Homogeneity and Monotonicity of Distance-Rationalizable Voting Rules

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ABSTRACT

Distance rationalizability is a framework for classifying voting rules by interpreting them in terms of distances and consensus classes. It can also be used to design new voting rules with desired properties. A particularly natural and versatile class of distances that can be used for this purpose is that of *votewise* distances [12], which "lift" distances over individual votes to distances over entire elections using a suitable norm. In this paper, we continue the investigation of the properties of votewise distance-rationalizable rules initiated in [12]. We describe a number of general conditions on distances and consensus classes that ensure that the resulting voting rule is homogeneous or monotone. This complements the results of [12], where the authors focus on anonymity, neutrality and consistency. We also introduce a new class of voting rules, that can be viewed as "majority variants" of classic scoring rules, and have a natural interpretation in the context of distance rationalizability.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Theory

Keywords

voting, distance rationalizability, monotonicity, homogeneity

1. INTRODUCTION

In collaborative environments, agents often need to make joint decisions based on their preferences over possible outcomes. Thus, social choice theory emerges as an important tool in the design and analysis of multiagent systems [13]. However, voting procedures that have been developed for human societies are not necessarily optimal for artificial agents and vice versa. For instance, there are voting rules that allow for polynomial-time winner determination (and thus are suitable for autonomous agents), yet have been deemed too complicated to be comprehended by an average voter in many countries; an example is provided by Single Transferable Vote. Further, unlike an electoral committee in a human society, the designer of a multi-agent voting system is usually unencumbered by legacy issues or the need to appeal to the general public, and can choose a voting rule that is most suitable for the application at hand, or, indeed, design a brand-new voting rule that satisfies the axioms that he deems important.

A recently proposed *distance rationalizability* framework [17, 10, 12, 11] is ideally suited for such settings. Under this framework, one can define a voting rule by a class of consensus elections and a distance over elections; the winners of an election are defined as the winners in the nearest consensus. In other words, for any election this rule seeks the most similar election with an obvious winner (where the similarity is measured by the given distance), and outputs its winner. Examples of natural consensus classes include *strong unanimity consensus*, where all voters agree on the ranking of all candidates, and *Condorcet consensus*, where there is a candidate that is preferred by a majority of voters to every other candidate. Combined with the *swap distance* (defined as the number of swaps of adjacent candidates that transforms one election into the other), these consensus classes produce, respectively, the Kemeny rule and the Dodgson rule.

The examples above illustrate that the distance rationalizability framework can be used to interpret (rationalize) existing voting rules in terms of a search for consensus (see [17] for a comprehensive list of results in this vein). It can also be applied to design new voting rules: for instance, in [10] the authors investigate the rule obtained by combining the Condorcet consensus with the Hamming distance. Further, by decomposing a voting rule into a consensus class and a distance we can hope to gain further insights into the structure of the rule. This decomposition is especially useful when the distance reflects changes in voters' opinions in a simple and transparent way. This is the case for the so-called *votewise* distances introduced in [12]. These are distances over elections that are obtained by aggregating distances between individual votes using a suitable norm, such as ℓ_1 or ℓ_∞. Indeed, paper [12] shows that one can derive conclusions about anonymity, neutrality and consistency of votewise rules (i.e., rules rationalized via votewise distances) from the basic properties of the underlying distances on votes, norms, and consensus classes.

In this paper we pick up this thread of research and study two important properties of voting rules not considered in [12], namely, monotonicity and homogeneity. Briefly put, monotonicity ensures that providing more support to a winning candidate cannot turn him into a loser, and homogeneity ensures that the result of an election depends on the proportions of particular votes and not on their absolute counts. Both properties are considered highly desirable for reasonable voting rules. We focus on the four standard consensus classes considered in the previous work (strong unanimity $\mathcal{S}$, una-

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nimity $\mathcal{U}$, majority $\mathcal{M}$ and Condorcet $\mathcal{C}$) and ℓ_1- and ℓ_∞-norms. Our aim is to identify distances on votes that, combined with these norms and consensus classes, produce homogeneous and/or monotone rules.

Of the four consensus classes considered in this paper, the majority consensus $\mathcal{M}$ received relatively little attention in the existing literature. Thus, in order to study the homogeneity and monotonicity of the rules that are distance-rationalizable with respect to $\mathcal{M}$, we need to develop a better understanding of such rules. Our main result here is a characterization of all voting rules that are rationalizable with respect to $\mathcal{M}$ via a neutral distance on votes and the ℓ_1-norm. It turns out that such rules have a very natural interpretation: they are "majority variants" of classic scoring rules. This characterization enables us to analyze the homogeneity of the rules in this class, leading to a dichotomy result.

As argued above, a votewise distance-rationalizable rule can be characterized by three parameters: a distance on votes, a norm, and a consensus class. From this perspective, it is interesting to ask how much the voting rule changes if we vary one or two of these parameters. We provide two results that contribute to this agenda. First, we show that essentially any rule that is votewise-rationalizable with respect to $\mathcal{M}$ can also be rationalized with respect to $\mathcal{U}$, by modifying the norm accordingly. This enables us to answer a question left open in [11]. Second, we show that, for any consensus class and any distance on votes, replacing the ℓ_1-norm with the ℓ_∞-norm produces a voting rule that is an n-approximation of the original rule, where n is the number of voters. For the Dodgson rule, this transformation produces a rule that is polynomial-time computable and homogeneous. This line of work also emphasizes the constructive aspect of the distance rationalizability framework: we are able to derive new voting rules with attractive properties by combining a known consensus class with a known distance measure in a novel way.

Related work. The formal theory of distance rationalizability was put forward by Meskanen and Nurmi [17], though the idea, in one shape or another, appeared in earlier papers as well (see, e.g., [18, 2, 16, 15]). The goal of Meskanen and Nurmi was to seek best possible distance-rationalizations of classic voting rules. This research program was advanced by Elkind, Faliszewski, and Slinko [10, 12, 11], who, in addition to further classification work, also suggested studying general properties of distance-rationalizable voting rules. In particular, in [11] they identified an interesting and versatile class of distances—which they called votewise distances—that lead to rules whose properties can be meaningfully studied.

The study of distance rationalizability is naturally related to the study of another—much older—framework, which is based on interpreting voting rules as maximum likelihood estimators (the MLE framework). This framework could be dated back to Condorcet and has been pursued by Young [21], and, more recently, in [8], [7], and [19]. To date, most of the research on the MLE framework was concerned with determining which of the existing voting rules can be interpreted as maximum likelihood estimators; however, paper [19] also shows that the MLE approach can be used to deduce new useful voting rules.

This paper is loosely related to the work of Caragiannis et al. [6], where the authors give a monotone, homogeneous voting rule that calculates scores which approximate candidates' Dodgson scores up to an $O(m \log m)$ multiplicative factor, where m is the number of candidates. The relation to our work is twofold. First, we also focus on monotonicity and homogeneity, although our goal is to come up with a general method of constructing monotone and homogeneous rules and not to approximate particular rules. Second, in the course of our study we discover a homogeneous and polynomial-time computable voting rule that approximates the scores of candidates in Dodgson elections up to a multiplicative factor of n, where n is the number of voters. While the number of voters is usually much bigger than the number of candidates, and thus our algorithm is usually inferior to that of [6], it illustrates the power of the distance rationalizability framework.

