Worst-Case Optimal Redistribution of VCG Payments in Heterogeneous-Item Auctions with Unit Demand

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ABSTRACT

Many important problems in multiagent systems involve the allocation of multiple resources among the agents. For resource allocation problems, the well-known VCG mechanism satisfies a list of desired properties, including efficiency, strategy-proofness, individual rationality, and the non-deficit property. However, VCG is generally not budget-balanced. Under VCG, agents pay the VCG payments, which reduces social welfare. To offset the loss of social welfare due to the VCG payments, VCG redistribution mechanisms were introduced. These mechanisms aim to redistribute as much VCG payments back to the agents as possible, while maintaining the aforementioned desired properties of the VCG mechanism.

We continue the search for worst-case optimal VCG redistribution mechanisms – mechanisms that maximize the fraction of total VCG payment redistributed in the worst case. Previously, a worst-case optimal VCG redistribution mechanism (denoted by WCO) was characterized for multi-unit auctions with nonincreasing marginal values [7]. Later, WCO was generalized to settings involving heterogeneous items [4], resulting in the HETERO mechanism. [4] *conjectured* that HETERO is feasible and worst-case optimal for heterogeneous-item auctions with unit demand. In this paper, we propose a more natural way to generalize the WCO mechanism. We prove that our generalized mechanism, though represented differently, actually coincides with HETERO. Based on this new representation of HETERO, we prove that HETERO is indeed feasible and worst-case optimal in heterogeneous-item auctions with unit demand. Finally, we conjecture that HETERO remains feasible and worst-case optimal in the even more general setting of combinatorial auctions with gross substitutes.

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1. INTRODUCTION

1.1 VCG Redistribution Mechanisms

Many important problems in multiagent systems involve the allocation of multiple resources among the agents. For resource allocation problems, the well-known VCG mechanism satisfies the following list of desired properties:

- *Efficiency*: the allocation maximizes the agents' total valuation (without considering payments).
- *Strategy-proofness*: for any agent, reporting truthfully is a dominant strategy, regardless of the other agents' types.
- *(Ex post) individual rationality*: Every agent's final utility (after deducting her payment) is always nonnegative.
- *Non-deficit*: the total payment *from* the agents is nonnegative.

However, VCG is generally not budget-balanced. Under VCG, agents pay the VCG payments, which reduces social welfare. To offset the loss of social welfare due to the VCG payments, VCG redistribution mechanisms were introduced. These mechanisms still allocate the resources using VCG. On top of VCG, these mechanisms try to redistribute as much VCG payments back to the agents as possible. We require that *an agent's redistribution be independent of her own type*. This is sufficient for maintaining strategy-proofness and efficiency (an agent has no control over her own redistribution). For smoothly connected domains (including multi-unit auctions with nonincreasing marginal values and heterogeneous-item auctions with unit demand), the above requirement is also necessary for maintaining strategy-proofness and efficiency [8]. A VCG redistribution mechanism is *feasible* if it maintains all the desired properties of the VCG mechanism. That is, we also require that the redistribution process maintains individual rationality and the non-deficit property.

Let n be the number of agents. Since all VCG redistribution mechanisms start by allocating according to the VCG mechanism, a VCG redistribution mechanism is characterized by its redistribution scheme $\vec{r} = (r_1, r_2, \ldots, r_n)$. Under VCG redistribution mechanism $\vec{r}$, agent i's redistribution equals $r_i(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$, where θ_j is agent j's type. (We do not have to differentiate between an agent's true type and her reported type, since all VCG redistribution mechanisms are strategy-proof.) For the mechanism design objective studied in this paper, it is without loss of generality to only consider VCG redistribution mechanisms that are anonymous (we defer the proof of this claim to the appendix). An anonymous VCG redistribution mechanism is characterized by a single function r. Under (anonymous) VCG redistribution mechanism r, agent i's redistribution equals $r(\theta_{-i})$, where θ_{-i} is the *multiset* of the types of the agents other than i.

We use $\vec{\theta}$ to denote the type profile. Let $VCG(\vec{\theta})$ be the total VCG payment for this type profile. A VCG redistribution mechanism r satisfies the non-deficit property if the total redistribution never exceeds the total VCG payment. That is, for any type profile $\vec{\theta}$, $\sum_i r(\theta_{-i}) \leq VCG(\vec{\theta})$. A VCG redistribution mechanism r is (ex post) individually rational if every agent's final utility is always nonnegative. Since VCG is individually rational, we have that a sufficient condition for r to be individually rational is for any $\vec{\theta}$ and any i, $r(\theta_{-i}) \geq 0$ (on top of VCG, every agent also receives a redistribution amount that is always nonnegative). On the other hand, when agent i is not interested in any item (her valuation on any item bundle equals 0), under VCG, i's utility always equals 0. After redistribution, agent i's utility is exactly her redistribution $r(\theta_{-i})$. That is, $r(\theta_{-i}) \geq 0$ for all θ_{-i} (hence for all $\vec{\theta}$ and all i) is also necessary for individual rationality.

We want to find VCG redistribution mechanisms that maximize *the fraction of total VCG payment redistributed in the worst-case*. This mechanism design problem is equivalent to the following functional optimization model:

Variable function: r
Maximize: α (worst-case redistribution fraction)
Subject to:
Non-deficit: $\forall\vec{\theta}$, $\sum_i r(\theta_{-i}) \leq VCG(\vec{\theta})$
Individual rationality: $\forall\theta_{-i}$, $r(\theta_{-i}) \geq 0$
Worst-case guarantee: $\forall\vec{\theta}$, $\sum_i r(\theta_{-i}) \geq \alpha VCG(\vec{\theta})$

In this paper, we will analytically characterize one worst-case optimal VCG redistribution mechanism for heterogeneous-item auctions with unit demand.[1]

We conclude this subsection with an example VCG redistribution mechanism in the simplest setting of single-item auctions. In a single-item auction, an agent's type is a nonnegative real number representing her utility for winning the item. Without loss of generality, we assume that $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0$. In single-item auctions, the Bailey-Cavallo VCG redistribution mechanism [2, 3] works as follows:

- Allocate the item according to VCG: Agent 1 wins the item and pays θ_2. The other agents win nothing and do not pay.
- Every agent receives a redistribution that equals $\frac{1}{n}$ times the second highest *other* type: Agent 1 and 2 each receives $\frac{1}{n}\theta_3$. The other agents each receives $\frac{1}{n}\theta_2$.

