Communication Complexity of Approximating Voting Rules*

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ABSTRACT

This paper considers the communication complexity of approximating common voting rules. Both upper and lower bounds are presented. For n voters and m alternatives, it is shown that for all $\epsilon \in (0, 1)$, the communication complexity of obtaining a $1-\epsilon$ approximation to Borda is $O(\log(\frac{1}{\epsilon})nm)$. A lower bound of $\Omega(nm)$ is provided for fixed small values of ϵ. The communication complexity of computing the true Borda winner is $\Omega(nm \log(m))$ [5]. Thus, in the case of Borda, one can obtain arbitrarily good approximations with less communication overhead than is required to compute the true Borda winner.

For other voting rules, no such $1\pm\epsilon$ approximation scheme exists. In particular, it is shown that the communication complexity of computing any constant factor approximation, ρ, to Bucklin is $\Omega(\frac{nm}{\rho^2})$. Conitzer and Sandholm [5] show that the communication complexity of computing the true Bucklin winner is $O(nm)$. However, we show that for all $\delta \in (0, 1)$, the communication complexity of computing a m^δ approximate winner in Bucklin elections is $O(nm^{1-\delta} \log(m))$. For $\delta \in (\frac{1}{2}, 1)$, a lower bound of $\Omega(nm^{1-2\delta})$ is also provided.

Similar lower bounds are presented on the communication complexity of computing approximate winners in Copeland elections.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent systems

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voting, communication complexity, approximation

1. INTRODUCTION

The naive application of a voting rule requires each voter to send their entire preference order over the set of alternatives. If there are n voters and m alternatives communicating every voter's preference order requires $\Theta(nm \log(m))$ bits be communicated. However, using an intelligent elicitation protocol can result in significant savings. Conitzer and Sandholm [5] show that a number of voting rules can be computed with low communication overhead. For example, the communication complexity of plurality is $O(n \log(m))$ and the communication complexity of single transferable vote is $O(n \log^2(m))$. Conitzer and Sandholm further characterized the communication complexity of a number of common voting rules by presenting lower bounds on the communication complexity of each.

Not every voting rule can be computed with a low amount of communication overhead [5]. Determining the true winner in Borda and Copeland elections requires communication complexity $\Theta(nm \log(m))$. Likewise, computing the winner in a Bucklin election requires communication complexity $\Theta(nm)$.

Conitzer and Sandholm [5] suggest that when, for example, voting over issues of relatively low importance:

> Knowing which voting rules require little communication is especially important when the issue to be voted on is of low enough importance that the following is true: the parties involved are willing to accept a rule that tends to produce outcomes that are slightly less representative of the voters' preferences, if this rule reduces the communication burden on the voters significantly.

A more natural approach in such situations is to use a low communication complexity approximation to the desired voting rule. For example, rather than selecting plurality (with communication complexity $\Theta(n \log(m))$) over the preferred voting rule Borda (with communication complexity $\Theta(nm \log(m))$), it is more natural to obtain a $1-\epsilon$ approximation to Borda, for some $\epsilon \in (0, 1)$, using a reduced amount of communication.

This paper considers the communication complexity of approximating common score-based rules. It is shown that it is possible to obtain arbitrarily good approximations to some voting rules using less communication than is required to compute the actual winner. For example, an approximation scheme for Borda voting is presented that, for all $\epsilon \in (0, 1)$, obtains a $1-\epsilon$ approximation to Borda with communication complexity $O(\log(\frac{1}{\epsilon})nm)$. It is shown that, up to constant factors, this approximation scheme is optimal. That is, it is shown that for all $\delta \in (0, 1 - \frac{1}{\sqrt{2}})$, the communication

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complexity of computing a $\frac{1}{\sqrt{2}} + \delta$ approximation to Borda is $\Omega\left(\frac{\delta^3}{\log(1/\delta)} nm - \log(\log(1/\delta))\right)$.

While some voting rules, such as Borda, admit $1 \pm \epsilon$ approximation schemes, others do not. We show that for any $\rho > 1$, the communication complexity of computing a ρ approximate winner in Bucklin elections is $\Omega\left(\frac{nm}{\rho^2}\right)$. Conitzer and Sandholm [5] show that $O(nm)$ bits of communication are sufficient to determine the true Bucklin winner.

For sufficiently good approximations, lower bounds on the communication complexity of $\Omega(nm)$ for computing approximate winners in Copeland elections are presented as well.

Both incremental preference elicitation [6, 8, 9] and approximation of voting rules [2, 3, 10] has seen increasing attention. Most work on approximation of voting rules considers rules under which it is NP-hard to compute the true winner [2, 3, 10], or attempts to employ approximation in an effort to guarantee strategy-proofness [11]. However, to the best of the authors' knowledge, this paper is the first to provide bounds on the communication complexity of computing approximate winners under voting rules.

The remainder of this paper is structured as follows. Preliminary definitions and background are presented in Section 2. Section 3 presents the upper and lower bounds on the communication complexity of approximating Borda, Bucklin, and Copeland. Section 4 presents a discussion of the results and some future work.

2. PRELIMINARIES

This section provides basic background related to voting rules, communication complexity, and the proof methods employed in this paper.

2.1 Voting Rules

Let V be a set of n voters and let A be a set of m alternatives. Each voter, v_i, has a strict preference order, $\succ_i$, over the m alternatives. A preference profile is a vector of voter preference orders.

A voting rule is a mapping from preference profiles to winning alternatives. In the interest of obtaining results on approximating voting rules, this paper restricts its attention to rules that assign each alternative a score-based on the voters' preference. That is, if $sc : A \to \mathbb{R}$ is a function assigning a score to each alternative, then the winner is the alternative that maximizes/minimizes its score. Score-based voting rules allow for natural measures of approximation.

This paper considers the following voting rules.