The rest of the paper is organized as follows. Section 2 contains preliminary definitions regarding voting rules in general and the distance-rationalizability framework specifically. In Section 3 we provide a detailed study of rules that are votewise rationalizable with respect to the majority consensus. Sections 4 and 5 present our results on, respectively, homogeneity and monotonicity of votewise rules. We conclude in Section 6. We omit most proofs.

2. PRELIMINARIES

2.1. Basic notation. An *election* is a pair $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ is the set of *candidates* and $V = (v_1, \dots, v_n)$ is the set of *voters*. Voter v_i is identified with a total order $\succ_i$ over C, which we will refer to as v_i's *preference order*, or *ranking*. We write $c_j \succ_i c_\ell$ to denote that voter v_i prefers c_j to c_ℓ. We denote by $\mathcal{P}(C)$ the set of all preference orders over C. For a voter v, we denote by $\mathrm{top}(v)$ the candidate ranked first by v, and set $\mathcal{P}(C, c) = \{v \in \mathcal{P}(C) \mid \mathrm{top}(v) = c\}$. For any voter $v_i \in V$ and a candidate $c \in C$, we denote by $\mathrm{rank}(v_i, c)$ the position of c in v_i's ranking. For example, if $\mathrm{top}(v_i) = c$ then $\mathrm{rank}(v_i, c) = 1$. A *voting rule* is a mapping $\mathcal{R}$ that for any election (C, V) outputs a non-empty subset of candidates $W \subseteq C$ called the *election winners*. Given an election $E = (C, V)$ and $s \in \mathbb{N}$, we denote by sE the election (C, sV), where sV is obtained by concatenating s copies of V.

Two important properties of voting rules that will be studied in this paper are homogeneity and monotonicity.

Homogeneity. A voting rule $\mathcal{R}$ is *homogeneous* if for each election $E = (C, V)$ and each positive $s \in \mathbb{N}$ we have $\mathcal{R}(E) = \mathcal{R}(sE)$.

Monotonicity. A voting rule $\mathcal{R}$ is *monotone* if for every election $E = (C, V)$, every $c \in \mathcal{R}(E)$ and every $E' = (C, V')$ obtained from E by moving c up in some voters' rankings (but not changing their rankings in any other way) we have $c \in \mathcal{R}(E')$.

2.2. Voting rules. We will now define the classic voting rules discussed in this paper, namely, scoring rules, (Simplified) Bucklin, and Dodgson.

Scoring rules In this paper, we will use a somewhat nonstandard definition of a scoring rule. Any vector $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{R}_+ \cup \{0\})^m$ defines a partial voting rule $\mathcal{R}_\alpha$ for elections with a fixed number m of candidates. Under this rule, for each preference order $u \in \mathcal{P}(C)$, $|C| = m$, a candidate $c \in C$ gets $\alpha_{\mathrm{rank}(u,c)}$ points (as is standard) and these values are summed up to obtain the score of c. However, we define the winners to be the candidates with the lowest score (rather than the highest, as is typical when discussing scoring rules). A sequence of scoring vectors $(\alpha^{(m)})_{m \in \mathbb{N}}$, where $\alpha^{(m)} \in (\mathbb{R}_+ \cup \{0\})^m$, defines a voting rule $\mathcal{R}_{(\alpha^{(m)})}$ which is applicable for any number of alternatives.

For example, in this notation the Borda rule is defined by a family of scoring vectors $\alpha^{(m)} = (0, 1, \dots, m-1)$ and the k-approval is the family of scoring vectors given by $\alpha_i^{(m)} = 0$ for $i \le k$, $\alpha_i^{(m)} = 1$ for $i > k$. The 1-approval rule is also known as Plurality. The traditional model, where the winners are the candidates with the highest score, can be converted to our notation by setting $\alpha'_i = \alpha_{\max} - \alpha_i$, where $\alpha_{\max} = \max_{i=1}^m \alpha_i$. The reason for this deviation is that in the context of this paper it will be much more convenient to speak of minimizing one's score. Note that, in general, we do not require $\alpha_1 \leq \cdots \leq \alpha_m$, although this assumption is obviously required for monotonicity.

Note that vectors $(\alpha_1, \ldots, \alpha_m)$ and $(\beta\alpha_1, \ldots, \beta\alpha_m)$ define the same voting rule for any $\beta > 0$; the same is true for $(\alpha_1, \ldots, \alpha_m)$ and $(\alpha_1+\gamma, \ldots, \alpha_m+\gamma)$ for any $\gamma \geq 0$. Thus, in what follows, we normalize the scoring vectors by requiring their smallest coordinate to be 0, and the smallest non-zero coordinate to be 1.

Bucklin Under the *Bucklin rule*, we first determine the smallest value of k such that some candidate is ranked in top k positions by more than half of the voters. The winner(s) are the candidates that are ranked in the top k positions the maximum number of times. Under the *Simplified Bucklin rule* $\mathcal{R}_{sB}$, the winners are all candidates ranked in top k positions by a majority of voters.

Dodgson To define the Dodgson rule, we need to introduce the concept of a *Condorcet winner*. A Condorcet winner is a candidate that is preferred to any other candidate by a majority of voters. The *Dodgson score* of a candidate c is the smallest number of swaps of adjacent candidates that have to be performed on the votes to make c the Condorcet winner. The winner(s) under the Dodgson rule are the candidates with the lowest Dodgson score.

2.3. Norms and Metrics. A *norm* on $\mathbb{R}^n$ is a mapping $N : \mathbb{R}^n \to \mathbb{R}$ that has the following properties for all $x, y \in \mathbb{R}^n$: (1) $N(\alpha x) = |\alpha| N(x)$ for all $\alpha \in \mathbb{R}$; (2) $N(x) \geq 0$ and $N(x) = 0$ if and only if $x = (0, \ldots, 0)$; (3) $N(x + y) \leq N(x) + N(y)$.

Two important properties of norms that will be of interest to us are symmetry and monotonicity. We say that a norm N is *symmetric* if for each permutation $\sigma : [1, n] \to [1, n]$ it holds that $N(x_1, \ldots, x_n) = N(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. For monotonicity, we make use of the definition proposed in [3]. Specifically, we say that a norm N is *monotone in the positive orthant*, or $\mathbb{R}^n_+$*-monotone*, if for any two vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ such that $x_i \leq y_i$ for all $i \leq n$ we have $N(x_1, \ldots, x_n) \leq N(y_1, \ldots, y_n)$.

A well-studied class of norms are the ℓ_p*-norms* given by

$$\ell_p(x_1, \ldots, x_n) = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

for $p \in \mathbb{N}$. This definition can be extended to $p = +\infty$ by setting $\ell_\infty(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$. Observe that for any $p \in \mathbb{N} \cup \{+\infty\}$ the ℓ_p norm is, in fact, a family of norms, i.e., it is well-defined on $\mathbb{R}^i$ for any $i \in \mathbb{N}$. Also, any such norm is clearly symmetric and monotone in the positive orthant.

