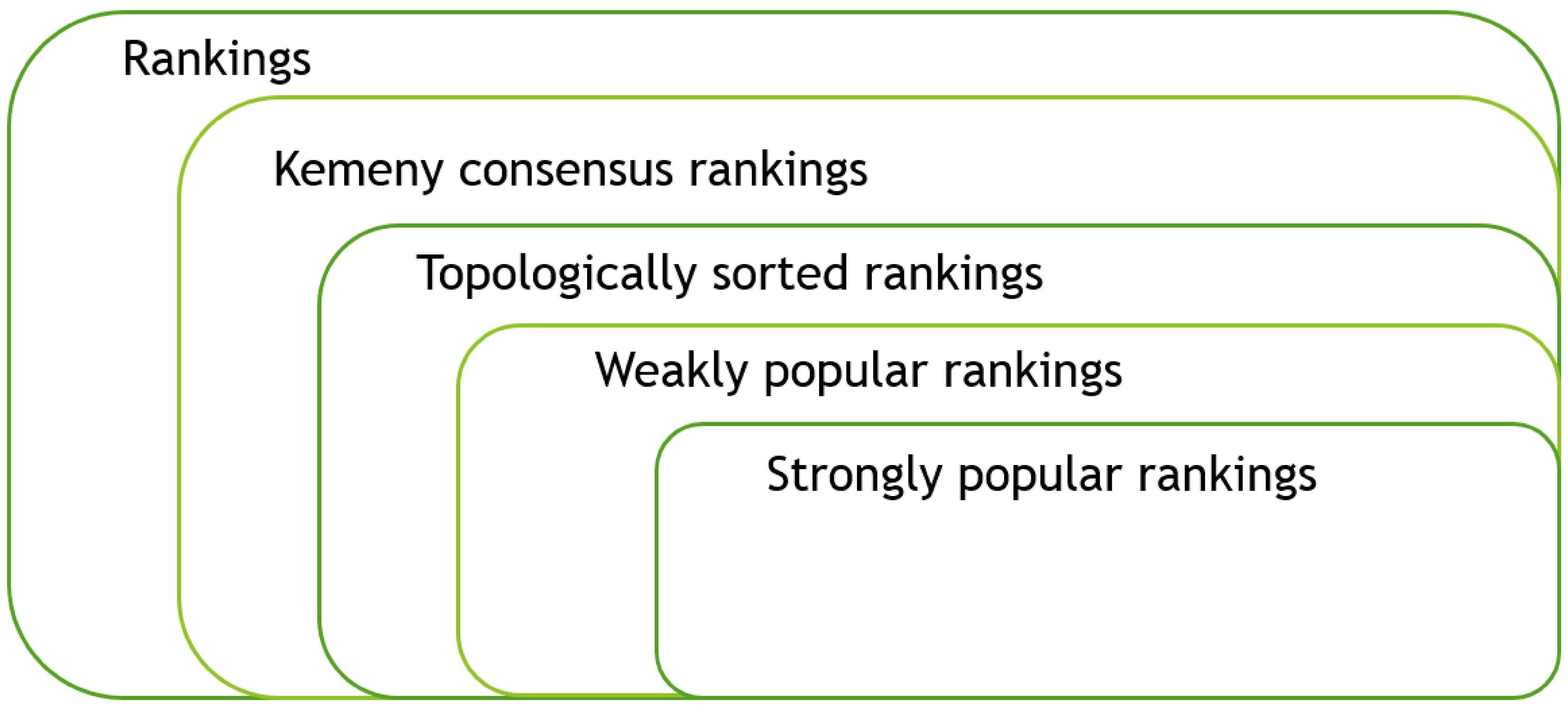
Strong and weak popularity

If many voters abstain, the devil is in the non-abstaining voters.

Say we have 100 *voters and* 10 *candidates, and we have two rankings* π *and* σ *where* 51 *voters abstain (i.e., they like* π *equally as much as* σ*) but* 49 *prefer* π *to* σ*. Intuitively, we want to prevent* σ *from being popular!*

Solution: A ranking π is **strongly popular** if compared against any other ranking σ, it is preferred by at least half of the *non-abstaining* voters.

Popular rankings in the hierarchy of rankings



Discovering unpopularity

Question: Given a ranking π, find a ranking σ that is preferred by a majority of voters or output that it does not exist.