The above mechanism obviously maintains strategy-proofness and efficiency (an agent's redistribution does not depend on her own type). It also maintains individual rationality because all redistributions are nonnegative. The total redistribution equals $\frac{2}{n}\theta_3 + \frac{n-2}{n}\theta_2$. This is never more than the total VCG payment θ_2. That is, the above mechanism maintains the non-deficit property. Finally, the total redistribution $\frac{2}{n}\theta_3 + \frac{n-2}{n}\theta_2 \geq \frac{n-2}{n}\theta_2$. That is, for single-item auctions, this example mechanism's worst-case redistribution fraction is $\frac{n-2}{n}$ (the worst-case is reached when $\theta_3 = 0$).

1.2 Previous Research on Worst-Case Optimal VCG Redistribution Mechanisms

In this subsection, we review existing results on worst-case optimal VCG redistribution mechanisms. Besides high-level discussions, we also choose to include a certain level of technical details, as they are needed for later sections.

[1]The problem of assigning heterogeneous items to unit demand agents is also often called the assignment problem.

We organize existing results by their settings.

Worst-Case Optimal Redistribution in Multi-Unit Auctions with Unit Demand [7, 12]: In multi-unit auctions with unit demand, the items for sale are identical. Each agent wants at most one copy of the item. (Single-item auctions are special cases of multi-unit auctions with unit demand.) Let m be the number of items. *Throughout this paper, we only consider cases where* $m \leq n-2$.[2] Here, an agent's type is a nonnegative real number representing her valuation for winning one copy of the item. It is without loss of generality to assume that $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0$. [7] showed that for multi-unit auctions with unit demand, any VCG redistribution mechanism's worst-case redistribution fraction is at most

$$\alpha^* = 1 - \frac{\binom{n-1}{m}}{\sum_{j=m}^{n-1}\binom{n-1}{j}}$$

If we switch to a more general setting, then α^* is still an upper bound: if there exists a VCG redistribution mechanism whose worst-case redistribution fraction is strictly larger than α^* in a more general setting, then this mechanism, when applied to multi-unit auctions with unit demand, has a worst-case redistribution fraction that is strictly larger than α^*, which contradicts with the meaning of α^*.

[7] also characterized a VCG redistribution mechanism for multi-unit auctions with unit demand, called the WCO mechanism.[3] WCO's worst-case redistribution fraction is exactly α^*. That is, it is worst-case optimal.

WCO was obtained by optimizing within the family of *linear* VCG redistribution mechanisms. A linear VCG redistribution mechanism r takes the following form:

$$r(\theta_{-i}) = \sum_{j=1}^{n-1} c_j[\theta_{-i}]_j$$

Here, the c_i are constants. (We only consider the c_i that correspond to feasible VCG redistribution mechanisms.) $[\theta_{-i}]_j$ is the j-th highest type among θ_{-i}. Linear mechanism r is characterized by the values of the c_i. The optimal values the c_i are as follows:

$$c_i^* = \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i\sum_{j=m}^{n-1}\binom{n-1}{j}} \frac{1}{\binom{n-1}{i}} \sum_{j=i}^{n-1}\binom{n-1}{j}$$

for $i = m+1, \ldots, n-1$, and $c_i^* = 0$ for $i = 1, 2, \ldots, m$.

The characterization of WCO then follows:

$$r(\theta_{-i}) = \sum_{j=1}^{n-1} c_j^*[\theta_{-i}]_j = \sum_{j=m+1}^{n-1} c_j^*[\theta_{-i}]_j$$

Worst-Case Optimal Redistribution in Multi-Unit Auctions with Nonincreasing Marginal Values [7]: Multi-unit auctions with non-

[2][7] showed that for multi-unit auctions with unit demand, when $m = n-1$, the worst-case redistribution fraction (of any feasible VCG redistribution mechanism) is at most 0. Since the setting studied in this paper is more general (heterogeneous-item auctions with unit demand), we also have that the worst-case redistribution fraction is at most 0 when $m = n-1$. Since heterogeneous-item auctions with x units are special cases of heterogeneous-item auctions with $x+1$ units, we have that for our setting the worst-case redistribution fraction is at most 0 when $m \geq n-1$. That is, not redistributing anything is worst-case optimal when $m \geq n-1$.

[3]WCO has also been independently derived in [12], under a slightly different objective of maximizing worst-case efficiency ratio. Also, for [12]'s objective, the optimal mechanism coincides with WCO only when the individual rationality constraint is enforced.

increasing marginal values are more general than multi-unit auctions with unit demand. In this more general setting, the items are still identical, but an agent may demand more than one copy of the item. An agent's valuation for winning the first copy of the item is called her initial/first marginal value. Similarly, an agent's additional valuation for winning the i-th copy of the item is called her i-th marginal value. An agent's type contains m nonnegative real numbers (i-th marginal value for $i = 1, \ldots, m$). In this setting, it is further assumed that the marginal values are nonincreasing.

As discussed earlier, in this more general setting, any VCG redistribution mechanism's worst-case redistribution fraction is still bounded above by α^*. [7] generalized WCO to this setting, and proved that its worst-case redistribution fraction remains the same. Therefore, WCO (after generalization) is also worst-case optimal for multi-unit auctions with nonincreasing marginal values.