A *metric*, or *distance*, on a set X is a mapping $d : X^2 \to \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$: (1) $d(x, y) \geq 0$; (2) $d(x, y) = 0$ if and only if $x = y$; (3) $d(x, y) = d(y, x)$; (4) $d(x, z) \leq d(x, y) + d(y, z)$. A function that satisfies conditions (1), (3) and (4), but not (2), is called a *pseudodistance*.

Given a distance d on X and a norm N on $\mathbb{R}^n$, we can define a distance $N \circ d$ on X^n by setting

$$(N \circ d)(\mathbf{x}, \mathbf{y}) = N(d(x_1, y_1), \ldots, d(x_n, y_n))$$

for all vectors $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in X^n$. A distance defined in this manner is called a *product metric*.

In this paper, we will study distances over votes and their extensions to distances over elections via product metrics. Some examples of distances over votes are given by the *discrete distance* d_{discr}, the *swap distance* d_{swap}, and the *Sertel distance* d_{ser}, defined as follows. For any set of candidates C and any $u, v \in \mathcal{P}(C)$, we set $d_{\mathrm{discr}}(u, v) = 0$ if $u = v$ and $d_{\mathrm{discr}}(u, v) = 1$ otherwise. The swap distance d_{swap} is given by $d_{\mathrm{swap}}(u, v) = \frac{1}{2}|\{(c, c') \in C^2 \mid c \succ_u c', c' \succ_v c\}|$, where $\succ_u$ and $\succ_v$ are the preference orders associated with u and v, respectively. The Sertel distance between u and v is defined as the smallest value of i such that for all $j > i$ voters u and v rank the same candidate in position j.

A distance d on $\mathcal{P}(C)$ is called *neutral* if for any $u, v \in \mathcal{P}(C)$ and any permutation $\pi : C \to C$ we have $d(u, v) = d(\pi(u), \pi(v))$, where $\pi(x)$ denotes the vote obtained from x by moving candidate c_i into position $\mathrm{rank}(x, \pi(c_i))$, for $i = 1, \ldots, |C|$. Clearly, all distances listed above are neutral.

2.4. Distance Rationalizability. Intuitively, a consensus class is a collection of elections with an obvious winner. Formally, a *consensus class* is a pair $(\mathcal{E}, \mathcal{W})$ where $\mathcal{E}$ is a set of elections and $\mathcal{W} \colon \mathcal{E} \to C$ is a function that for each election $E \in \mathcal{E}$ outputs the alternative called the *consensus winner*. The following four consensus classes have been considered in the previous work on distance rationalizability:

Strong unanimity. Denoted $\mathcal{S}$, contains elections $E = (C, V)$ where all voters report the same preference order. The consensus winner is the candidate ranked first by all voters.

Unanimity. Denoted $\mathcal{U}$, contains all elections $E = (C, V)$ where all voters rank the same candidate first. The consensus winner is the candidate ranked first by all voters.

Majority. Denoted $\mathcal{M}$, contains all elections $E = (C, V)$ where more than half of the voters rank the same candidate first. The consensus winner is the candidate ranked first by the majority of voters.

Condorcet. Denoted $\mathcal{C}$, contains all elections $E = (C, V)$ with a Condorcet winner. The consensus winner is the Condorcet winner.

We say that a voting rule $\mathcal{R}$ is *compatible* with a consensus class $\mathcal{K}$ if for any consensus election $E \in \mathcal{K}$ it holds that $\mathcal{W}(E) = \mathcal{R}(E)$. Similarly, $\mathcal{R}$ is said to be *weakly compatible* with $\mathcal{K}$ if for any $E \in \mathcal{K}$ we have $\mathcal{W}(E) \in \mathcal{R}(E)$. Essentially all well-known voting rules are weakly compatible with $\mathcal{S}, \mathcal{U}$ and $\mathcal{M}$, but there are rules that are not compatible with any of these consensus classes (e.g., k-approval for $k > 1$). The rules that are compatible with $\mathcal{C}$ are also known as *Condorcet-consistent* rules; we use the term "compatibility" rather than "consistency" to avoid confusion with the consistency property of voting rules.

We are now ready to define the concept of distance rationalizability. Our definition below is taken from [12], which itself was inspired by [17, 10].

DEFINITION 2.1. *Let d be a distance over elections and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class. The* $(\mathcal{K}, d)$-score *of a candidate c in an election E is the distance (according to d) between E and a closest election $E' \in \mathcal{E}$ such that $c \in \mathcal{W}(E')$. A voting rule $\mathcal{R}$ is* distance-rationalizable *via a consensus class $\mathcal{K}$ and a distance d over elections (is* $(\mathcal{K}, d)$-rationalizable*) if for each election E the set $\mathcal{R}(E)$ consists of all candidates with the smallest $(\mathcal{K}, d)$-score.*

A particularly useful class of distances to be used in distance rationalizability constructions is that of *votewise* distances, which are obtained by combining a distance over votes with a suitable norm. Formally, given a set of candidates C, consider a distance d over $\mathcal{P}(C)$ and a family of norms $\mathcal{N} = (N_i)_{i=1}^{\infty}$, where N_i is a norm over $\mathbb{R}^i$. We define a distance $\widehat{d^{\mathcal{N}}}$ over elections with the set of candidates C as follows: for any $E = (C, V)$, $E' = (C, V')$, we set $\widehat{d^{\mathcal{N}}}(E, E') = (N_i \circ d)(V, V')$ if $|V| = |V'| = i$, and $\widehat{d^{\mathcal{N}}}(E, E') = +\infty$ if $|V| \neq |V'|$. A voting rule $\mathcal{R}$ is said to be $\mathcal{N}$*-votewise distance-rationalizable* (or simply $\mathcal{N}$*-votewise*) with respect to a consensus class $\mathcal{K}$ if there exists a distance d over votes such that $\mathcal{R}$ is $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$-rationalizable. When $\mathcal{N}$ is the ℓ_p-norm for some $p \in \mathbb{N} \cup \{+\infty\}$, we write $\widehat{d^p}$ instead of $\widehat{d^{\ell_p}}$, and when $\mathcal{N} = \ell_1$, we omit the index altogether and write $\widehat{d}$. It is known that any voting rule is distance-rationalizable with respect to any consensus class that it is compatible with [12]. However, some voting rules are not $\mathcal{N}$-votewise distance-rationalizable with respect to standard consensus classes for any reasonable norm $\mathcal{N}$ [11].

Let us now consider some examples of distance-rationalizations of voting rules. Nitzan [18] was the first to show that Plurality is $(\mathcal{U}, \widehat{d}_{\rm discr})$-rationalizable and Borda is $(\mathcal{U}, \widehat{d}_{\rm swap})$-rationalizable. It is easy to see that Dodgson is $(\mathcal{C}, \widehat{d}_{\rm swap})$-rationalizable and Kemeny is $(\mathcal{S}, \widehat{d}_{\rm swap})$-rationalizable. The distance $\widehat{d^{\infty}_{\rm ser}}$, combined with the majority consensus, yields the Simplified Bucklin rule [12].