1. Borda: Each alternative a is awarded $m - k$ points for every voter that ranks a in its k-th position. The Borda winner is the alternative with the greatest score. If w is the Borda winner and a any other alternative, then the approximation ratio obtained by a is $\frac{sc(a)}{sc(w)} \leq 1$.

2. Bucklin: The Bucklin score of each alternative a is the minimum value of k such that a strict majority of voters rank a in one of the top k positions. The Bucklin winner is the alternative with the least score. If w is the Bucklin winner and a any other alternative, then the approximation ratio obtained by a is $\frac{sc(a)}{sc(w)} \geq 1$.

3. Copeland: An alternative a is said to defeat an alternative b in a pairwise election if a strict majority of voters prefer a to b. Under Copeland every alternative a receives one point for every alternative that a defeats in a pairwise election and half a point for every alternative a ties. The Copeland winner is the alternative with the greatest score. If w is the Copeland winner and a any other alternative, then the approximation ratio obtained by a is $\frac{sc(a)}{sc(w)} \leq 1$.

For Borda and Copeland, the approximation ratio obtained by a communication protocol, f, is $\rho \in [0, 1]$ if for every preference profile, P, $\frac{sc(f(P))}{sc(w)} \leq \rho$, where $f(P)$ is the alternative selected by f under P and w is the winning alternative in P. Likewise, f obtains a $\rho \geq 1$ approximation in Bucklin elections if for every preference profile, P, $\frac{sc(f(P))}{sc(w)} \geq \rho$.

For each voter $v \in V$ and alternative $a \in A$, let $v(a)$ be the rank of a in v's preference order. For example, if v has the preference order

$$x \succ y \succ a \succ z,$$

then $v(a) = 3$.

2.2 Communication Complexity

This paper employs the standard model of communication complexity [5, 7, 12]. The objective is to compute the outcome of a voting rule $f(\succ_1, \cdots, \succ_n)$. However, each piece of the input $\succ_i$ is known only to a single voter v_i. A protocol for computing f consists of a number of rounds. During each round, a single voter announces a single bit to all other voters. In a deterministic protocol, the next voter to announce a bit and the bit to be announced are completely determined by the preceding rounds and that voter's preference order. A communication pattern is the sequence of bits announced. The winner elected by the voters is then a function of the particular communication pattern observed. Note that in a k round deterministic protocol there are at most 2^k possible communication patterns.

During each round of the protocol all voters have observed the same sequence of communicated bits. The protocol terminates when sufficient information has been communicated for every voter to compute f, or in our case to determine an approximate winner. The communication complexity of approximating a voting rule f is the worst case number of bits sent by the best approximation protocol.

In order to compute lower bounds on the amount of communication required to approximate certain voting rules, a slight generalization of the standard lower bound technique of constructing a *fooling set* is employed. The definition for rules that select alternatives that maximize their score is presented. The definition for rules that select alternatives that minimize their score is analogous.

Definition 1 (Fooling set)**.** *Let sc be a score-based voting rule. A ρ-fooling set S for sc is a set of preference profiles with the following properties:*

1. *$w \in A$ is the winning alternative in every preference profile in S under the score-based voting rule sc.*

2. *In every preference profile $P \in S$, every $a \in A \setminus \{w\}$ does not obtain a ρ-approximate solution in P. That is $\frac{sc(a)}{sc(w)} < \rho$, for every $a \in A \setminus \{w\}$.*

3. *For every N preference profiles $P_1 = (v_1^1, \cdots, v_n^1), \cdots, P_N = (v_1^N, \cdots, v_n^N)$ in S, there exists a vector of indices $(r_1, \cdots, r_n) \in \{1, \cdots, N\}^n$ such that w does not obtain a ρ-approximation in $P_r = (v_1^{r_1}, \cdots, v_n^{r_n})$ (i.e., it is possible to mix voters from the N preference profiles such that w no longer obtains a ρ-approximation). That is, there exists some alternative $a \in A$ such that $\frac{sc(w)}{sc(a)} < \rho$.*

Theorem 1. *If sc is a score-based voting rule and S a fooling set, then the deterministic communication complexity of computing a ρ-approximation to sc is $\Omega\left(\log\left(\frac{|S|}{N}\right)\right)$.*

Proof. Suppose there is a deterministic protocol D that computes a ρ-approximation to sc in $\left\lfloor \log\left(\frac{|S|}{N}\right)\right\rfloor - 1$ bits. Thus, there are at most $2^{\left\lfloor \log\left(\frac{|S|}{N}\right)\right\rfloor - 1} < \frac{|S|}{N}$ possible communication patterns. By the pigeonhole principle, there exists N preference profiles $P_i = (v_1^i, \cdots, v_n^i)$ for $i = 1, \cdots, N$ in S that have the same communication pattern under D.

Since all the P_i's exhibit the same communication pattern, the protocol must select the same alternative w under each. Let $r = (r_1, \cdots, r_n) \in \{i, j\}^n$ such that w does not obtain a ρ-approximation in P_r. By assumption, such an r exists. It is known that D produces the same communication pattern on P_r as it does on all the P_i's [1]. Since D exhibits the same communication pattern on P_r as it does all the P_i's, D must select w as the winning alternative in P_r. However, w does not obtain a ρ-approximation in P_r; a contradiction. Hence, the communication complexity of obtaining a ρ approximation is $\Omega\left(\log\left(\frac{|S|}{N}\right)\right)$. □

A natural question is whether Conitzer and Sandholm's [5] fooling set constructions already provide lower bounds on the communication complexity of computing approximate winners for the score-based voting rules considered in this paper. However, on inspection, it is observed that Conitzer and Sandholm's lower bound proofs for Borda, Bucklin, and Copeland construct fooling sets in which the single winning alternative a has a constant number of points more than the next highest alternative(s). Unfortunately, in our setting, Contizer and Sandholm's constructions are not strong enough to lower bound the deterministic communication requirements of approximately computing these rules. That is, the ratio of the scores of the winning alternative to the scores of the other alternatives need to be bounded away from 1. With Contizer and Sandholm's constructions, the ratio of the score of any alternative to that of the winning alternative tends towards 1 for increasingly large elections.