6/7 voters: This is NP-hard for 7 voters (Van Zuylen et al. [1]). It is also NP-hard for 6 voters and the version, in which we seek σ preferred by a majority of non-abstaining voters for 6/7 voters.

4/5 voters: Theorem: If it was poly-time solvable, then the Kemeny consensus ranking problem for 3 voters, currently an open problem, is too!

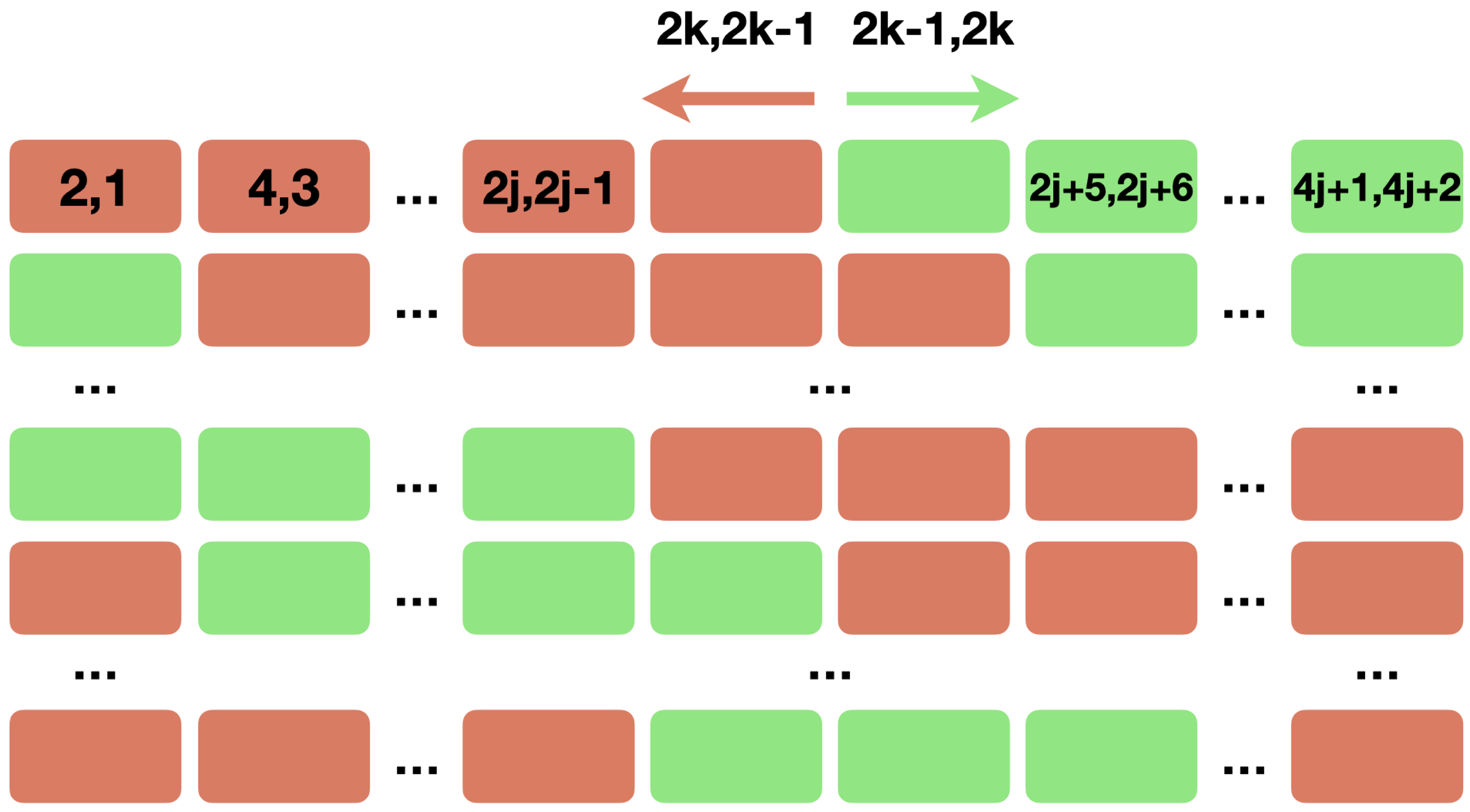
2/3 voters: Easy, as topologically sorted rankings are weakly/strongly popular.