For any set of candidates C with $|C| = m$ and a scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, paper [12] defines a (pseudo)distance $d_\alpha(u, v)$ on $\mathcal{P}(C)$ as as $d_\alpha(u, v) = \sum_{j=1}^{m} |\alpha_{{\rm rank}(u,c_j)} - \alpha_{{\rm rank}(v,c_j)}|$, and shows that if—in our notation—$\alpha_1 \le \alpha_k$ for all $k > 1$ then $\mathcal{R}_\alpha$ is $(\mathcal{U}, \widehat{d}_\alpha)$-(pseudo)distance-rationalizable.

3. $\mathcal{M}$-SCORING RULES

The majority consensus is a very natural notion of agreement in the society. However, it has received little attention in the literature so far. Here we will show that it leads to a series of interesting rules with nice properties.

DEFINITION 3.1. *For any scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, let $\mathcal{M}$-$\mathcal{R}_\alpha$ be a partial voting rule defined on the profiles with m alternatives as follows. Given an election $E = (C, V)$ with $|C| = m$ and $V = (v_1, \dots, v_n)$, for each candidate $c \in C$, we define the $\mathcal{M}$-score of c as the sum of $\lfloor \frac{n}{2} \rfloor + 1$ lowest values among $\alpha_{{\rm rank}(v_1,c)}, \dots, \alpha_{{\rm rank}(v_n,c)}$. The winners are the candidates with the lowest $\mathcal{M}$-$\mathcal{R}_\alpha$ scores. As in the classic case, a family of scoring vectors $(\alpha^{(i)})_{i\in\mathbb{N}}$ defines an $\mathcal{M}$-scoring rule $\mathcal{M}$-$\mathcal{R}_{(\alpha^{(i)})}$.*

We will refer to voting rules from Definition 3.1 as $\mathcal{M}$-*scoring rules*. Such rules (or their slight modifications) are often used for score aggregation in real-life settings; for example, it is not unusual for a professor to grade the students on the basis of their five best assignments out of six or in some sport competitions to select winners on the basis of one or more of their best attempts.

It is not hard to see that $\mathcal{M}$-Plurality is equivalent to Plurality: under both rules, the winners are the candidates with the maximum number of first-place votes. However, essentially all other scoring rules differ from their $\mathcal{M}$-counterparts.

PROPOSITION 3.2. *Consider a normalized scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$. The rule $\mathcal{M}$-$\mathcal{R}_\alpha$ coincides with $\mathcal{R}_\alpha$ if and only if (i) $\alpha_1 = \dots = \alpha_m$ or (ii) $\alpha_i = 0$ for some $i \in \{1, \dots, m\}$ and $\alpha_j = 1$ for all $j \ne i$.*

The $\mathcal{M}$-scoring rules tend to ignore extremely negative opinions. Therefore, intuitively, they are less susceptible to manipulation: if a voter v ranks a candidate c lower than the majority of other voters, v cannot manipulate against c by moving her to the bottom of their ranking. In this section we will show that these rules are also very interesting from the distance rationalizability point of view: it turns out that they essentially coincide with the class of rules that are ℓ_1-votewise rationalizable with respect to $\mathcal{M}$.

We will first need to generalize a result from [12] to pseudodistances and weak compatibility.

PROPOSITION 3.3. *Any voting rule that is pseudodistance-rationalizable with respect to a consensus class $\mathcal{K}$ is weakly compatible with $\mathcal{K}$.*

Now, we can characterize $\mathcal{M}$-scoring rules that are (pseudo)distance-rationalizable with respect to $\mathcal{M}$.

PROPOSITION 3.4. *Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a normalized scoring vector. The rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is ℓ_1-votewise distance-rationalizable with respect to $\mathcal{M}$ if and only if $\alpha_1 = 0$, $\alpha_j > 0$ for all $j \ne 1$. Further, $\mathcal{M}$-$\mathcal{R}_\alpha$ is ℓ_1-votewise pseudodistance-rationalizable with respect to $\mathcal{M}$ if and only if $\alpha_1 = 0$.*

We remark that our proof generalizes to scoring rules and $\mathcal{U}$, thus answering a question left open in [10], namely, whether scoring rules with $\alpha_i = \alpha_j$ for $i, j > 1$ can be distance-rationalized (rather than pseudodistance-rationalized). Further, in [10] the authors consider only monotone scoring rules, i.e., rules that satisfy—in our notation—$\alpha_1 \le \dots \le \alpha_m$, while our result holds for all scoring vectors.

The following lemma explains how to find an $\mathcal{M}$-consensus that is nearest to a given election with respect to a given ℓ_1-votewise distance.

LEMMA 3.5. *Let $\mathcal{R}$ be a voting rule that is $(\mathcal{M}, \widehat{d})$-rationalized. Let $E = (C, V)$ be an arbitrary election where $V = (v_1, \dots, v_n)$ and let $E' = (C, U)$ be an $\mathcal{M}$-consensus such that $\widehat{d}(E, E')$ is minimal among all n-voter $\mathcal{M}$-consensuses over C. Let $c \in C$ be the consensus winner of (C, U). Then, for each $i = 1, \dots, n$, either $u_i \in \arg\min_{x\in\mathcal{P}(C,c)} d(x, v_i)$ or $u_i = v_i$.*

Combining Lemma 3.5 with the argument in the proof of Theorem 4.9 in [12], we can show that the converse of Proposition 3.4 is also true: any voting rule that can be pseudodistance-rationalized via $\mathcal{M}$ and a neutral ℓ_1-votewise pseudodistance is, in fact, an $\mathcal{M}$-scoring rule. Also, any $\mathcal{M}$-scoring rule is obviously neutral. We can summarize these observations in the following theorem.

THEOREM 3.6. *Let $\mathcal{R}$ be a voting rule. There exists a neutral ℓ_1-votewise pseudodistance $\widehat{d}$ such that $\mathcal{R}$ is $(\mathcal{M}, \widehat{d})$-rationalizable if and only if $\mathcal{R}$ can be defined as an $\mathcal{M}$-scoring rule $\mathcal{M}$-$\mathcal{R}_{(\alpha^{(i)})}$ such that $\alpha_1^{(i)} \le \alpha_j^{(i)}$ for all $j > 1$ and all $i \in \mathbb{N}$.*

The discussion above suggests that using the majority consensus to rationalize a voting rule is similar to using the unanimity consensus, except that we only take into account the best "half-plus-one" votes. In fact, it turns out that under very weak assumptions we can translate a votewise rationalization of a rule with respect to $\mathcal{M}$ to a votewise rationalization of that rule with respect to $\mathcal{U}$.