2.3 Probabilistic Method

The stronger requirements on the fooling sets needed to lower bound the deterministic communication complexity of approximately computing voting rules complicates the construction of fooling sets. However, the fooling set need not actually be constructed. Showing the *existence* of such a set is sufficient for the lower bound proofs.

Instead of explicit constructions, the lower bound proofs in this paper employ a powerful method from combinatorics to show the existence of objects satisfying certain properties. The probabilistic method proves the existence of a combinatorial object satisfying certain properties as follows:

1. First, construct a probability distribution over the objects of interest.
2. Second, show that an object drawn from that distribution possesses the desired properties with strictly positive probability.

Since with probability greater than zero the object drawn from the distribution satisfies the requirements, it is assured to exist. Using the probabilistic method, an appropriate fooling set can be shown to exist without providing an explicit construction.

The probabilistic method is employed in our lower bound proofs by constructing a distribution over preference profiles. This distribution over preference profiles then implicitly defines a distribution over sets of preference profiles (i.e., potential fooling sets). It is shown that a set of k preference profiles drawn from this distribution satisfies the fooling set properties with strictly positive probability. It can be concluded that a fooling set of size k exists, which implies a $\Omega(\log(\frac{k}{N}))$ lower bound on the communication complexity.

All of the presented lower bound results hinge on Chernoff bounds [4].

Theorem 2 (Chernoff [4])**.** *Let $X_1, \cdots, X_n$ be n independent random variables taking on values 0 or 1, such that $Pr(X_i = 1) = p$, for each $i = 1, \cdots, n$. Let $X = \sum_{i=1}^n X_i$ and let $\delta \in (0, 1)$ then*

$$Pr(X < (1-\delta)\mathbb{E}(X)) = Pr(X < (1-\delta)pn) < e^{\frac{-pn\delta^2}{2}}.$$

3. RESULTS

Upper bounds on the communication complexity of obtaining approximations to Borda and Bucklin are presented first. It is shown that arbitrarily good approximations to Borda can be obtained with less communication overhead than computing the true Borda winner. For Bucklin, it is shown that a number of non-constant approximations can be achieved with less communication complexity than computing the true Bucklin winner.

A number of lower bounds are then presented on the communication complexity of obtaining a number of approximation ratios with respect to Borda, Bucklin, and Copeland. In particular, it is shown that the Borda and Copeland voting rules require $\Omega(nm)$ communication complexity to compute sufficiently good constant factor approximations. For Bucklin, it is shown that for any constant ρ, the communication complexity of computing a *rho*-approximate winner in Bucklin elections is $\Omega(\frac{1}{\rho^2}nm)$, which, for fixed ρ, matches the upper bound given by Conitzer and Sandholm for computing the true Bucklin winner.

3.1 Upper Bounds

Conitzer and Sandholm [5] show that the communication complexity of determining the Borda winner is $\Theta(nm\log(m))$. Theorem 3 shows that a $(1-\epsilon)$ approximation to Borda can be obtained by a protocol with communication complexity $O(\log(\frac{1}{\epsilon})nm)$.

Informally, in the protocol presented in Theorem 3, each voter announces an approximate rank of each alternative in its preference order using a $O(\log(\frac{1}{\epsilon}))$ bits. That is, the preference order of each voter is divided into k equally sized segments and each voter indicates which of the k segments each alternative falls into using only $O(\log(k))$ bits per alternative. With this information, upper and lower bounds can be inferred for the Borda score of each alternative. The alternative with the greatest upper bound is selected as the winner. The proof of Theorem 3 shows that arbitrarily good approximation ratios can be obtained given an appropriate choice of k.

Theorem 3. *For all $\epsilon \in (0,1)$, there is a deterministic communication protocol that approximates Borda to within a factor of $1-\epsilon$ with communication complexity $O(\log(\frac{1}{\epsilon})nm)$.*

Proof. Given $\epsilon \in (0,1)$, let $k = \lceil \frac{4}{\epsilon} \rceil$. Let $a_1, a_2, \cdots, a_m$ be a fixed ordering on the alternatives in A. For each a_i, every voter, v, announces in order the value $l \in (0, k-1)$ such that

$$v(a_i) \in \left[\left\lceil \frac{lm}{k} \right\rceil, \left\lceil \frac{(l+1)m}{k} \right\rceil + 1 \right].$$

This procedure requires $\lceil \log(k) \rceil nm = O(\log(\frac{1}{\epsilon})nm)$ bits for fixed ϵ.

For a voter v and alternative a, let $v^l(a)$ be the value of l returned by voter v for a. For each alternative $a \in A$, define the following lower and upper bounds, $lb(a)$ and $ub(a)$, respectively, on a's true Borda score:

$$\begin{aligned} lb(a) &= \sum_{v \in V} \left(m - \left\lceil \frac{(v^l(a)+1)m}{k} \right\rceil + 1 \right), \\ ub(a) &= \sum_{v \in V} \left(m - \left\lceil \frac{v^l(a)m}{k} \right\rceil \right). \end{aligned}$$

In essence, the upper bound on a's true Borda score is obtained by assuming that a's true rank in each voter's preference order falls at the lower end of the range of ranks reported by each voter. The lower bound is obtained by assuming that a's true rank falls at the upper end of the range reported by each voter.

