Coalitional Affinity Games

(Extended Abstract)

Simina Brânzei
Cheriton School of Computer Science
University of Waterloo
Waterloo, ON, Canada
sbranzei@uwaterloo.ca

Kate Larson
Cheriton School of Computer Science
University of Waterloo
Waterloo, ON, Canada
klarson@uwaterloo.ca

ABSTRACT

We present and analyze *coalitional affinity games*, a family of hedonic games that explicitly model the value that an agent receives from being associated with other agents. We provide a characterization of the social-welfare maximizing coalition structures, and study the stability properties of affinity games, using the core solution concept. Interestingly, we observe that members of the core do not necessarily maximize social welfare. We introduce a new measure, the *stability-gap* to capture this difference. Using the stability gap, we show that for an interesting class of coalitional affinity games, the difference between the social welfare of a stable coalition structure and a social-welfare maximizing coalition structure is bounded by a factor of 2, and that this bound is tight.

Categories and Subject Descriptors

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General Terms

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1. INTRODUCTION

Imagine the following scenario. You are organizing a party and have to come up with a seating arrangement, but this arrangement should take into consideration the relationships between the guests. For example, Alice may be good friends with Bob and would like to sit with him. However, Alice may also be feuding with Chris, Bob's best friend, and will refuse to remain at any table with Chris. You want to make all the guests as happy as possible, but you also do not want guests changing the seating arrangement when they arrive. How should you assign the guests to tables?

In this paper we propose a model which explicitly captures the value, which we call the *affinity*, that an agent receives from being associated with another agent. In particular we study situations where an agent is interested in belonging to groups (coalitions) that contain agents for which it has high affinity, while avoiding groups that contain agents for which it has negative affinity. By placing our affinity model into the context of non-transferable utility coalitional games [3], we are able to characterize and compare cooperative structures (*i.e.* coalition structures) that maximize social welfare with those that are stable (*i.e.* in the core). We argue that our affinity model is a rich representation that can be used to model many situations, while at the same time provides enough structure for interesting characterizations.

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2. THE MODEL

Let there be a set of agents $N = \{x_1, \ldots, x_n\}$. For any pair of agents, i and j, we denote the *affinity* that agent x_i has for x_j as $a(x_i, x_j) \in \mathbb{R}$ which represents the value that agent x_i receives from being associated with agent x_j. We represent the agents and their affinities with an *affinity graph.*

Definition 1 *An* affinity graph*, $A = (N, E)$, is a weighted directed graph where N is a set of agents, $N = \{x_1, \ldots, x_n\}$, edge $(x_i, x_j) \in E$ represents an affinity relation between agents x_i and x_j, and weight $a(x_i, x_j) \in \mathbb{R}$ is the value that agent x_i receives from being associated with agent x_j.*[1] *If for all $(x_i, x_j), (x_j, x_i) \in E$, $a(x_i, x_j) = a(x_j, x_i)$ then the affinity graph is* symmetric.

Given an affinity graph, we are interested in understanding how agents will choose to interact with each other by forming coalitions. We define the *utility* of agent x_i from belonging in coalition $S \subseteq N$ as follows:

$$u(S, x_i) = \begin{cases} 0 & \text{if } |S| = 1 \\ \sum_{(x_i, x_j) \in A | x_j \in S} a(x_i, x_j) & \text{otherwise} \end{cases}$$

It is now possible to define a *coalitional affinity game.* which is modeled as a characteristic function game with non-transferable utility [3].

Definition 2 *Given an affinity graph $A = (N, E)$, the* coalitional affinity game*, $G(A)$, is the pair $\langle N, v \rangle$ where N is the set of agents defined by A, and for any $S \subseteq N$, $v(S) \subset \mathbb{R}^{|S|}$, such that for $x_i \in S$, $v_i(S) = u(S, x_i)$.*

While the value function of a coalitional affinity game returns a vector given a coalition, where entry i is the value that agent x_i receives from being in the coalition, we will

[1] If $(x_i, x_j) \notin E$ then $a(x_i, x_j) = 0$.

sometimes abuse notation and say that the value of coalition S is $V(S) = \sum_{x_i \in S} v_i(S) = \sum_{x_i, x_j \in S} a(x_i, x_j)$.

As is standard, we define a *coalition structure*, P, to be a partition of the set of agents into coalitions. We are interested in properties of different coalition structures, such as the *social welfare* of a coalition structure and whether it is *stable* (*i.e* in the core).

Definition 3 *The* social welfare *of coalition structure* $P = (S_1, \ldots, S_m)$ *is* $SW(P) = \sum_{i=1}^{m} \sum_{x_j \in S_i} u(S_i, x_j)$.

Definition 4 *A coalition structure* $P = (S_1, \ldots, S_m)$ *is in the* core *if there is no coalition* $B \subseteq N$ *such that* $\forall x \in B$*, if* $x \in S_i$ *then* $u(B, x) \geq u(S_i, x)$ *and for some* j*,* $1 \leq j \leq m$*,* $\exists y \in S_j$ *such that* $u(B, y) > u(S_j, y)$.