DEFINITION 3.7. *Let $\mathcal{N} = (N_i)_{i=1}^{\infty}$ be a family of functions where for each i, $i \ge 1$, N_i is a mapping from $\mathbb{R}^i$ to $\mathbb{R}$. We define a family $\mathcal{N}^{\mathcal{M}} = (N_i^{\mathcal{M}})_{i=1}^{\infty}$ as follows. For each $i \ge 1$, $N_i^{\mathcal{M}}$ is a mapping from $\mathbb{R}^i$ to $\mathbb{R}$ given by*

$$N_i^{\mathcal{M}}(x_1, \dots, x_i) = N_{\lfloor \frac{i}{2} \rfloor + 1}(|x_{\pi(1)}|, \dots, |x_{\pi(\lfloor \frac{i}{2} \rfloor + 1)}|),$$

where π is a permutation of $[1, i]$ such that $|x_{\pi(1)}| \ge |x_{\pi(2)}| \ge \dots \ge |x_{\pi(i)}|$.

For a family of symmetric norms $\mathcal{N} = (N_i)_{i=1}^{\infty}$ that are monotone in the positive orthant, the family $\mathcal{N}^{\mathcal{M}}$ is also a family of norms, which we will call the *majority variant* of $\mathcal{N}$.

PROPOSITION 3.8. *Let $\mathcal{N} = (N_i)_{i=1}^{\infty}$ be a family of norms, where each N_i is a symmetric norm on $\mathbb{R}^i$ that is monotone in the positive orthant. Then the family $\mathcal{N}^{\mathcal{M}} = (N^{\mathcal{M}})_{i=1}^{\infty}$ is also a family of symmetric norms that are monotone in the positive orthant.*

As an immediate corollary we get the following result.

COROLLARY 3.9. *Let $\mathcal{N}$ be a family of symmetric norms that are monotone in the positive orthant and let d be a distance over votes. Let $\mathcal{R}$ be a voting rule that is $(\mathcal{M}, \widehat{d^{\mathcal{N}}})$-rationalizable. Then $\mathcal{R}$ is $(\mathcal{U}, \widehat{d^{\mathcal{N}\mathcal{M}}})$-rationalizable.*

This discussion illustrates that when a rule can be rationalized in several different ways, the right choice of a consensus class plays an important role, as it may greatly simplify the underlying norm and hence the distance. This is why it pays to keep a variety of consensus classes available and search for the best distance rationalizations possible. Corollary 3.9 also has a useful application: Paper [11] shows that STV[1] cannot be rationalized with respect to $\mathcal{S}, \mathcal{C}$ or $\mathcal{U}$ by any neutral $\mathcal{N}$-votewise distance, where $\mathcal{N}$ is a family of symmetric norms monotone in the positive orthant. Corollary 3.9 allows us to extend this result to $\mathcal{M}$, thus showing that STV cannot be rationalized by a "reasonable" votewise distance with respect to any of the standard consensus classes.

4. HOMOGENEITY

Homogeneity is a very natural property of voting rules. It can be interpreted as a weaker form of another appealing property, namely, consistency. Recall that a voting rule $\mathcal{R}$ is said to be *consistent* if for any two elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ with $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$ it holds that $\mathcal{R}(C, V_1 + V_2) = \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, where $V_1 + V_2$ denotes the concatenation of V_1 and V_2. Thus, loosely speaking, homogeneity imposes the same requirement as consistency, but only for the restricted case $V_1 = V_2$. Now, consistency is known to be hard to achieve: by Young's theorem [20], the only voting rules that are simultaneously anonymous, neutral and consistent are the scoring rules (or their compositions). In contrast, we will now argue that for many consensus classes and many values of $p \in \mathbb{N} \cup \{+\infty\}$, the rules that are ℓ_p-votewise rationalizable with respect to these classes are homogeneous. We start by showing that this is the case for ℓ_p, $p \in \mathbb{N}$, and consensus classes $\mathcal{S}$ and $\mathcal{U}$.

THEOREM 4.1. *For any distance d on votes, the voting rule $\mathcal{R}$ that is $(\mathcal{K}, \widehat{d^p})$-rationalizable for $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and $p \in \mathbb{N}$ is homogeneous.*

For $\mathcal{M}$, the conclusion of Theorem 4.1 is no longer true. However, we can fully characterize homogeneous rules that can be rationalized via $\mathcal{M}$ and a neutral ℓ_1-votewise pseudodistance (recall that by Theorem 3.6 all such rules are necessarily $\mathcal{M}$-scoring rules). For convenience, we state the following theorem for scoring vectors that satisfy $\alpha_1 \leq \cdots \leq \alpha_m$; it is not hard to show that this can be done without loss of generality.

THEOREM 4.2. *A voting rule $\mathcal{M}$-$\mathcal{R}_\alpha$ with a normalized scoring vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ that satisfies $\alpha_1 \leq \cdots \leq \alpha_m$ is homogeneous if and only if $\alpha_m = 1$ or $\alpha_{\lceil \frac{m}{2} \rceil} = 0$.*

PROOF SKETCH. Set $h = \lceil \frac{m}{2} \rceil$. We skip the easy proof of the case when $\alpha_m = 1$ (remember that the smallest non-zero coordinate is also 1). When $\alpha_h = 0$, then by the pigeonhole principle either there exists a candidate that is ranked in top h positions by a majority of voters (and its score is 0), or each candidate is ranked in top h positions by exactly half of the voters. In both cases, it is easy to show that the rule is homogeneous; we omit the details.

We will now show that if $\alpha_m > 1$ and $\alpha_h > 0$, the rule $\mathcal{M}$-$\mathcal{R}_\alpha$ is not homogeneous. We will only consider the case $\alpha_3 > 1$ (note that this implies $\alpha_2 = 1$); by careful padding, the construction in this proof can be modified to work for the general case.

Set $\alpha = \alpha_3$; we have $\alpha_1 = 0$, $\alpha_2 = 1$. We start by considering the case $m = 3$; later, we will generalize our construction to $m > 3$. Suppose first that $\alpha = \frac{p}{q}$ is a rational number written in its lowest terms. We construct an election $E = (C, V)$, where $C = \{a, b, c\}$ and V consists of the following votes:

1. $2p + q + 1$ votes $a \succ b \succ c$,
2. $2q + p + 1$ votes $b \succ c \succ a$, and
3. $p + q - 2$ votes $c \succ b \succ a$.

We observe that $|V| = 4(p + q)$, and the $\mathcal{M}$-scores of a and b are equal to p, and the $\mathcal{M}$-score of c is at least $p + q + 3$. Hence, both a and b are winners of E. On the other hand, in the election $2E = (C, 2V)$, the $\mathcal{M}$-scores of candidates a and b are, respectively, $(2q - 1)\alpha = 2p - \alpha$ and $2p - 1$. Since $\alpha > 1$, it cannot be the case that both a and b are winners of $2E$. Thus, in this case $\mathcal{M}$-$\mathcal{R}_\alpha$ is not homogeneous.