Then, for each $a \in A$

$$\begin{aligned} ub(a) - lb(a) &= \sum_{v \in V} \left(\left\lceil \frac{(v^l(a)+1)m}{k} \right\rceil - \left\lceil \frac{v^l(a)m}{k} \right\rceil - 1 \right) \\ &\leq \sum_{v \in V} \left(\frac{(v^l(a)+1)m}{k} + 1 - \frac{v^l(a)m}{k} - 1 \right) \\ &= \frac{nm}{k}. \end{aligned}$$

The protocol selects the alternative a with the greatest $ub(a)$ value. Now it is shown that a is a $1-\epsilon$ approximate winner.

Let w be the true Borda winner. Since the sum of all the alternatives scores is $\frac{nm(m-1)}{2}$, $sc(w) \geq \frac{n(m-1)}{2}$. Also since a was selected, $sc(w) \leq ub(w) \leq ub(a)$. The approximation ratio is

$$\begin{aligned} \frac{sc(a)}{sc(w)} &\geq \frac{lb(a)}{sc(w)} \\ &\geq \frac{ub(a) - \frac{nm}{k}}{sc(w)} \\ &= \frac{ub(a)}{sc(w)} - \frac{\frac{nm}{k}}{sc(w)} \\ &\geq \frac{ub(a)}{ub(a)} - \frac{2m}{k(m-1)} \\ &\geq 1 - \frac{4}{k} \\ &\geq 1 - \epsilon. \end{aligned}$$

□

It will be shown that obtaining any constant factor approximation to Bucklin requires $\Omega(nm)$ communication complexity. Since, the true Bucklin winner can be determined using $O(nm)$ bits of communication complexity, there does not exists any asymptotically better communication protocol to obtain a constant factor approximation to Bucklin. However, non-constant factor approximations can be easily obtained in Bucklin elections.

Theorem 4. *For every $\delta \in (0,1)$, there is a deterministic communication protocol that obtains a m^δ approximation to Bucklin with communication complexity $O(nm^{(1-\delta)}\log(m))$.*

Proof. Consider the following protocol. Every voter broadcasts the top $m^{(1-\delta)} - 1$ entries in its preference order using $O(nm^{(1-\delta)}\log(m))$ bits. If any alternative appears in the top $m^{(1-\delta)}$ positions in a strict majority of voters, then the true Bucklin winner must also appear in the top $m^{(1-\delta)}$ positions by a strict majority of the voters as well. Hence, the true Bucklin winner can be computed given the partial lists of preferences reported by each voter.

Otherwise, if no alternative appears in the top $m^{(1-\delta)}$ positions in a strict majority of voters, the Bucklin score of any alternative is at least $m^{(1-\delta)}$. Since the Bucklin score of every alternative is at most m, every alternative is a $\frac{m}{m^{(1-\delta)}} = m^\delta$ approximation solution. Hence, an arbitrary alternative may be selected in this case. □

3.2 Lower Bounds

All of the lower bound proofs will employ the same parameterized distribution over preference profiles.

Let the set of m alternatives be $A = \{a_1, \cdots, a_m\}$. Fix $w \in A$ and partition $A \setminus \{w\}$ into the following sets

1. $X = \{x_i : i = 1, \cdots, |X|\}$,
2. $Y = \{y_i^r : i = 1, \cdots, \frac{|Y|}{2}$ and $r \in \{0,1\}\}$
3. $Z = \{z_i^r : i = 1, \cdots, \frac{|Z|}{2}$ and $r \in \{0,1\}\}$.

The sizes of the sets X, Y, and Z will depend upon the particular voting rule and desired approximation ratio. Informally, we construct a distribution over preference profiles that satisfies the following properties:

1. w is preferred to every alternative in X by every voter,
2. for every alternative in $a \in Z \cup Y$, half of the voters prefer w to a and the other half prefer a to w, and

3. for every two alternatives $a, b \in A \setminus \{w\}$, half of the voters prefer a to b and the other half prefer b to a.

Thus, every alternative in $A \setminus \{w\}$ obtains a roughly average score under the considered voting rules.

We then show that for any N preference profiles drawn from this distribution with high probability it is possible to mix and match the voters from the N preference profiles such that, there is some alternative $z \in Z$ that an overwhelming majority of the voters prefer to all members of $Y \cup X$. That is, it is possible to mix and match the voters such that the score of some $z \in Z$ is significantly higher than z's score in any of the individual preference profiles. In particular, z's score will be significantly higher than that of w.

For $r \in \{0,1\}$, let $Y_r = \{y_i^r : i = 1, \cdots, \frac{|Y|}{2}\}$, similarly for Z_r. Let π_X be a fixed permutation over X. Let $n = 2n'$.

Construct a distribution $\mathcal{P}$ over preference profiles as follows. Every preference profile from $\mathcal{P}$ is constructed using the following random procedure. For each $i \in \{1, \cdots, n'\}$, select $r \in \{0,1\}^{|Z_0|}$ and $j \in \{0,1\}^{|Y_0|}$ uniformly at random. Voter v_{2i} has preference order:

$$\begin{aligned}
&z_1^{r_1} \succ \cdots \succ z_{|Z_0|}^{r_{|Z_0|}}\\
&\succ y_1^{j_1} \succ \cdots \succ y_{|Y_0|}^{j_{|Y_0|}}\\
&\succ w \succ \pi_X(1) \succ \cdots \succ \pi_X(|X|)\\
&\succ y_1^{\neg j_1} \succ \cdots \succ y_{|Y_0|}^{\neg j_{|Y_0|}}\\
&\succ z_{|Z_0|}^{\neg r_{|Z_0|}} \succ \cdots \succ z_1^{\neg r_1}
\end{aligned}$$

Voter v_{2i-1} has preference order:

$$\begin{aligned}
&z_1^{\neg r_1} \succ \cdots \succ z_{|Z_0|}^{\neg r_{|Z_0|}}\\
&\succ y_1^{\neg j_1} \succ \cdots \succ y_{|Y|}^{\neg j_{|Y|}}\\
&\succ w \succ \pi_X(|X|) \succ \cdots \succ \pi_X(1)\\
&\succ y_1^{j_1} \succ \cdots \succ y_{|Y|}^{j_{|Y|}}\\
&\succ z_{|Z_0|}^{r_{|Z_0|}} \succ \cdots \succ z_1^{r_1}
\end{aligned}$$

Notice that w is preferred to every alternative in X and is ranked among the upper half of the alternatives by every voter. Thus, w has a strictly better than average score under Borda, Bucklin and Copeland. However, every other alternative $a \in A \setminus \{w\}$ obtains a score that is roughly average, since if voter v_{2i} ranks a in position k, then v_{2i-1} ranks a in position $m - k$. Also notice that each $z \in Z_0$ is placed among the top $|Z_0|$ positions in voter v_{2i} with probability $\frac{1}{2}$, independent of the rankings of the other members of Z_0.

