Consensus Games
(Extended Abstract)

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ABSTRACT

Consensus Games (CGs) are a novel approach to modelling coalition formation in multi-agent systems inspired by threshold models in sociology. In a CG, each agent's degree of commitment to the coalitions in which it may participate is expressed as a quorum function. Agents are willing to form a coalition only if a quorum consensus can be achieved amongst all agents of the coalition.

Categories and Subject Descriptors

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General Terms

Algorithms, Theory

Keywords

Multi-agent, Consensus, Threshold, Coalition Formation

1. INTRODUCTION

Coalition formation has traditionally been modelled using game theoretic techniques. Such models often necessitate strong economic assumptions, including that utility is transferable, and that coalitional valuations are known and can be fairly distributed. The multi-agent community in particular have investigated coalition formation in situations where these assumptions cannot easily be applied, for example, [1, 7]. A common assumption in this work is that all member-agents must somehow 'agree'; in other words, for a coalition to form it is necessary that there is a *consensus* among the members of the coalition regarding its formation.

In this extended abstract we propose *consensus games* (CGs), a novel model of consensual coalition formation for multi-agent systems inspired by threshold models in sociology. Threshold models have been used to describe a variety of social phenomena. For example, [3] presents a model in which n individuals face a binary decision, e.g., regarding whether to participate in a riot. Each individual has an idiosyncratic threshold representing the minimum proportion of others which must participate in order that the given individual will also participate. It is shown that the number of agents that ultimately decide to participate, e.g., number of agents that decide to riot, is critically dependent on the distribution of thresholds. Similar models have been used to investigate segregation in urban housing [6], and the adoption of consumer trends [4].

We extend the model proposed in [3] beyond binary choice decisions to the more general problem of coalition formation. For each coalition of which it may be a member, each agent has a threshold representing the proportion of agents that must support the formation of the coalition in order that the agent will also support the formation of the coalition. We focus on the special case of consensus: there is consensus about the formation of a particular coalition only where all agents support the formation of the coalition.

2. CONSENSUS GAMES

Definition 1. *A consensus game (CG) is a tuple* $\Gamma = \langle G, q \rangle$ *where:*

G *is a finite set of agents,* $\{1, \ldots, n\}, n \geq 2$.

q *is a quorum function.* $q : G \times 2^G \rightarrow [0, 1]$ *is a partial function which takes an agent* $i \in G$ *and a coalition* $H \subseteq G$ *where* $i \in H$*, and returns a number in the interval* $[0, 1]$.

For each coalition of which it may be a member, the value of the quorum function indicates the agent's 'degree of support' for the formation of that coalition. For an agent $i \in H \subseteq G$ the quorum function $q(i, H)$ gives the minimum proportion of agents in H that must support the formation of the coalition H in order that i will support the formation of H. Where $q(i, H) = 0$ agent i unconditionally supports the formation of the coalition H, where $0 < q(i, H) \leq \frac{|H|-1}{|H|}$ i conditionally supports the formation of the coalition H; where $\frac{|H|-1}{|H|} < q(i, H) \leq 1$ i does not support the formation of the coalition H. We use the abbreviation $q\#(i, H)$ to denote the number of other agents in H that must support H in order for i to support H. Formally, $q\#(i, H)$ is the minimal natural number k such that $q(i, H) \leq k/|H|$. We will denote by $n_k(H)$ the number of agents $i \in H$ with $q\#(i, H) = k$.

3. STRONG CONSENSUS

A key solution concept for CGs is the strong consensus coalition. A *strong consensus coalition* H is a coalition

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where for each agent $i \in H$ the quorum threshold $q(i, H)$ is satisfied in the sense that H contains at least $q\#$ other agents with strictly lower $q\#$ values.

Definition 2. *A coalition H is a strong consensus coalition if the following conditions hold:*

- $n_0(H) \neq 0$
- *if* $n_k(H) \neq 0$*, then* $\Sigma_{j<k} n_j(H) \geq k$

Note that the definition implies that if H is a strong consensus coalition, then $n_{|H|}(H) = 0$.

Consider the following example.