Now, if α is irrational, consider its continued fraction expansion $\alpha = (a_0, a_1, \ldots)$, and the successive convergents $\frac{h_i}{k_i}$, $i = 0, 1, \ldots$, where $h_0 = a_0$, $k_0 = 1$, $h_1 = a_1 h_0 + 1$, $k_1 = a_1$, and $h_i = a_i h_{i-1} + h_{i-2}$, $k_i = a_i k_{i-1} + k_{i-2}$ for $i \geq 2$. We know that for even values of i we have $\frac{h_i}{k_i} < \alpha$ and $|\alpha - \frac{h_i}{k_i}| < \frac{1}{k_i k_{i+1}}$. Also, it is not hard to show that for any $N > 0$ there exists an even value of i such that $k_{i+1} > N$. Thus, we pick an even i such that $k_{i+1} > \frac{2}{\alpha - 1}$ (recall that $\alpha > 1$). We obtain

$$0 < \alpha - \frac{h_i}{k_i} < \frac{1}{k_i k_{i+1}} < \frac{\alpha - 1}{2k_i}.$$

Now, set $p = h_i$, $q = k_i$, let $\varepsilon = \alpha - \frac{p}{q}$, and use the same construction as above. In E, the $\mathcal{M}$-score of a is $q\alpha$, the $\mathcal{M}$-score of b is $p < q\alpha$, and the $\mathcal{M}$-score of c exceeds that of a and b, so b is the unique winner. On the other hand, in $2E$ the $\mathcal{M}$-score of a is $(2q - 1)\alpha = 2p + 2q\varepsilon - \alpha$, while the $\mathcal{M}$-score of b is $2p - 1$. We have $\varepsilon < \frac{\alpha - 1}{2q}$, so a has a lower $\mathcal{M}$-score than b, and hence b cannot be the winner of $2E$. Thus, in this case, too, our rule is not homogeneous.

For $m > 3$, we modify this construction by adding $m - 3$ dummy candidates that each voter ranks last (in some arbitrary order). □

We have seen that many voting rules that are ℓ_1-votewise distance-rationalizable with respect to $\mathcal{M}$ are not homogeneous. However, homogeneity appears to be easier to achieve if we use the ℓ_∞-norm instead of ℓ_1. For example, Simplified Bucklin has been shown to be $(\mathcal{M}, \widehat{d^\infty_{\mathrm{ser}}})$-rationalizable [12] and it can be shown to be homogeneous. Indeed, this follows from a more general result stating that ℓ_∞-votewise rules are homogeneous as long as they are rationalized via a consensus class that satisfies a fairly weak requirement.

DEFINITION 4.3. *A consensus class $\mathcal{K}$ is* split-homogeneous *if the following two conditions hold:*

(a) If U is a $\mathcal{K}$-consensus then for every positive integer s it holds that sU is a $\mathcal{K}$-consensus with the same winner;

(b) If U and W are two profiles, with n votes each, such that $U + W$ is a $\mathcal{K}$-consensus, then at least one of U and W is a $\mathcal{K}$-consensus with the same winner as $U + W$.

It turns out that combining a split-homogeneous consensus class with an ℓ_∞-votewise distance produces a homogeneous rule.

[1]We skip the description of STV due to space,but we mention that STV is one of the very few nontrivial voting rules used in real-life political systems.

THEOREM 4.4. *For any split-homogeneous consensus class $\mathcal{K}$ and any pseudodistance d on votes, the voting rule that is rationalized via $\mathcal{K}$ and $\widehat{d^\infty}$ is homogeneous.*

It is not hard to see that the consensus classes $\mathcal{S}$, $\mathcal{U}$ and $\mathcal{M}$ are split-homogeneous. Thus, we obtain the following corollary.

COROLLARY 4.5. *For any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}\}$ and any pseudodistance d on votes, the voting rule that is rationalized via $\mathcal{K}$ and $\widehat{d^\infty}$ is homogeneous.*

In contrast, the Condorcet consensus is not split-homogeneous.

EXAMPLE 4.6. Consider the following election $E = (C, V)$ with $C = \{a, b, c, d, e\}$ and $V = (v_1, \dots, v_{12})$:

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
a	b	c	d	e	c	e	a	b	c	d	c
b	c	d	e	a	a	d	e	a	b	c	a
c	d	e	a	b	b	c	d	e	a	b	b
d	e	a	b	c	d	b	c	d	e	a	d
e	a	b	c	d	e	a	b	c	d	e	e

Voters $v_1, \dots, v_5$ form a Condorcet cycle, and voters $v_7, \dots, v_{11}$ are obtained from voters $v_1, \dots, v_5$ by reversing their preferences. Voters v_6 and v_{12} are identical and rank c first. It is not hard to verify that c is the Condorcet winner in E. On the other hand, in elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$, where $V_1 = (v_1, \dots, v_6)$ and $V_2 = (v_7, \dots, v_{12})$, c is not a Condorcet winner both in E_1 and in E_2.

Indeed, we can construct an ℓ_∞-votewise distance that combined with $\mathcal{C}$ yields a nonhomogeneous rule.

PROPOSITION 4.7. *There exists a distance d on votes such that that rule rationalized by $\mathcal{C}$ and $\widehat{d^\infty}$ is not homogeneous.*

The combination of $\mathcal{C}$ and an ℓ_1-votewise distance does not necessarily lead to a homogeneous rule either: it is well known that the Dodgson rule is not homogeneous (see, e.g., [4] for a recent survey of Dodgson rule deficiencies), yet it is $(\mathcal{C}, \widehat{d}_{\rm swap})$-rationalizable. In fact, we are not aware of any homogeneous voting rule that is ℓ_1-votewise distance-rationalizable with respect to $\mathcal{C}$. In contrast, we can construct a homogeneous rule that is ℓ_∞-votewise distance-rationalizable with respect to $\mathcal{C}$ by replacing ℓ_1 with ℓ_∞ in the rationalization of the Dodgson rule. We will call the resulting rule Dodgson$^\infty$; the next section will explain the name of the rule. To prove that Dodgson$^\infty$ is homogeneous, we will first explain how to determine the winners under this rule. It turns out that, in contrast to the Dodgson rule itself, Dodgson$^\infty$ admits a polynomial-time winner determination algorithm.

PROPOSITION 4.8. *The problem of computing the $(\mathcal{C}, \widehat{d^\infty}_{\rm swap})$-score of a given candidate c in an election $E = (C, V)$ is in* P.

PROOF. It can be verified that the following algorithm runs in polynomial time and computes the $(\mathcal{C}, \widehat{d^\infty}_{\rm swap})$-score of c.

1. Set $k = 0$.
2. If c is a Condorcet winner of E then return k.
3. For each vote where c is not ranked first, swap c and its predecessor.
4. Increase k by 1 and go to Step 2. □

Using the algorithm given in the proof of Proposition 4.8, it is not hard to show that Dodgson$^\infty$ is homogeneous.

PROPOSITION 4.9. *Dodgson$^\infty$ is homogeneous.*

The Dodgson$^\infty$ rule has some desirable properties that the Dodgson rule itself is lacking. Thus, it is interesting to ask if the fomer can be used to approximate the latter, in the sense of Caragiannis et al. [5, 6]. It turns out that the answer is "yes": each ℓ_∞-votewise rule approximates the corresponding ℓ_1-votewise rule. However, the approximation ratio is often quite large.