Lemma 1 shows that with high probability, given any N preference profiles $P_1, \cdots, P_N$, it is possible to mix and match voters from the N profiles in such a way that w no longer obtains a good approximation ratio.

Lemma 1. *Let $P_i = (v_1^i, \cdots, v_n^i)$ for $i = 1, \cdots, N$ be random preference profiles drawn from $\mathcal{P}$ and let $\delta \in (0,1)$. There exists a $z \in Z_0$ and a $r \in \{1, \cdots, N\}^n$ such that in $P_r = (v_1^{r_1}, \cdots, v_n^{r_n})$, z is ranked among the top $|Z_0|$ positions by at least $(1-\delta)(1-\frac{1}{2}^{N-1})n$ voters with probability at least $1 - e^{-\frac{|Z_0|(1-\frac{1}{2}^{N-1})n'\delta^2}{2}}$.*

Proof. Recall that for each $i \in \{1, \cdots, n'\}$ and $z \in Z_0$, exactly one of v_{2i}^1 and v_{2i-1}^1 rank z among the top $|Z_0|$ positions. Without loss of generality, assume that v_{2i}^1 ranks z among the top $|Z_0|$ positions.

The probability that for each $j \in \{2, \cdots, N\}$, v_{2i}^j also ranks z among the top $|Z_0|$ positions (rather than v_{2i-1}^j) is $\frac{1}{2}^{N-1}$. Hence, independently for each $i \in \{1, \cdots, n'\}$, with probability $p = 1 - \frac{1}{2}^{N-1}$ there exists indices $j, k \in \{1, \cdots, N\}$ such that v_{2i}^j and v_{2i-1}^k both rank z among the top $|Z_0|$ positions.

By Chernoff bounds, for each $\delta \in (0,1)$, there are fewer than $(1-\delta)pn'$ indices $i \in \{1, \cdots, n'\}$ such that there exists $j, k \in \{1, \cdots, N\}$, where v_{2i}^j and v_{2i-1}^k both rank z among the top $|Z_0|$ positions with probability at most $e^{-\frac{pn'\delta^2}{2}}$.

If at least $(1-\delta)pn'$ such indices i exists, then there exists an $r \in \{1, \cdots, N\}$ such that z is ranked among the top $|Z_0|$ positions by $2(1-\delta)pn' = (1-\delta)pn$ voters, since each such i contributes 2 voters that rank z among the top $|Z_0|$ positions.

As each $z \in Z_0$ is placed independently of the other members of Z_0, the probability that for every $z \in Z_0$ there exist fewer than $(1-\delta)pn'$ indices $i \in \{1, \cdots, n'\}$, such that there exist $j, k \in \{1, \cdots, N\}$ where v_{2i}^j and v_{2i-1}^k both rank z among the top $|Z_0|$ positions, is at most $e^{-\frac{|Z_0|pn'\delta^2}{2}}$. □

Theorem 5 will provide the basis for all of the lower bound proofs.

Theorem 5. *Let $\epsilon \in (0, \frac{1}{2})$, $\delta = 1 - (1-\epsilon)^{\frac{1}{2}}$, $n = 2n'$, and $N = 2 + \lceil \log(\frac{1}{\delta}) \rceil = O(\log(\frac{1}{\epsilon}))$. There exists a set of*

$$2^{\Omega\left(\frac{\epsilon^2}{\log(1/\epsilon)} n|Z_0| - \log(\log(1/\epsilon))\right)}$$

preference profiles S such that

1. *w is ranked in position $|Z_0| + |Y_0| + 1$ by all voters in every preference profile in S,*
2. *For every N preference profiles $P_i = (v_1^i, \cdots, v_n^i)$, $i = 1, \cdots, N$, from S, there exists an $r \in \{1, \cdots, N\}^n$ and $z \in Z_0$, such that in $P_r = (v_1^{r_1}, \cdots, v_n^{r_n})$, z is ranked among the top $|Z_0|$ positions by at least $(1-\epsilon)n$ voters.*

Proof. The proof of Theorem 5 employs the probabilistic method. Let S be a collection of

$$\left[e^{\frac{|Z_0|(1-\frac{\delta}{2})n'\delta^2}{8}}\right]^{\frac{1}{N}} - 1 = 2^{\Omega\left(\frac{\epsilon^2}{\log(1/\epsilon)} n|Z_0|\right)}$$

random preference profiles drawn from $\mathcal{P}$. Note that the collection S may not contain distinct preference profiles, since we sample from $\mathcal{P}$ with replacement.

Clearly, by construction, every preference profile in S satisfies property (1).