The original definition of WCO does not directly generalize to multi-unit auctions with nonincreasing marginal values. When it comes to multi-unit auctions with nonincreasing marginal values, an agent's type is no longer a single value, which means that there is no such thing as "the j-th highest type among θ_{-i}". To address this, [7] replaced $[\theta_{-i}]_j$ by $\frac{1}{m}R(\theta_{-i}, j-m-1)$ for $j = m+1, \ldots, n-1$. Basically, $R(\theta_{-i}, j-m-1)$ is the generalization of $[\theta_{-i}]_j$: it is identical to $[\theta_{-i}]_j$ in the unit demand setting, and it remains well-defined for multi-unit auctions with nonincreasing marginal values. We abuse notation by not differentiating the agents and their types. For example, θ_{-i} is equivalent to the set of agents other than i. Let S be a set of agents. $R(S, i)$ is formally defined as follows (this definition is included for completeness; we will not use it anywhere):

- $R(S, 0) = VCG(S)$ (the total VCG payment when only those in S participate in the auction).
- For $i = 1, \ldots, |S|-m-1$, $R(S, i) = \frac{1}{m+i}\sum_{j=1}^{m+i} R(U(S, j), i-1)$. Here, $U(S, j)$ is the new set of agents, after removing the agent with the j-th highest initial marginal value in S from S.

The general form of WCO is as follows:

$$r(\theta_{-i}) = \frac{1}{m}\sum_{j=m+1}^{n-1} c_j^* R(\theta_{-i}, j-m-1)$$

Worst-Case Optimal Redistribution in Heterogeneous-Item Auctions with Unit Demand [4]: In heterogeneous-item auctions with unit demand, the items for sale are different. Each agent demands at most one item. Here, an agent's type consists of m nonnegative real numbers (her valuation for winning item i for $i = 1, \ldots, m$). Heterogeneous-item auctions with unit demand is the main focus of this paper.

Since heterogeneous-item auctions with unit demand is more general than multi-unit auctions with unit demand, α^* is still an upper bound on the worst-case redistribution fraction. [4] proposed the HETERO mechanism, by generalizing WCO. The authors *conjectured* that HETERO is feasible and has a worst-case redistribution fraction that equals α^*. That is, the authors conjectured that HETERO is worst-case optimal in this setting. The main contribution of this paper is a proof of this conjecture.

Redistribution in Combinatorial Auctions with Gross Substitutes [6]: The gross substitutes condition was first proposed in [9]. Like unit demand, the gross substitutes condition is a condition on an agent's type (does not depend on the mechanism under discussion). In words, an agent's type satisfies the gross substitutes condition if her demand for an item does not decrease when the prices of the other items increase. Both multi-unit auctions with nonincreasing marginal values and heterogeneous-item auctions with unit demand are special cases of combinatorial auctions with gross substitutes [5, 9]. [6] showed that for this setting, the worst-case redistribution fraction of the Bailey-Cavallo mechanism [2, 3] is exactly $\frac{n-m-1}{n}$ (when $n \geq m+1$), and it is possible to construct mechanisms with even higher worst-case redistribution fractions. The authors did not find a worst-case optimal mechanism for this setting. At the end of this paper, we conjecture that HETERO is optimal for combinatorial auctions with gross substitutes.

Finally, Naroditskiy *et al.* [13] proposed a numerical technique for designing worst-case optimal redistribution mechanisms. The proposed technique only works for single-parameter domains. It does not apply to our setting (multi-parameter domain).

1.3 Our contribution

We generalize WCO to heterogeneous-item auctions with unit demand. We prove that the generalized mechanism, though represented differently, coincides with the HETERO mechanism proposed in [4]. That is, what we proposed is not a new mechanism, but a new representation of an existing mechanism. Based on our new representation of HETERO, we prove that HETERO is indeed feasible and worst-case optimal when applied to heterogeneous-item auctions with unit demand, thus confirming the conjecture raised in [4]. We conclude with a new conjecture that HETERO remains feasible and worst-case optimal in the even more general setting of combinatorial auctions with gross substitutes.

2. NEW REPRESENTATION OF HETERO

We recall that WCO was obtained by optimizing within the family of linear VCG redistribution mechanisms. The original representation of HETERO was obtained using a similar approach [4]. The authors focused on the following family of mechanisms:

$$r(\theta_{-i}) = \sum_{j=1}^{n-m-1} \beta_j t(\theta_{-i}, j-1)$$

Here, the β_i are constants. $t(S, j)$ is the *expected* total VCG payment when we remove j agents uniformly at random from S, and allocate all the items to the remaining agents. It is easy to see that all member mechanisms of the above family are well-defined for general combinatorial auctions. Not every member mechanism is feasible though.

[4] did not attempt optimizing over the family. Instead, the β_i are chosen so that the corresponding mechanism coincides with WCO when it comes to multi-unit auctions with unit demand. It turns out that the choice is *unique*, and the corresponding mechanism is called HETERO. [4] *conjectured* that HETERO is feasible and worst-case optimal for heterogeneous-item auctions with unit demand.

In this section, we propose another way to generalize WCO. We will show that the generalized WCO actually coincides with HETERO. That is, what we derive is a new representation of HETERO. This new representation will prove itself useful in later discussions.

We recall that the characterization of WCO for multi-unit auctions with nonincreasing marginal values is based on a series of functions $R(S, i)$. These functions do not directly generalize to settings involving heterogeneous items, because, for $i > 0$, $R(S, i)$ is defined explicitly based on the agents' initial marginal values. Fortunately, there is an easy way to rewrite $R(S, i)$, so that it becomes well-defined for settings involving heterogeneous items.

[7] proved that for $0 \le j \le |S| - m - 2$,

$$\sum_{a \in S} R(S-a, j) = (|S|-m-1-j)R(S,j)+(m+1+j)R(S,j+1) \quad (1)$$

Based on Equation 1, WCO can be rewritten into the following form (the only changes are that for $i > 0$, $R(S,i)$'s definition no longer mentions "initial marginal values"):

Definition 1. Heterogeneous WCO (new representation of HETERO):

$$r(\theta_{-i}) = \frac{1}{m} \sum_{j=m+1}^{n-1} c_j^* R(\theta_{-i}, j-m-1)$$

- $R(S, 0) = VCG(S)$
- For $i = 1, \ldots, |S| - m - 1$, $R(S,i)$ equals:

$$\frac{1}{m+i}\left(\sum_{a \in S} R(S-a, i-1) - (|S| - m - i)R(S, i-1)\right)$$

Heterogeneous WCO is well-defined for general combinatorial auctions, so we can directly apply it to heterogeneous-item auctions with unit demand. Of course, we still have the burden to prove that it remains feasible and worst-case optimal. We will do so in the next section.