THEOREM 4.10. *For any consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ and any distance d on votes, let $\mathcal{R}$ and $\mathcal{R}^\infty$ be the voting rules rationalized via $\mathcal{K}$ and $\widehat{d}$ and $\widehat{d^\infty}$, respectively. Let ${\rm score}^{\mathcal{R}}_E(c)$ (respectively, ${\rm score}^{\mathcal{R}^\infty}_E(c)$) denote the $(\mathcal{K}, \widehat{d})$-score (respectively, $(\mathcal{K}, \widehat{d^\infty})$-score) of a candidate c in an election $E = (C, V)$. Then for each election $E = (C, V)$ and each candidate $c \in C$ we have*

$$
{\rm score}^{\mathcal{R}^\infty}_E(c) \le {\rm score}^{\mathcal{R}}_E(c) \le |V| \cdot {\rm score}^{\mathcal{R}^\infty}_E(c).
$$

For the majority consensus we can strengthen the approximation guarantee from $|V|$ to $\lceil \frac{|V|}{2} + 1 \rceil$ using the fact that we only need the majority of the voters to rank a candidate first for him to be the $\mathcal{M}$-winner.

Of course, these approximations are very weak as they depend linearly on the number of voters; their appeal is in their generality. Further, since for the Dodgson rule its ℓ_∞-variant is homogeneous and polynomial-time computable, an appealing conjecture is that replacing ℓ_1 with ℓ_∞ in the rationalization of a voting rule is a general recipe for designing voting rules that are homogeneous and admit an efficient winner determination algorithm. It is unlikely that this conjecture holds unconditionally, but it would be very interesting to identify sufficient conditions for it to hold.

5. MONOTONICITY

Monotonicity is a very desirable property of voting rules: it stipulates that campaigning in favor of a candidate should not hurt him. While homogeneity seems to be essentially a function of the norm and the consensus class (as illustrated by Theorem 4.1 and Theorem 4.4, which hold for any distance d on votes), monotonicity seems to be most closely related to the properties of the distance on votes. Therefore, in this section we propose several notions of monotonicity for distances on votes that, combined with appropriate norms and consensus classes, produce a monotone rule. We do not consider the Condorcet consensus in this section: even a very well-behaved distance such as $\widehat{d}_{\rm swap}$ may produce a non-monotone rule when combined with $\mathcal{C}$ (recall that the resulting rule is Dodgson, which is known to be non-monotone (see, e.g., [4]). Also, for simplicity, we focus on ℓ_1-votewise rules and ℓ_∞-votewise rules.

Let C be a set of candidates and let d be a distance on votes. How can we specify a condition on d so that voting rules rationalized using this distance are monotone? Consider an election with a winner c, a vote y, a vote $x \in \mathcal{P}(C, c)$ and a vote $z \in \mathcal{P}(C, a)$ for some $a \neq c$. It is tempting to require that for any vote y' obtained from y by pushing c forward it holds that $d(y', x) \le d(y, x)$ and $d(y', z) \ge d(y, z)$. However, this condition turns out to be so strong that no reasonable distance can satisfy it. Indeed, suppose that y ranks c in position three or lower, and y' is obtained from y by shifting c by one position. Then y does not rank c in the first position, and our condition should hold for $z = y'$, implying $d(y, y') \le 0$, which is clearly impossible.

Thus, we need to relax the condition above. There are two ways of doing so. First, we can require that when we move c forward in the vote, the distance to x declines faster than the distance to z. Alternatively, instead of imposing this condition for all $x \in \mathcal{P}(C, c)$

and $z \in \mathcal{P}(C, a)$, we can require that it holds for the closest vote that ranks c first, and the closest vote that ranks a first, respectively. We will now show that both relaxations, which we call, respectively, *relative monotonicity* and *min-monotonicity*, lead to meaningful conditions that are satisfied by some natural distances, and, combined with appropriate consensus classes, result in monotone voting rules. We consider relative monotonicity first.

DEFINITION 5.1. *Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is* relatively monotone *if for each $c \in C$, every two preference orders y and y' such that y' is identical to y except that y' ranks c higher than y, and every two preference orders x and z such that x ranks c first and z does not, it holds that*

$$d(x, y) - d(x, y') \geq d(z, y) - d(z, y').$$

As a quick sanity check, we note that the swap distance, d_{swap}, satisfies the relative monotonicity condition. Indeed, let $d = d_{\mathrm{swap}}$ and let C be a set of candidates, c be a candidate in C, and let y, y', x, and z be as in the definition of relative monotonicity. In addition, let k be a positive integer such that y' is identical to y except in y' candidate c is ranked k positions higher. We need k swaps to transform y into y' so $d(y, y') = k$. We first note that $d(x, y) - d(x, y') = k$. This is so because the swap distance measures the number of inverses between two preference orders. As x ranks c on top and y' ranks it k positions higher than y does (without any other changes), the number of inverses between x and y' is the same as that between x and y less k. By the triangle inequality $d(z, y) \leq d(z, y') + d(y', y) = d(z, y') + k$, hence $d(z, y) - d(z, y') \leq k$ and this completes the proof.

Relative monotonicity of a distance on votes naturally translates to the monotonicity of the resulting voting rule, provided we use ℓ_1 as a norm and either $\mathcal{S}$ or $\mathcal{U}$ as a consensus.

THEOREM 5.2. *Let $\mathcal{R}$ be a voting rule rationalized by $(\mathcal{K}, \widehat{d})$, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and d is a relatively monotone distance on votes. Then $\mathcal{R}$ is monotone.*

However, relative monotonicity is a remarkably strong condition, not satisfied even by very natural distances that are, intuitively, monotone.

EXAMPLE 5.3. Consider a scoring vector $\alpha = (0, 1, 2, 3, 4, 5)$ that corresponds to the 6-candidate Borda rule and a candidate set $C = \{c, d, x_1, x_2, x_3, x_4\}$. Consider the following four votes:

$$\begin{aligned} x &: c > d > x_1 > x_2 > x_3 > x_4, \\ z &: x_1 > c > x_2 > x_3 > x_4 > d, \\ y &: x_1 > x_2 > d > c > x_3 > x_4, \\ y' &: x_1 > x_2 > c > d > x_3 > x_4. \end{aligned}$$

Note that y and y' are identical except that in y' candidate c is ranked one position higher, and that c is ranked on top in x and is not ranked on top in z. We verify that $d_\alpha(x, y) - d_\alpha(x, y') = 0$ but $d_\alpha(z, y) - d_\alpha(z, y') = 2$. Thus, d_α is not relatively monotone.

Our second approach to monotone distances, i.e., min-monotonicity, captures the intuition that d_α in the example above should be classified as monotone. We first define min-monotonicity formally.

DEFINITION 5.4. *Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is* min-monotone *if for every candidate $c \in C$ and every two preference orders y and y' such that y' is the same as y except that it ranks c higher, for each $a \in C \setminus \{c\}$ we have:*

$$\begin{aligned} \min_{x \in \mathcal{P}(C,c)} d(x, y) &\geq \min_{x' \in \mathcal{P}(C,c)} d(x', y'), \\ \min_{z \in \mathcal{P}(C,a)} d(z, y) &\leq \min_{z' \in \mathcal{P}(C,a)} d(z', y'). \end{aligned}$$

We will now argue that for any non-decreasing scoring vector α the distance d_α is min-monotone.