Example 1. *Alice (A) and Bob (B) are considering whether to get married. Bob, tired of bachelorhood, is keen to be married. Alice is not opposed to marrying Bob provided that Bob also wants to marry her, otherwise Alice will happily continue to be single. Alice's and Bob's positions can be formalised as the consensus game $\Gamma = \langle G, q \rangle$ where:*

$$G = \{A, B\}$$

$$q(i, H) = \begin{cases} 0 & \text{if } i = B \text{ and } H = \{A, B\} \\ 0.5 & \text{if } i = A \text{ and } H = \{A, B\} \\ 1 & \text{if } i = B \text{ and } H = \{B\} \\ 0 & \text{if } i = A \text{ and } H = \{A\} \end{cases}$$

In the example, Bob unconditionally supports the formation of the grand coalition (of all agents); Alice conditionally supports formation of this coalition provided that one other agent (Bob) also supports its formation. Alice also unconditionally supports formation of the singleton coalition {A}, whereas Bob does not support formation of the singleton coalition {B}. The grand coalition in this example is a strong consensus coalition.

We now show that there is an alternative definition of a strong consensus coalition as a fixed point of a function that intuitively corresponds to agents indicating their support for a coalition.

Consider the function $f_H : 2^G \rightarrow 2^G$ defined relative to $H \subseteq G$:

$$i \in f_H(Q) \text{ iff } i \in H \text{ and } |Q \cap H \setminus \{i\}| \geq q(i, H) \times |H|$$

This function takes as its input a set $Q \subseteq G$ and returns the set of agents in H whose quorum thresholds are satisfied by the membership of $Q \cap H$. If $Q = \emptyset$, f_H will contain only the agents i with $q(i, H) = 0$, if Q is the set of agents which have unconditional support for H, then $f_H(Q)$ will contain the agents i with $q\#(i, H) \leq |Q|$, and so on.

A coalition H is a strong consensus coalition if and only if it is the least fixed point of f_H. First we need the following auxiliary result:

Proposition 1. *The function f_H is guaranteed to possess at least one fixed point.*

We omit the proof due to lack of space.

The least fixed point of f_H can be established by recursive calls to the function starting with the empty set of agents as an argument. We refer to each invocation of f_H as a *round*. If H can achieve strong consensus, then it will be achieved in at most $|H|$ rounds.

We can now show that:

Theorem 1. *H is a strong consensus coalition if and only if it is the least fixed point of f_H.*

We omit the proof due to lack of space.

In characterising the computational complexity of CGs, a natural decision problem is given a CG $\Gamma = \langle G, q \rangle$ and a coalition $H \subseteq G$, can H reach strong consensus? Algorithm 1, which runs in time linear in the number of agents, can be used to determine if H is the least fixed point of f_H.

Algorithm 1 Can H reach strong consensus.

```
function SCC(q, H)
    array support[|H| + 1] ← {0, ..., 0}
    for all i ∈ H do
        k ← ⌈q(i, H) × |H|⌉
        support[k] ← support[k] + 1
    s ← support[0]
    for k from 1 to |H| do
        if k ≤ s then
            s ← s + support[k]
        else
            return false
    return true
```

4. DISCUSSION

The key idea of CGs is that agents' choices are conditioned by the number of other agents also making the same choice. This has some similarities with anonymous games [2], in which the individual utility of participation in a coalition can be dependant on factors including the size of the coalition, and with imitation games [5], in which an agent's behaviour may influence that of other agents.

CGs as presented here treat the problem of coalition formation in an abstract sense. It is often the case that coalition formation in multi-agent systems is directed toward the achievement of the agents' goals. It would therefore be interesting to extend the model of CGs to include representations of collective action and heterogeneous goals.

5. REFERENCES

[1] J. Abdou and H. Keiding. *Effectivity functions in social choice.* Kluwer Academic, 1991.

[2] C. Daskalakis and C. Papadimitriou. Computing equilibria in anonymous games. In *Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07)*, pages 83–93. IEEE, 2007.

[3] M. Granovetter. Threshold models of collective behavior. *American Journal of Sociology*, 83(6):1420–1443, 1978.

[4] M. Granovetter and R. Soong. Threshold models of interpersonal effects in consumer demand. *Journal of Economic Behavior & Organization*, 7(1):83–99, 1986.

[5] A. McLennan and R. Tourky. Simple complexity from imitation games. *Games and Economic Behavior*, 68(2):683 – 688, 2010.

[6] T. C. Schelling. Models of segregation. *The American Economic Review*, 59(2):488–493, 1969.

[7] M. Wooldridge and P. E. Dunne. On the computational complexity of qualitative coalitional games. *Artificial Intelligence*, 158(1):27–73, 2004.