Heterogeneous WCO is not a new mechanism. It turns out that it coincides with HETERO for general combinatorial auctions. That is, Definition 1 is a new representation of the existing mechanism HETERO.

PROPOSITION 1. *Heterogeneous WCO coincides with HETERO for general combinatorial auctions.*

Proof omitted since it is based on pure algebraic manipulation.

3. FEASIBILITY AND WORST-CASE OPTIMALITY OF HETERO

In this section, we prove that HETERO, as represented in Definition 1, is feasible and worst-case optimal for heterogeneous-item auctions with unit demand.

We first define the *redistribution monotonicity* condition:

Definition 2. An auction setting satisfies *redistribution monotonicity* if for any set of agents S, we have that

$$R(S, 0) \ge R(S, 1) \ge \ldots \ge R(S, |S| - m - 1) \ge 0$$

R was defined in Definition 1. That is, $R(S, 0) = VCG(S)$, and for $i = 1, \ldots, |S| - m - 1$, $R(S,i)$ equals

$$\frac{1}{m+i}\left(\sum_{a \in S} R(S-a, i-1) - (|S| - m - i)R(S, i-1)\right).$$

For example, the setting of single-item auctions satisfies redistribution monotonicity. In a single-item auction, $R(S, 0) = VCG(S) = [S]_2$ ($[S]_i$ is the i-th highest type from the agents in S).

$$R(S, 1) = \frac{1}{2}\left(\sum_{a \in S} R(S-a, 0) - (|S| - 2)R(S, 0)\right)$$

$$= \frac{1}{2}\left(2[S]_3 + (|S| - 2)[S]_2 - (|S| - 2)[S]_2\right) = [S]_3.$$

Similarly, $R(S, 2) = [S]_4$, $R(S, 3) = [S]_5, \ldots$, and finally $R(S, |S| - m - 1) = R(S, |S| - 2) = [S]_{|S|}$ (lowest type from the agents in S). It is clear that redistribution monotonicity holds here.

More generally, redistribution monotonicity holds for multi-unit auctions with nonincreasing marginal values: Claim 17 of [7] proved that $R(S,i)$ is nonincreasing in i for multi-unit auctions with nonincreasing marginal values; $R(S,i)$'s original definition as described in Subsection 1.2 makes it clear that the $R(S,i)$ are nonnegative.

The following proposition greatly simplifies our task:

PROPOSITION 2. *If the setting satisfies redistribution monotonicity, then HETERO is feasible (strategy-proof, efficient, individually rational, and non-deficit), and its worst-case redistribution fraction is at least α^*. If the setting is also more general than multi-unit auctions with unit demand, then HETERO is worst-case optimal.*

PROOF. We first prove that HETERO is feasible given redistribution monotonicity. According to Definition 1, under HETERO, an agent's redistribution does not depend on her own type. That is, HETERO is strategy-proof and efficient in all settings. We only need to prove that HETERO is individually rational and non-deficit given redistribution monotonicity.

Individual rationality: As discussed in Subsection 1.1, individual rationality is equivalent to redistributions being nonnegative. We recall that for multi-unit auctions with unit demand, under WCO, agent i's redistribution equals

$$r(\theta_{-i}) = \sum_{j=m+1}^{n-1} c_j^* [\theta_{-i}]_j$$

WCO is known to be individually rational. That is, for all θ_{-i},

$$\sum_{j=m+1}^{n-1} c_j^* [\theta_{-i}]_j \ge 0$$

This is equivalent to for all $x_0 \ge \ldots \ge x_{n-m-2} \ge 0$,

$$\sum_{j=m+1}^{n-1} c_j^* x_{j-m-1} \ge 0 \quad (2)$$

Under HETERO, agent i's redistribution equals

$$\frac{1}{m} \sum_{j=m+1}^{n-1} c_j^* R(\theta_{-i}, j-m-1) \quad (3)$$

Redistribution monotonicity implies that

$$R(\theta_{-i}, 0) \ge R(\theta_{-i}, 1) \ge \ldots \ge R(\theta_{-i}, n-m-2) \ge 0 \quad (4)$$

Based on (2) and (4) (substituting $R(\theta_{-i}, j)$ for x_j for all j), we have that (3) is nonnegative. Therefore, redistribution monotonicity implies individual rationality.

Non-deficit and worst-case optimality: For multi-unit auctions with unit demand, under WCO, the total VCG payment is $m\theta_{m+1}$. The total redistribution is

$$\sum_{i=1}^{n} \sum_{j=m+1}^{n-1} c_j^* [\theta_{-i}]_j = \sum_{j=m+1}^{n-1} c_j^* \sum_{i=1}^{n} [\theta_{-i}]_j$$

$$= \sum_{j=m+1}^{n-1} c_j^* (j\theta_{j+1} + (n-j)\theta_j)$$

WCO is known to be non-deficit and have worst-case redistribution fraction α^*. That is, for all $\theta_{m+1} \ge \ldots \ge \theta_n \ge 0$,

$$\alpha^* m\theta_{m+1} \le \sum_{j=m+1}^{n-1} c_j^* (j\theta_{j+1} + (n-j)\theta_j) \le m\theta_{m+1}$$

That is, for all $x_0 \geq x_1 \geq \ldots \geq x_{n-m-1} \geq 0$,

$$\alpha^* m x_0 \leq \sum_{j=m+1}^{n-1} c_j^*(j x_{j-m} + (n-j) x_{j-m-1}) \leq m x_0 \quad (5)$$

Under HETERO, the total redistribution is

$$\frac{1}{m}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1} c_j^* R(\theta_{-i}, j-m-1)$$

$$= \frac{1}{m}\sum_{j=m+1}^{n-1} c_j^*(jR(\vec{\theta}, j-m) + (n-j)R(\vec{\theta}, j-m-1)) \quad (6)$$

The total VCG payment equals $VCG(\vec{\theta}) = R(\vec{\theta}, 0)$. Redistribution monotonicity implies that

$$R(\vec{\theta}, 0) \geq R(\vec{\theta}, 1) \geq \ldots \geq R(\vec{\theta}, n-m-1) \geq 0 \quad (7)$$

Given (5) and (7) (substituting $R(\vec{\theta}, j)$ for x_j for all j), we have that (6) is between α^* times the total VCG payment and the total VCG payment. Therefore, redistribution monotonicity implies the non-deficit property and also worst-case optimality. $\square$

In the remaining of this section, we prove that heterogeneous-item auctions with unit demand satisfies redistribution monotonicity, which would then imply that HETERO is feasible and worst-case optimal for heterogeneous-item auctions with unit demand.