Popular rankings may not exist

Van Zuylen et al. [1]: A popular ranking has to be topologically sorted, but even without a Condorcet paradox, a popular ranking may not exist.

Definition: c-sorted ranking: If candidate a is ranked above candidate b, at least a c-fraction of the voters prefer a to b. *topologically sorted rankings are* $\frac{1}{2}$*-sorted*

Theorem: $\frac{3}{4}$ **is the smallest constant** c **such that any** c**-sorted ranking is weakly-popular.**



This is a construction with $4j$ voters to show that for any $\epsilon > 0$, the $\frac{3}{4} - \epsilon$-sorted ranking may not be weakly popular.

When the abstaining voters are not in your favour :(

Context: Two rankings, σ and π and a majority of all voters who don't abstain prefers π to σ. So σ is not strongly popular.

Theorem: If the abstaining voters are themselves Condorcet-paradox-free, then σ is not even weakly popular!

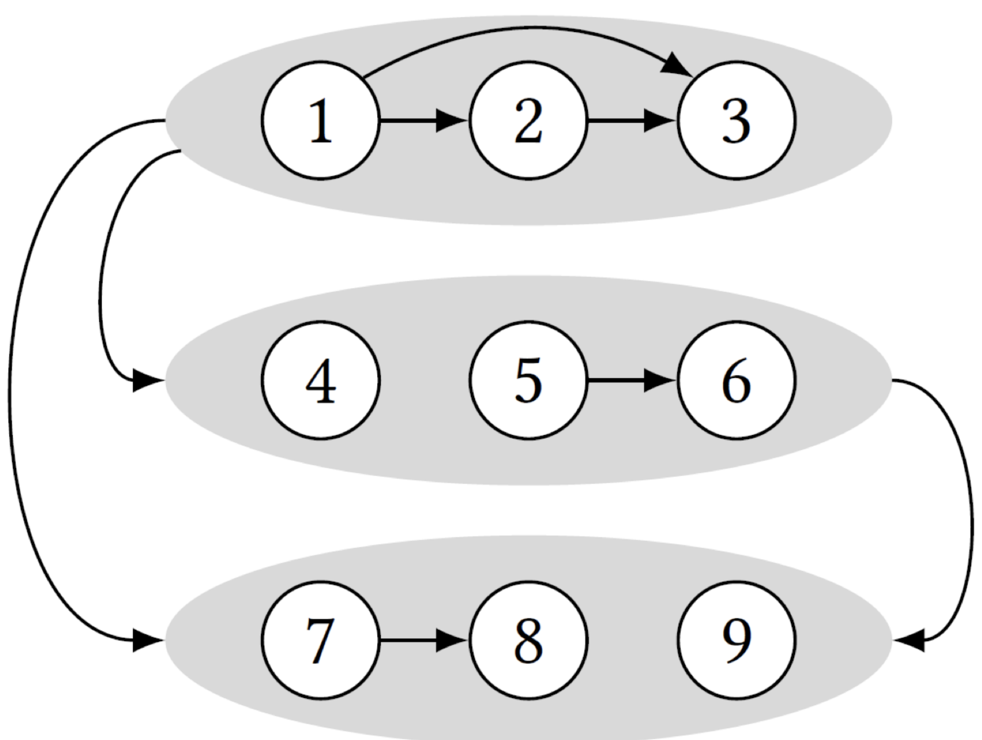
Consequences:

1. **Weak and strong popularity are the same if** $n \leq 5$.
2. For at least 6 voters, they are not in general the same.

The following instance has a ranking that is weakly popular but not strongly popular.

$$\pi_{v_1} = [1,2,3],[6,4,5],[8,9,7], \pi_{v_2} = [2,3,1],[4,5,6],[9,7,8]$$
$$\pi_{v_3} = [3,1,2],[5,6,4],[7,8,9], \pi_{v_4} = [1,2,3],[4,5,6],[7,8,9]$$
$$\pi_{v_5} = [1,2,3],[5,4,6],[9,7,8], \pi_{v_6} = [1,2,3],[5,6,4],[7,9,8]$$

The majority graph of the instance:



[1] Anke van Zuylen, Frans Schalekamp, and David P. Williamson. 2014. Popular ranking. *Discrete Applied Mathematics* 165 (2014), 312–316.

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