PROPOSITION 5.5. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a normalized scoring vector. (Pseudo)distance d_α is min-monotone if and only if α is nondecreasing.*

Proposition 5.5, combined with the proof of Theorem 4.9 of [12] gives the next corollary.

COROLLARY 5.6. *A voting rule $\mathcal{R}$ is $(\mathcal{U}, \widehat{d})$-rationalizable for some min-monotone neutral pseudodistance d on votes if and only if $\mathcal{R}$ can be defined via a family of nondecreasing scoring vectors (one for each number of candidates).*

In essence, Proposition 5.5 ensures that for every nondecreasing scoring vector α, $\mathcal{R}_\alpha$ is ℓ_1-votewise rationalizable with respect to $\mathcal{U}$ via a min-monotone distance over votes, and the definition of min-monotonicity ensures that the scoring vector derived in the proof of Theorem 4.9 of [12] is nondecreasing.

Min-monotonicity is also useful in the context of the majority consensus: for $\mathcal{M}$, we can show an analogue of Theorem 5.2 both for ℓ_1-votewise rules and for ℓ_∞-votewise rules.

THEOREM 5.7. *Let d be a min-monotone distance on votes, and let $\mathcal{R}$ be the voting rule rationalized by $(\mathcal{M}, \widehat{d^{\mathcal{N}}})$, where $\mathcal{N} \in \{\ell_1, \ell_\infty\}$. Then $\mathcal{R}$ is monotone.*

However, it is not clear how to apply the notion of min-monotonicity in the context of the strong unanimity consensus. The reason is that given a profile V of voters over some candidate set C, finding an $\mathcal{S}$-consensus closest to V requires finding a single preference order u that minimizes the aggregated distance from V to this order. However, it need not be the case that u is a preference order that minimizes the distance from some vote $v \in V$ to a preference order that ranks $\mathrm{top}(u)$ first.

Finally, we remark that we can combine both relaxations considered in this section, obtaining a class of distances that includes both relatively monotone distances and min-monotone distances.

DEFINITION 5.8. *Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is* relatively min-monotone *if for each candidate $c \in C$ and each two preference orders y and y' such that y' is identical to y except that y' ranks c higher than y, for each candidate $a \in C \setminus \{c\}$ it holds that*

$$\min_{x \in \mathcal{P}(C,c)} d(x, y) - \min_{x' \in \mathcal{P}(C,c)} d(x', y') \geq \min_{z \in \mathcal{P}(C,a)} d(z, y) - \min_{z' \in \mathcal{P}(C,a)} d(z', y').$$

PROPOSITION 5.9. *Each distance on votes that is relatively monotone or min-monotone is relatively min-monotone.*

PROOF. Due to lack of space, we only give the proof for relatively monotone distances. Let C be a set of candidates, $c, a \in C$, and let $y, y' \in \mathcal{P}(C)$ be identical, except y' ranks c higher than y. Pick $\hat{x} \in \arg\min_{x' \in \mathcal{P}(C,c)} d(x', y)$, $\hat{z} \in \arg\min_{z' \in \mathcal{P}(C,a)} d(z', y')$. Then

$$\min_{x \in \mathcal{P}(C,c)} d(x, y) - \min_{x' \in \mathcal{P}(C,c)} d(x', y') \geq d(\hat{x}, y) - d(\hat{x}, y') \geq d(\hat{z}, y) - d(\hat{z}, y') \geq \min_{z \in \mathcal{P}(C,a)} d(z, y) - \min_{z' \in \mathcal{P}(C,a)} d(z', y').$$

Thus, d is relatively min-monotone. □

For $\mathcal{U}$ the proof of Theorem 5.2 extends to relatively min-monotone distances (and hence to min-monotone distances).

COROLLARY 5.10. *Any voting rule rationalized by $\mathcal{U}$ and $\widehat{d}$, where d is relatively min-monotone distance on votes, is monotone.*

6. CONCLUSIONS

We have discussed homogeneity and monotonicity of voting rules that are distance-rationalizable via votewise distances, focusing on ℓ_p-votewise rules, $p \in \mathbb{N} \cup \{+\infty\}$. A quick summary of our results is given in Tables 1 and 2.

	$\mathcal{S}$	$\mathcal{U}$	$\mathcal{M}$	$\mathcal{C}$
ℓ_1	Y (Th. 4.1)	Y (Th. 4.1)	Y/N (Th. 4.2)	n (Dodgson)
ℓ_∞	Y (Th. 4.4)	Y (Th. 4.4)	Y (Th. 4.4)	y (Prop. 4.9)/ n (Prop. 4.7)

Table 1: (Homogeneity) Y at the intersection of column $\mathcal{K}$ and row $\mathcal{N}$ indicates that for any distance d on votes the $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$-rationalizable rule is homogeneous. Y/N refers to a dichotomy result, and y/n refer to examples of homogeneous/non-homogeneous rules.

	$\mathcal{S}$	$\mathcal{U}$	$\mathcal{M}$
ℓ_1	rel-mon (Th. 5.2)	rel-min-mon (Cor. 5.10)	min-mon (Th. 5.7)
ℓ_∞	?	?	min-mon (Th. 5.7)

Table 2: (Monotonicity) At the intersection of column $\mathcal{K}$ and row $\mathcal{N}$, we indicate a sufficient condition on d (relative monotonicity, min-monotonicity, relative min-monotonicity) for the $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$-rationalizable rule to be monotone.

Motivated by our goal, we obtained a number of results, that, while not directly related to the primary topic of our study, contribute to the general understanding of votewise rationalizable rules. In particular, we identified a natural family of voting rules, which we called $\mathcal{M}$-scoring rules. These rules constitute a (provably distinct) variant of scoring rules that, when counting points for a given candidate, ignore the less favorable half of the votes. We have shown that $\mathcal{M}$-scoring rules have a natural interpretation in the context of distance rationalizability. By establishing a relationship between rules that are rationalizable with respect to $\mathcal{U}$ and $\mathcal{M}$, we resolved (in the negative) an open question about votewise rationalizability of STV posed in [11]. Also, our study of monotonicity allowed us to refine a result of [12] characterizing the class of scoring rules in terms of distance-rationalizability (our Corollary 5.6).

Our work leads to several open problems. First, we are far from having a complete understanding of homogeneity of the rules that are votewise distance-rationalizable with respect to the Condorcet consensus; even less is known about the monotonicity of such rules. Also, it would be interesting to know whether there are distances $d \neq d_{\mathrm{swap}}$ for which the winner determination for the $(\mathcal{C}, \widehat{d^{\infty}})$-rationalizable rule is easier than for the $(\mathcal{C}, \widehat{d})$-rationalizable rule; the same question can be asked for the consensus class $\mathcal{S}$. We are also very much interested in finding less demanding, yet practically useful, conditions on distances that lead to monotone rules.

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