We define $R^j(S, i)$ by modifying the definition of $R(S, i)$ in Definition 1.

- $R^j(S, 0) = VCG^j(S)$. $VCG^j(S)$ is the VCG price of item j (the VCG payment from the agent winning item j) when we allocate all the items to the agents in S using VCG.
- For $i = 1, \ldots, |S| - m - 1$, $R^j(S, i)$ equals

$$\frac{1}{m+i}\left(\sum_{a \in S} R^j(S-a, i-1) - (|S|-m-i)R^j(S, i-1)\right).$$

PROPOSITION 3. *For any set of agents S, for $i = 0, \ldots, |S| - m - 1$, we have*

$$\sum_{j=1}^{m} R^j(S, i) = R(S, i)$$

PROOF. We prove by induction. When $i = 0$, by definition, for any S,

$$\sum_{j=1}^{m} R^j(S, 0) = R(S, 0)$$

Now let us assume that for $0 \leq k < |S| - m - 1$,

$$\sum_{j=1}^{m} R^j(S, k) = R(S, k)$$

We have that

$$\sum_{j=1}^{m} R^j(S, k+1)$$

$$= \sum_{j=1}^{m} \frac{1}{m+k+1}(\sum_{a \in S} R^j(S-a, k) - (|S|-m-k-1)R^j(S, k))$$

$$= \frac{1}{m+k+1}(\sum_{a \in S} R(S-a, k) - (|S|-m-k-1)R(S, k))$$

$$= R(S, k+1)$$

$\square$

We want to prove that for heterogeneous-item auctions with unit demand, the following redistribution monotonicity condition holds.

$$R(S, 0) \geq R(S, 1) \geq \ldots \geq R(S, |S|-m-1) \geq 0$$

By Proposition 3, it suffices to prove that for all j,

$$R^j(S, 0) \geq R^j(S, 1) \geq \ldots \geq R^j(S, |S|-m-1) \geq 0.$$

Without loss of generality, we will prove

$$R^1(S, 0) \geq R^1(S, 1) \geq \ldots \geq R^1(S, |S|-m-1) \geq 0.$$

To prove the above inequality, we need the following definitions and propositions. *From now on to the end of this section, the setting by default is heterogeneous-item auctions with unit demand, unless specified.*

We use $E(T, S)$ to denote the efficient total valuation when we allocate all the items in T to the agents in S.

PROPOSITION 4. *Submodularity in both items and agents [14]: For any T_1, T_2, S, we have*

$$E(T_1, S) + E(T_2, S) \geq E(T_1 \cup T_2, S) + E(T_1 \cap T_2, S).$$

For any T, S_1, S_2, we have

$$E(T, S_1) + E(T, S_2) \geq E(T, S_1 \cup S_2) + E(T, S_2 \cap S_2).$$

[14] showed that the proposition is true when gross substitutes condition holds. Heterogeneous-item auctions with unit demand satisfies gross substitutes.

We use $\{1\} \oplus \{1, \ldots, m\}$ to denote the item set that contains not only item 1 to m, but also an additional duplicate of item 1.

PROPOSITION 5. *Let S be any set of agents. Let a be the agent who wins item 1 when we allocate the items $\{1, \ldots, m\}$ to the agents in S. We have that $E(\{1\} \oplus \{1, \ldots, m\}, S) = E(\{1\}, a) + E(\{1, \ldots, m\}, S - a)$. That is, after we add an additional duplicate of item 1 to the auction, there exists an efficient allocation under which agent a still wins item 1.*

The above proposition was proved in [11].

PROPOSITION 6. *For any set of agents S, for any $a \in S$, we have $VCG^1(S) \geq VCG^1(S - a)$. That is, the VCG price of item 1 is nondecreasing as the set of agents expands.*

PROOF. Let w_1 be the winner of item 1 when we allocate the items $\{1, \ldots, m\}$ to the agents in S using VCG. $VCG^1(S) = E(\{1, \ldots, m\}, S - w_1) - E(\{2, \ldots, m\}, S - w_1)$. a could be either w_1 or some other agent. We discuss case by case.

Case $a = w_1$: Let w_1' be the new winner of item 1 when we allocate the items $\{1, \ldots, m\}$ to the agents in $S - w_1$ using VCG. $VCG^1(S - w_1) = E(\{1, \ldots, m\}, S - w_1 - w_1') - E(\{2, \ldots, m\}, S - w_1 - w_1')$. We need to prove that $E(\{1, \ldots, m\}, S - w_1) - E(\{2, \ldots, m\}, S - w_1) \geq E(\{1, \ldots, m\}, S - w_1 - w_1') - E(\{2, \ldots, m\}, S - w_1 - w_1')$. We construct a new agent x. Let x's valuation for item 1 be extremely high so that she wins item 1. The above inequality can be rewritten as $E(\{1, \ldots, m\}, S - w_1) - E(\{2, \ldots, m\}, S - w_1) - E(\{1\}, x) \geq E(\{1, \ldots, m\}, S - w_1 - w_1') - E(\{2, \ldots, m\}, S - w_1 - w_1') - E(\{1\}, x)$. This is, $E(\{1,\ldots,m\}, S-w_1) - E(\{1,\ldots,m\}, S-w_1+x) \geq E(\{1,\ldots,m\}, S-w_1-w_1') - E(\{1,\ldots,m\}, S-w_1-w_1'+x)$. We rearrange the terms, and get $E(\{1,\ldots,m\}, S-w_1) + E(\{1,\ldots,m\}, S-w_1-w_1'+x) \geq E(\{1,\ldots,m\}, S-w_1+x) + E(\{1,\ldots,m\}, S-w_1-w_1')$. This inequality can be proved based on Proposition 4.