3. MAXIMIZING SOCIAL WELFARE

Given an affinity graph one question we are interested in is how the agents should be assigned to coalitions so as to maximize the social welfare. We note the relationship between the social-welfare maximizing coalition structure and the *minimal cut* of the affinity graph. This observation provides us with a complete characterization of the structure of social-welfare maximizing coalition structures.

Lemma 1 *Given an affinity graph* $A = (N, E)$*, and a fixed value* k*,* $1 \leq k \leq n$*, the minimal* k*-cut has the highest social welfare amongst all coalition structures of size* k.

Theorem 1 *Let* P_k^* *be the minimal* k*-cut of affinity graph* $A = (N, E)$*. Then the social welfare maximizing coalition is* $P^* = max_k[P_1^*, \ldots, P_n^*]$.

Theorem 1 provides a characterization of the social-welfare maximizing coalition structure for an arbitrary affinity graph. If the affinity graph is *symmetric* then we are able to describe some additional properties of the social-welfare maximizing coalition structure, P^*.

Theorem 2 *Let* $A = (N, E)$ *be a symmetric affinity graph, and let* $P^* = (S_1^*, \ldots, S_k^*)$ *be the social-welfare maximizing coalition structure. Then, for any* $S_i^* \in P^*$*, any cut of* S_i^* *is non-negative.*

4. STABILITY AND AFFINITY GAMES

In this section we study the stability of different coalition structures for affinity games, and in particular how the existence of stable coalition structures (where we use the core as our benchmark for stability) depend on the underlying affinity graph. We first note that for a general affinity graph, the core may be empty. While for general affinity graphs, the core may be empty, there are interesting affinity-graph structures for which we can obtain positive results. Our first two results follow immediately from the definition of the utility functions of the agents.

Theorem 3 *Given affinity graph* $A = (N, E)$ *such that for all* $(x_i, x_j) \in E$*,* $a(x_i, x_j) \geq 0$*, the grand coalition is in the core.*

Theorem 4 *Given affinity graph* $A = (N, E)$ *such that for all* $(x_i, x_j) \in E$*,* $a(x_i, x_j) \leq 0$ *then the coalition structure* $P = (\{x_1\}, \{x_2\}, \ldots, \{x_n\})$ *is in the core.*

We contrast Theorem 4 with the transferable-utility case. In particular, if agents have transferable utility then the core is non-empty if and only if there is no negative cut [1].

5. STABILITY AND SOCIAL WELFARE

In this section we study the relationship between the core and social welfare. Our first observation is that for affinity games, if a coalition structure, P, is in the core, then this does not imply that it must also be a social-welfare maximizing coalition structure. While the core members may not maximize social welfare, and the social welfare maximizing coalition structure may not be in the core, we are still interested in understanding the relationship between the two concepts. In particular, we are interested in understanding the potential loss of social welfare that comes from being in the core. We call this loss the *stability gap*.

Definition 5 *Let* $A = (N, E)$ *be an affinity graph with a non-empty core. Let* P^* *be the social-welfare maximizing coalition structure, and let* P^C *be a member of the core. The* stability gap *of* P^C *is*

$$\mathrm{Gap}(P^C) = \frac{SW(P^*)}{SW(P^C)}.$$

If $\mathrm{Gap}(P^C) = 1$ then P^C is a social-welfare maximizing coalition structure. If $\mathrm{Gap}(P^C) > 1$ then P^C sacrifices social welfare in exchange for stability. For a given affinity graph A we are particularly interested in measuring the stability gap of the member of the core with the *lowest* social welfare,

$$\mathrm{Gap_{min}(A)} = \frac{\mathrm{SW(P^*)}}{\mathrm{min_{P \in Core(A)}\, SW(P)}}$$

and the stability gap of the member of the core with the *highest* social welfare

$$\mathrm{Gap_{max}(A)} = \frac{\mathrm{SW(P^*)}}{\mathrm{max_{P \in Core(A)}\, SW(P)}}.$$

Clearly, $\mathrm{Gap_{min}}(A) \geq \mathrm{Gap_{max}}(A)$ for any affinity graph with a non-empty core. We also note that $\mathrm{Gap_{min}}(A)$ has parallels with the *price of anarchy* while $\mathrm{Gap_{max}}(A)$ has parallels with the *price of stability* [2].

Theorem 5 *Let* $A = (N, E)$ *be an affinity graph with a non-empty core. Then,* $\mathrm{Gap_{max}}(A)$ *can be unbounded.*

While Theorem 5 is distressing since it states that even core members with the highest social welfare can still be arbitrarily worse that the maximum social welfare, if we place *some* restrictions on the affinity graph, then the sacrifice in terms of social welfare is significantly reduced.

Theorem 6 *Let* $A = (N, E)$ *be a* symmetric *affinity graph with a non-empty core. Then* $\mathrm{Gap_{min}}(A)$ *is bounded by 2, and this bound is tight.*

6. REFERENCES

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