Notice that $\frac{1}{2}^{N-1} \leq \frac{\delta}{2}$. Consider N random preference profiles $P_1, \cdots, P_N$ drawn from $\mathcal{P}$. By Lemma 1, the probability that there is no $z \in Z_0$ and $r \in \{1, \cdots, N\}^n$ such that, in P_r there are at least $(1-\frac{\delta}{2})(1-\frac{1}{2}^{N-1})n > (1-\delta)^2 n = (1-\epsilon)n$ voters that rank z among the top $|Z_0|$ positions is

$$e^{-\frac{|Z_0|(1-\frac{1}{2}^{N-1})n'\delta^2}{8}} < e^{-\frac{|Z_0|(1-\frac{\delta}{2})n'\delta^2}{8}}.$$

The probability that the collection S satisfies property (2) is

$$
\begin{aligned}
Pr[S \text{ satisfies } (2)] &\geq 1 - Pr[S \text{ fails } (2)] \\
&\geq 1 - \binom{|S|}{N} Pr[P_1, \cdots, P_N \text{ fails } (2)] \\
&> 1 - |S|^N Pr[P_1, \cdots, P_N \text{ fails } (2)] \\
&\geq 1 - |S|^N \cdot e^{-\frac{|Z_0|(1-\frac{\delta}{2})n'\delta^2}{8}} \\
&> 0,
\end{aligned}
$$

where, in the second and third lines, $Pr[P_1, \cdots, P_N \text{ fails } (2)]$ is the probability that N randomly selected preference profiles from $\mathcal{P}$ fails to satisfy property (2).

Since with probability strictly greater than 0, S satisfies property (2), it is concluded that such a collection S exists.

Notice that if a given preference profile P appears in S more than $N-1$ times, then S does not satisfy property (2) (because mixing and matching voters from N copies of P results in another copy of P). Hence, there are necessarily a set of

$$
\frac{|S|}{N} = 2^{\Omega\left(\frac{\epsilon^2}{\log(1/\epsilon)} n|Z_0| - \log(\log(1/\epsilon))\right)}
$$

distinct preference profiles that satisfies (1) and (2). $\square$

For a given $\epsilon \in (0,1)$, let S_ϵ be the set shown to exist in Theorem 5 and let N_ϵ be the corresponding value of N.

Theorem 5 can be used to prove lower bounds on the communication complexity of approximating Borda, Bucklin, and Copeland.

Theorem 6. *Let $\rho \in (\frac{1}{\sqrt{2}}, 1)$ and $\frac{1}{\sqrt{2}} + \delta = \rho$. The communication complexity of obtaining a rho-approximation to Borda is $\Omega\left(\frac{\delta^3}{\log(1/\delta)} nm - \log(\log(1/\delta))\right)$.*

Proof. Let $c = \frac{1}{2}(1 + \frac{1}{\sqrt{2}\rho})$. Let $\alpha = 1 - \frac{1}{2c\rho}$ and let $\beta = 1 - 2(1-\alpha)^2$. Notice that $\frac{1}{1-\alpha} = 2c\rho$ and $\frac{1-\alpha}{1-\beta} = \frac{1}{2(1-\alpha)}$.

Let m be sufficiently large so that $\left(\frac{m-1}{m-1/(1-\alpha)}\right) < \frac{1}{c}$. Let $\epsilon = 1 - c$. Thus, $\frac{1}{1-\epsilon} = \frac{1}{c}$.

Employing Theorem 5 requires that we must specify how A is partitioned into X, Y, and Z. It suffices to specify the sizes of each set, as we are indifferent to the particular alternatives in each. Let Z_0 and Z_1 each contain βm alternatives and let Y_0 and Y_1 each contain $(\alpha - \beta)m$ alternatives. Thus, $|Z| + |Y| = 2\alpha m$ and $|X| = m - 2\alpha m - 1$. Notice that $\beta < \alpha < \frac{1}{2}$, so A can be partitioned in this manner.

Let S_ϵ be the set shown to exist in Theorem 5. Then

$$
\begin{aligned}
|S_\epsilon| &= 2^{\Omega\left(\frac{\epsilon^2}{\log(1/\epsilon)} n|Z_0| - \log(\log(1/\epsilon))\right)} \\
&= 2^{\Omega\left(\frac{\delta^3}{\log(1/\delta)} nm - \log(\log(1/\delta))\right)}.
\end{aligned}
$$

It will be shown that S_ϵ is a ρ-fooling set.

In every preference profile in S_ϵ, w is ranked in position $|Z_0| + |Y_0| + 1 = \alpha m + 1$. Hence, the Borda score of w is $n(m - \alpha m - 1)$ in every preference profile in S. The Borda score of every other alternative is at most $n\frac{m-1}{2}$. Thus, the approximation ratio obtained by any $x \in A \setminus \{w\}$ is

$$
\begin{aligned}
\frac{sc(x)}{sc(w)} &\leq \frac{n\frac{m-1}{2}}{n((1-\alpha)m - 1)} \\
&= \left(\frac{1}{1-\alpha}\right)\left(\frac{m-1}{m-1/(1-\alpha)}\right)\frac{1}{2} \\
&< (2c\rho) \cdot \frac{1}{c} \cdot \frac{1}{2} \\
&= \rho
\end{aligned}
$$

Thus, in every preference profile in S, no alternative other than w obtains a rho-approximation.

For any N_ϵ preference profiles, there exists a $z \in Z$ and $r \in \{1, \cdots, N_\epsilon\}$ such that in P_r, z is ranked among the top $|Z_0|$ positions by $(1-\epsilon)n$ of the voters. Hence, the Borda score of z is at least $(1-\epsilon)n(m - \beta m)$. In P_r the approximation obtained by w is then

$$
\begin{aligned}
\frac{sc(w)}{sc(x)} &\leq \frac{n((1-\alpha)m - 1)}{(1-\epsilon)n(1-\beta)m} \\
&< \frac{1-\alpha}{(1-\epsilon)(1-\beta)} \\
&= \frac{1}{2(1-\alpha)(1-\epsilon)} \\
&< \frac{1}{2} \cdot (2c\rho) \cdot \frac{1}{c} \\
&= \rho
\end{aligned}
$$

Therefore, in any $N_\epsilon = O(\log(1/\delta))$ preference profiles, it is possible to mix voters in such a way that w no longer obtains a rho-approximation. Therefore, the communication complexity of computing a rho-approximation to Borda is

$$
\log\left(\frac{|S_\epsilon|}{N_\epsilon}\right) = \Omega\left(\frac{\delta^3}{\log(1/\delta)} nm - \log(\log(1/\delta))\right).
$$

$\square$

Our construction shows that sufficiently good approximations to Borda have communication complexity $\Omega(nm)$. The lower bound for Bucklin is significantly stronger. It is shown that any deterministic communication protocol that computes any constant factor approximation to Bucklin has communication complexity $\Omega(nm)$. Further, non-trivial lower bounds are presented for a number of non-constant approximation ratios.