Case $a \neq w_1$: Let w_1' be the new winner of item 1 when we allocate all the items $\{1,\ldots,m\}$ to the agents in $S-a$ using VCG. $VCG^1(S-a) = E(\{1,\ldots,m\}, S-a-w_1') - E(\{2,\ldots,m\}, S-a-w_1')$. We need to prove that $E(\{1,\ldots,m\}, S-w_1) - E(\{2,\ldots,m\}, S-w_1) \geq E(\{1,\ldots,m\}, S-a-w_1') - E(\{2,\ldots,m\}, S-a-w_1')$. That is, we need to prove $E(\{1,\ldots,m\}, S-w_1) - E(\{2,\ldots,m\}, S-w_1) - E(\{1\}, w_1) - E(\{1\}, w_1') \geq E(\{1,\ldots,m\}, S-a-w_1') - E(\{2,\ldots,m\}, S-a-w_1') - E(\{1\}, w_1) - E(\{1\}, w_1')$. We simplify and rearrange terms, and get $E(\{1,\ldots,m\}, S-w_1) + E(\{1,\ldots,m\}, S-a) + E(\{1\}, w_1) \geq E(\{1,\ldots,m\}, S) + E(\{1,\ldots,m\}, S-a-w_1') + E(\{1\}, w_1')$. Proposition 4 says that $E(\{1,\ldots,m\}, S-a) + E(\{1,\ldots,m\}, S-w_1') \geq E(\{1,\ldots,m\}, S-a-w_1') + E(\{1,\ldots,m\}, S)$. So it suffices to prove $E(\{1,\ldots,m\}, S-w_1) + E(\{1\}, w_1) \geq E(\{1,\ldots,m\}, S-w_1') + E(\{1\}, w_1')$. By Proposition 5, the left-hand side is $E(\{1\} \oplus \{1,\ldots,m\}, S)$. The right-hand side is at most this. □

PROPOSITION 7. *Winners still win after we remove some other agents [4, 6]:*[4] *For any set of agents S and any set of items T, we use W to denote the set of winners when we allocate the items in T to the agents in S using VCG. After we remove some agents in S, those in W that have not been removed remain to be winners,* provided that a consistent tie-breaking rule exists.

It should be noted that there may not exist a consistent tie-breaking rule that satisfies the above proposition. Fortunately, we are able to prove that tie-breaking is irrelevant for the goal of proving redistribution monotonicity.

We say that a type profile is *tie-free* if it satisfies the following: Let $T_1 = \{1\} \oplus \{1,\ldots,m\}$. Let $T_2 = \{1,\ldots,m\}$. Basically, T_1 and T_2 are the only item sets that we will ever mention. A type profile is *tie-free* if for any set of agents S, when we allocate the items in T_1 (or T_2) to S, the set of VCG winners is unique. If we only consider tie-free type profiles, then we do not need to be bothered by tie-breaking. We notice that the set of tie-free type profiles is a *dense* subset of the set of all type profiles – any type profile can be perturbed infinitesimally to become a tie-free type profile.

Our ultimate goal is to prove that for any set of agents S,

$$R(S,0) \geq R(S,1) \geq \ldots \geq R(S,|S|-m-1) \geq 0$$

We notice that the $R(S,j)$ are continuous in the agents' types. Therefore, it suffices to prove the above inequality for tie-free type profiles only.

From now on, we simply assume that the set of VCG winners is always unique.

Definition 3. For any set of agents S with $|S| \geq m+1$, let $D(S)$ be the set of $m+1$ winners when we allocate $\{1\} \oplus \{1,\ldots,m\}$ to the agents in S. $D(S)$ is called the *determination set* of S.

PROPOSITION 8. *For any set of agents S and any $a \in S - D(S)$, we have $VCG^1(S) = VCG^1(S-a)$ and $D(S) = D(S-a)$.*

The above proposition says that for the purpose of calculating item 1's VCG price, only those agents in $D(S)$ are relevant.

[4]The proposition was originally introduced in [4]. A more rigorous proof of a more general claim was also given in [6].

PROOF. $D(S)$ is the set of VCG winners when we allocate $\{1\} \oplus \{1,\ldots,m\}$ to the agents in S. By Proposition 7, after removing $a \in S - D(S)$, every agent in $D(S)$ should still win. That is, $D(S-a) = D(S)$.

Let w_1 be the winner of item 1 when we allocate $\{1,\ldots,m\}$ to the agents in S. $VCG^1(S) = E(\{1,\ldots,m\}, S-w_1) - E(\{2,\ldots,m\}, S-w_1) = E(\{1,\ldots,m\}, S-w_1) + E(\{1\}, w_1) - E(\{1,\ldots,m\}, S) = E(\{1\} \oplus \{1,\ldots,m\}, S) - E(\{1,\ldots,m\}, S)$ (the last step is due to Proposition 5). The first term only depends on those in $D(S)$. The second term also only depends on those in $D(S)$ for the following reason: Let S' be the set of VCG winners when we allocate $\{1,\ldots,m\}$ to the agents in S. The second term only depends on those in S'. We introduce an agent x whose valuation for item 1 is extremely high so that she wins item 1. When we allocate $\{1\} \oplus \{1,\ldots,m\}$ to the agents in $S+x$, the set of VCG winners are then $x+S'$. $D(S)$ are the new set of VCG winners after we remove x. By Proposition 7, those in S' must still remain in $D(S)$. Overall, $VCG^1(S)$ only depends on those agents in $D(S)$. Similarly, $VCG^1(S-a)$ only depends on those agents in $D(S-a)$. For $a \in S - D(S)$, $D(S) = D(S-a)$. Therefore, we must have $VCG^1(S) = VCG^1(S-a)$. □

Definition 4. Let S be any set of agents. Let k be any integer from 1 to $|S|$. Let $a_1 \prec a_2 \prec \ldots \prec a_k$ be a sequence of k distinct agents in S. We say these k agents form a *winner sequence with respect to S* if

$$a_1 \in D(S); a_2 \in D(S-a_1); a_3 \in D(S-a_1-a_2);$$

$$\ldots; a_k \in D(S-a_1-\ldots-a_{k-1}).$$

Let S' be a subset of S of size k. We say that S' forms a winner sequence with respect to S if there exists an ordering of the agents in S' that forms a winner sequence with respect to S. When S' forms a winner sequence with respect to S, we call S' a *size-$|S'|$ winner sequence set* with respect to S.

Let $H(S',S) = 1$ if S' forms a winner sequence with respect to S, and let $H(S',S) = 0$ otherwise. For presentation purpose, we say that the empty set forms a winner sequence (of size 0) with respect to any set S. That is, $H(\emptyset, S) = 1$.

Now we are ready to prove that heterogeneous-item auctions with unit demand satisfies redistribution monotonicity. We recall that it suffices to prove that for any set of agents S,

$$R^1(S,0) \geq R^1(S,1) \geq \ldots \geq R^1(S,|S|-m-1) \geq 0.$$

Here, $R^1(S,0) = VCG^1(S)$, and for $i = 1,\ldots,|S|-m-1$, $R^1(S,i)$ equals

$$\frac{1}{m+i}\left(\sum_{a\in S} R^1(S-a,i-1) - (|S|-m-i)R^1(S,i-1)\right).$$

PROPOSITION 9. *For any set of agents S, $R^1(S,k)$ equals*

$$\frac{1}{\binom{m+k}{m}} \sum_{\substack{S'\subset S\\ |S'|=k\\ H(S',S)=1}} VCG^1(S-S').$$

We have that

$$|\{S' | S' \subset S; |S'| = k; H(S',S) = 1\}| = \binom{m+k}{m}.$$

That is, $R^1(S,k)$ is the average of $VCG^1(S-S')$ for all S' that is a size-k winner sequence set with respect to S. For any set of *agents S (it should be noted that for $R^1(S,k)$ to be well-defined, we need $|S| \geq k+m+1$), the total number of size-k winner sequence sets with respect to S is $\binom{m+k}{m}$.*

The following lemmas are needed for the proof of the above proposition. All these lemmas are implications of "winners still win after we remove some other agents". The proofs are omitted due to space constraints.

LEMMA 1. *Let S be any set of agents. Let S' be a subset of S that forms a winner sequence with respect to S. Let a be an arbitrary agent in $S - S'$. Then, S' must also form a winner sequence with respect to $S - a$.*

LEMMA 2. *Let S be any set of agents. Let a be an agent in S. Let S' be a subset of $S - a$ that forms a winner sequence with respect to $S - a$. If we have that $a \notin D(S - S')$, then S' also forms a winner sequence with respect to S.*

LEMMA 3. *Let S be any set of agents. Let a be an agent in S. Let S' be a subset of $S - a$ that forms a winner sequence with respect to $S - a$. We have that if $a \in D(S - S')$, then $S' + a$ forms a (longer) winner sequence with respect to S.*

LEMMA 4. *Let S be any set of agents. Let $S' + a$ be a subset of S that forms a winner sequence with respect to S. We must have that S' forms a winner sequence with respect to $S - a$ and $a \in D(S - S')$.*

Now we are ready to prove the proposition.

PROOF. We prove by induction.
Initial step: We have $R^1(S,0) = VCG^1(S)$. When $k = 0$,

$$\frac{1}{\binom{m}{m}} \sum_{\substack{S' \subset S \\ |S'|=0 \\ H(S',S)=1}} VCG^1(S - S') = VCG^1(S - \emptyset) = VCG^1(S)$$

Also, when $k = 0$,

$$|\{S' | S' \subset S; |S'| = 0; H(S',S) = 1\}| = |\{\emptyset\}| = 1 = \binom{m+0}{m}$$

Induction assumption: We assume that for $k \geq 0$, for any S ($|S| \geq k+m+1$), we have

$$R^1(S,k) = \frac{1}{\binom{m+k}{m}} \sum_{\substack{S' \subset S \\ |S'|=k \\ H(S',S)=1}} VCG^1(S - S')$$

Also, $|\{S' | S' \subset S; |S'| = k; H(S',S) = 1\}| = \binom{m+k}{m}$.
We need to prove that the results hold for $k+1$. That is, for any S ($|S| \geq k+m+2$),

$$R^1(S,k+1) = \frac{1}{\binom{m+k+1}{m}} \sum_{\substack{S' \subset S \\ |S'|=k+1 \\ H(S',S)=1}} VCG^1(S - S')$$

and $|\{S' | S' \subset S; |S'| = k+1; H(S',S) = 1\}| = \binom{m+k+1}{m}$.
Induction proof: By definition, $R^1(S,k+1)$ equals

$$\frac{1}{m+k+1} \left(\sum_{a \in S} R^1(S-a,k) - (|S| - m - k - 1) R^1(S,k) \right) \tag{8}$$

Now let us analyze the expression $\sum_{a \in S} R^1(S-a,k)$. By induction assumption, it can be rewritten as

$$\frac{1}{\binom{m+k}{m}} \sum_{a \in S} \sum_{\substack{S' \subset S-a \\ |S'|=k \\ H(S',S-a)=1}} VCG^1(S - a - S').$$