Theorem 7. *Let $\rho > 1$. The communication complexity of obtaining a ρ-approximation to Bucklin is $\Omega\left(\frac{nm}{\rho^2}\right)$.*

Proof. Let $\alpha = \frac{1}{2(\rho+1)}$ and $\beta = 2\alpha^2$. Let m be sufficiently large so that $\frac{m-1}{m} > \frac{\rho}{\rho+1}$. Let $\epsilon = \frac{1}{4}$.

Let Z_0 and Z_1 each contain βm alternatives and let Y_0 and Y_1 each contain $(\alpha - \beta)m$ alternatives. Thus, $|Z| + |Y| = 2\alpha m$ and $|X| = m - 2\alpha m - 1$. Notice that $\beta < \alpha < \frac{1}{2}$, so A can be partitioned in this manner.

Let S_ϵ be the set shown to exist in Theorem 5, then

$$
|S_\epsilon| = 2^{\Omega(n|Z_0|)} = 2^{\Omega\left(\frac{nm}{\rho^2}\right)}.
$$

In every preference profile in S_ϵ, w is ranked in position $|Z_0| + |Y_0| + 1 = \alpha m + 1$. Hence, the Bucklin score of w is $\alpha m + 1$. The Bucklin score of $x \in A \setminus \{w\}$ is at least $\frac{m-1}{2}$.

Hence, the approximation obtained by $x \in A \setminus \{w\}$ is

$$\frac{sc(x)}{sc(w)} \geq \frac{\frac{m-1}{2}}{\alpha m} = (\rho+1)\frac{m-1}{m} > \rho.$$

For any N_ϵ preference profiles, there exists a $z \in Z$ and $r \in \{1, \cdots, N_\epsilon\}$ such that in P_r, z is ranked among the top $|Z_0|$ positions by at least $(1-\epsilon)n = \frac{3n}{4}$ of the voters. Hence, the Bucklin score of z is at most βm. In P_r, the approximation obtained by w is then at least

$$\frac{sc(w)}{sc(x)} \geq \frac{\alpha m}{\beta m} = \frac{\alpha}{\beta} = \frac{1}{2\alpha} > \rho.$$

Therefore, in any N_ϵ preference profiles, it is possible to mix voters in such a way that w no longer obtains a ρ-approximation. By Theorem 1, the communication complexity of obtaining a ρ-approximation to Bucklin is

$$\log\left(\frac{|S_\epsilon|}{N_\epsilon}\right) = \Omega\left(\frac{nm}{\rho^2}\right).$$

□

The previous subsection showed that for each $\delta \in (0,1)$, the communication complexity of computing a m^δ approximate winner in Bucklin elections is $O(nm^{(1-\delta)}\log(m))$. The next result provides a lower bound of $\Omega(nm^{(1-2\delta)})$ for all $\delta \in (\frac{1}{2}, 1)$ on the communication complexity of computing a m^δ approximate winner in Bucklin elections.

Theorem 8. *Let $\delta \in (0, \frac{1}{2})$. The communication complexity of obtaining a m^δ approximation to Bucklin is $\Omega(nm^{(1-2\delta)})$.*

Proof Sketch. The Theorem follows from the proof of Theorem 7, by letting $\rho = m^\delta$. The fooling set S_ϵ in the proof of Theorem 7 contains

$$\begin{aligned} |S| &= 2^{\Omega\left(\frac{nm}{\rho^2}\right)} \\ &= 2^{\Omega\left(\frac{nm}{m^{2\delta}}\right)} \\ &= 2^{\Omega\left(nm^{(1-2\delta)}\right)}. \end{aligned}$$

All that needs to be observed is that the partition of A into X, Y, and Z, given the selection of sizes for each in the proof of Theorem 7, is still valid. However, this is easily determined to be true.

Therefore, the communication complexity of computing a m^δ approximation to Bucklin is

$$\log\left(\frac{|S|}{N_\epsilon}\right) = \Omega\left(nm^{(1-2\delta)}\right).$$

□

A lower bound on the communication complexity of computing ρ-approximate winners for Copeland is provided next.

Theorem 9. *Let $\rho \in (\frac{1}{\sqrt{2}}, 1)$ and $\frac{1}{\sqrt{2}} + \delta = \rho$. The communication complexity of obtaining a ρ-approximation to Copeland is $\Omega(\delta nm)$.*

Proof. Let $m = m' + 1$, $c = \frac{1}{2}(1 + \frac{1}{\sqrt{2}\rho})$. Let $\alpha = 1 - \frac{1}{2c\rho}$ and let $\beta = 1 - 2(1-\alpha)^2$. Notice that $\frac{1}{1-\alpha} = 2c\rho$ and $\frac{1-\alpha}{1-\beta} = \frac{1}{2(1-\alpha)}$. Let m' be sufficiently large so that $\frac{1}{(1-\beta)m'} < (1-c)\rho$. Let $\epsilon = \frac{1}{4}$.