By induction assumption, the above expression is the sum of $|S|\binom{m+k}{m}$ terms. Each term corresponds to one choice of a among S and one choice of S' among $S - a$. We divide these $|S|\binom{m+k}{m}$ terms into two groups:
Group A, terms with $a \notin D(S - S')$: By Lemma 2, S' must also form a winner sequence with respect to S. That is, there are at most $\binom{m+k}{k}$ choices of S'. For each choice of S', there are at most $|S - S' - D(S - S')| = |S| - k - m - 1$ choices of a. Overall, there are at most $\binom{m+k}{k}(|S| - k - m - 1)$ terms in Group A. On the other hand, for any S' that forms a winner sequence with respect to S, S' must also form a winner sequence with respect to $S - a$ by Lemma 1. For any $a \notin D(S - S')$, there must be a term in Group A that is characterized by a and S'. That is, there are at least $\binom{m+k}{k}(|S| - k - m - 1)$ terms in Group A. Hence, there are exactly $\binom{m+k}{k}(|S| - k - m - 1)$ terms in Group A. Since $a \notin D(S - S')$, we have that $VCG^1(S - a - S') = VCG^1(S - S')$ by Proposition 8. Therefore, the sum of all the terms in Group A equals

$$\frac{1}{\binom{m+k}{m}} (|S| - k - m - 1) \sum_{\substack{S' \subset S \\ |S'|=k \\ H(S',S)=1}} VCG^1(S - S')$$

This is exactly $|S| - k - m - 1$ times $R^1(S,k)$.
Group B, terms with $a \in D(S - S')$: There are exactly $|S|\binom{m+k}{m} - (|S| - k - m - 1)\binom{m+k}{m} = (k+m+1)\binom{m+k}{m} = \frac{(k+m+1)!(k+1)}{m!(k+1)!} = (k+1)\binom{m+k+1}{m}$ terms in Group B. Let X be the set of all size-$(k+1)$ winner sequence sets with respect to S. According to Lemma 3 and Lemma 4, every term in Group B must corresponds to an element in X, and every element in X must correspond to exactly $k+1$ terms in Group B (*e.g.*, a size-$(k+1)$ winner sequence set $Y = \{x_1, \ldots, x_{k+1}\}$ corresponds to the following $k+1$ terms: $a = x_i$ and $S' = Y - x_i$ for all i). Therefore, the total number of elements in X must be $\binom{m+k+1}{m}$.
The sum of the terms in Group B equals

$$\frac{k+1}{\binom{m+k}{m}} \sum_{\substack{S' \subset S \\ |S'|=k+1 \\ H(S',S)=1}} VCG^1(S - S')$$

Equation 8 can then be simplified as

$$\frac{1}{m+k+1} \left(\sum_{a \in S} R^1(S-a,k) - (|S| - m - k - 1) R^1(S,k) \right)$$

$$= \frac{1}{m+k+1} \left(\frac{k+1}{\binom{m+k}{m}} \sum_{\substack{S' \subset S \\ |S'|=k+1 \\ H(S',S)=1}} VCG^1(S - S') \right)$$

$$= \frac{1}{\binom{m+k+1}{m}} \sum_{\substack{S' \subset S \\ |S'|=k+1 \\ H(S',S)=1}} VCG^1(S - S')$$

□

Proposition 9 implies that function R^1 is always nonnegative. We still need to prove that

$$R^1(S,0) \geq R^1(S,1) \geq \ldots \geq R^1(S, |S| - m - 1).$$

Due to space constraint, we only present an outline of the proof of $R^1(S,3) \geq R^1(S,4)$, which highlights the main idea behind the full proof.

PROPOSITION 10. *$R^1(S,3) \geq R^1(S,4)$ for any S. (We need $4 \leq |S| - m - 1$ for $R^1(S,4)$ to be well-defined.)*

Proof sketch: By definition, $R^1(S,4) = \frac{1}{m+4}(\sum_{a \in S} R^1(S - a, 3) - (|S| - m - 4)R^1(S,3))$. To prove that $R^1(S,4) \leq R^1(S,3)$, it suffices to prove that $R^1(S,3) \geq R^1(S-a,3)$ for any $a \in S$.

Let a be an arbitrary agent in S. According to Proposition 9, we need to prove

$$\sum_{\substack{S' \subset S \\ |S'|=3 \\ H(S',S)=1}} VCG^1(S - S') \geq \sum_{\substack{S' \subset S-a \\ |S'|=3 \\ H(S',S-a)=1}} VCG^1(S - a - S').$$

The proof is outlined as follows:

- On both sides of the inequality, there are $\binom{m+3}{m}$ terms (Proposition 9). Every term is characterized by a size-3 winner sequence set S'.
- For every term on the right-hand side, we map it to a corresponding term on the left-hand side. The corresponding term on the left-hand side is larger or the same.
- We prove that the mapping is injective. That is, different terms on the right-hand side are mapped to different terms on the left-hand side.
- Therefore, the left-hand side must be greater than or equal to the right-hand side.

4. CONCLUSION

We conclude our paper with the following conjecture:

CONJECTURE 1. *Gross substitutes implies redistribution monotonicity. That is, HETERO remains feasible and worst-case optimal in combinatorial auctions with gross substitutes.*

The idea is that both multi-unit auctions with nonincreasing marginal values and heterogeneous-item auctions with unit demand satisfy redistribution monotonicity. A natural conjecture is that the "most restrictive joint" of these two settings also satisfies redistribution monotonicity. There are many well-studied auction settings that contain both multi-unit auctions with nonincreasing marginal values and heterogeneous-item auctions with unit demand (a list of which can be found in [10]). Among these well-studied settings, combinatorial auctions with gross substitutes is the most restrictive. To prove the conjecture, we need to prove that gross substitutes implies that for any set of agents S, $R(S,0) \geq R(S,1) \geq \ldots \geq R(S, |S| - m - 1) \geq 0$. So far, we have only proved $R(S,0) \geq R(S,1) \geq 0$.

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APPENDIX

PROPOSITION 11. *Let M be a feasible VCG redistribution mechanism that is possibly not anonymous. Let α^M be the worst-case redistribution fraction of M. If the agents' type spaces are identical, then there exists an anonymous feasible VCG redistribution mechanism, whose worst-case redistribution fraction is at least α^M.*

Proof sketch: M can be anonymized using the technique described in Section 3 of [1]. The resulting mechanism is anonymous, feasible, and its worst-case redistribution fraction is at least α^M.