Let Z_0 and Z_1 each contain $\beta m'$ alternatives and let Y_0 and Y_1 each contain $(\alpha - \beta)m' - 1$ alternatives. Thus, $|Z| + |Y| = 2\alpha m' - 2$ and $|X| = m' - 2\alpha m' + 2$. Notice that $\beta < \alpha < \frac{1}{2}$, so A can be partitioned in this manner.

Let S_ϵ be the set shown to exist in Theorem 5, then

$$|S_\epsilon| = 2^{\Omega(n|Z_0|)} = 2^{\Omega(\delta nm)}.$$

It will be shown that S_ϵ is a ρ-fooling set.

In every preference profile in S_ϵ, w defeats all members of X and ties all members of $Z \cup Y$ in pairwise elections. Hence, the Copeland score of w is $|X| + \frac{|Z|+|Y|}{2} = (1-2\alpha)m' + 2 + \alpha m' - 1 = (1-\alpha)m' + 1$ in every preference profile in S. The Copeland score of every other alternative is at most $\frac{m'}{2}$. Thus, the approximation ratio obtained by any $x \in A \setminus \{w\}$ is

$$\begin{aligned} \frac{sc(x)}{sc(w)} &= \frac{\frac{m'}{2}}{(1-\alpha)m' + 1} \\ &< \frac{1}{2(1-\alpha)} \\ &= c\rho \\ &< \rho. \end{aligned}$$

For any N_ϵ preference profiles, there exists a $z \in Z$ and $r \in \{1, \cdots, N_\epsilon\}$ such that in P_r, z is ranked among the top $|Z_0|$ positions by over half of the voters. Hence, in pairwise elections z defeats all alternatives in X and Y. By construction, if $z_i^0 \in Z$ defeats z in a pairwise election, then z necessarily defeats z_i^1 in a pairwise election. Hence, the Copeland score of z is at least $|X| + \frac{|Z|}{2} + |Y| = (1-2\alpha)m' + 2 + \beta m' + 2(\alpha - \beta)m' - 2 = (1-\beta)m'$.

Likewise, w defeats every alternative in X in a pairwise election. However, if w defeats z_i^0 then z_i^1 necessarily defeats w. Likewise, for the alternatives in Y. Thus, the Copeland score of w is at most $|X| + \frac{|Z|+|Y|}{2} = (1-2\alpha)m' + 2 + \alpha m' - 1 = (1-\alpha)m' + 1$.

Thus, in P_r, the approximation obtained by w is

$$\begin{aligned} \frac{sc(w)}{sc(x)} &\leq \frac{(1-\alpha)m' + 1}{(1-\beta)m'} \\ &= \frac{(1-\alpha)}{(1-\beta)} + \frac{1}{(1-\beta)m'} \\ &= \frac{1}{2(1-\alpha)} + \frac{1}{(1-\beta)m'} \\ &< c\rho + (1-c)\rho \\ &= \rho. \end{aligned}$$

Therefore, the communication complexity of obtaining a rho-approximation to Copeland is

$$\log\left(\frac{|S_\epsilon|}{N_\epsilon}\right) = \Omega(\delta nm).$$

□

4. CONCLUSIONS

This paper presents upper and lower bounds on the communication complexity for computing approximate winners in Borda, Bucklin, and Copeland elections. It is shown that for every $\epsilon > 0$ the communication complexity of computing a $1 - \epsilon$ approximate winner in a Borda election is

$O\left(\log(\frac{1}{\epsilon})nm\right)$. For $\delta \in (0, 1-\frac{1}{\sqrt{2}})$, we show that computing a $\frac{1}{\sqrt{2}}+\delta$ approximate winner in Borda elections has communication complexity $\Omega\left(\frac{\delta^3}{\log(1/\delta)}nm - \log(\log(1/\delta))\right)$. However, computing the true Borda winner has communication complexity $\Omega(nm\log(m))$.

In Bucklin elections, the communication complexity of computing the true winner is $\Theta(nm)$. We show that for all $\rho > 1$, computing a ρ approximate winner in a Bucklin election has communication complexity $\Omega\left(\frac{nm}{\rho^2}\right)$. Hence, fixed constant factor approximate winners in Bucklin elections cannot be computed with asymptotically less communication than computing the true Bucklin winner. However, we show that for all $\delta \in (0,1)$, computing a m^δ approximate Bucklin winner has communication complexity $O(nm^{1-\delta}\log(m))$. For $\delta \in (\frac{1}{2}, 1)$, a lower bound on the communication complexity of computing a m^δ approximate Bucklin winner of $\Omega(nm^{(1-2\delta)})$ is presented.

A $\Omega(\delta nm)$ lower bound is also presented for the communication complexity of computing $\frac{1}{\sqrt{2}}+\delta$, $\delta \in (0, 1-\frac{1}{\sqrt{2}})$ approximate winner in Copeland elections. However, as the communication complexity of determining the true Copeland winner is $\Theta(nm\log(m))$, this lower bound leaves open the possibility of an approximation scheme for Copeland, similar to the scheme presented for Borda.

The lower bounds on the communication complexity for computing approximate Borda and Copeland winners only hold for sufficiently good approximation ratios. It may be the case that worse constant factor approximation ratios can be obtained to these rules with a reduced communication complexity overhead. However, we conjecture that the communication complexity of obtaining any constant factor approximation to Borda and Copeland is $\Omega(nm)$. This is an interesting line of future work.

A second line of future work is the design of non-constant factor approximation protocols similar to the one presented for Bucklin. Along this theme, a construction that allows for non-constant factor lower bound proofs is desirable. The construction presented in this paper is limited to sufficiently small constant factor lower bound proofs for Borda and Copeland. It is also desirable to extend this construction to Maximin elections also.

Finally, a third, and potentially fruitful line of further work is the study of the randomized communication complexity of both exact and approximation winner determination. To the best of our knowledge, the randomized communication complexity of (approximate or exact) winner determination has not be studied.

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