The Undecidability of Group Announcements

Thomas Ågotnes
University of Bergen
thomas.agotnes@infomedia.uib.no

Hans van Ditmarsch
LORIA, CNRS - Université de Lorraine
hans.van-ditmarsch@loria.fr

Tim French
The University of Western Australia
tim.french@uwa.edu.au

ABSTRACT

This paper addresses and solves the long-standing open problem of whether Group Announcement Logic (GAL) is decidable. GAL is a dynamic epistemic logic for reasoning about which states of knowledge a group of agents can make come about by sharing their knowledge, with an operator for quantifying over all truthful public announcements that can be made by the group. We show that the satisfiability problem for the logic is undecidable; it is co-RE complete.

Categories and Subject Descriptors

F.4.1 [**Mathematical Logic and Formal Languages**]: Modal Logic

Keywords

Dynamic epistemic logic; Decision procedures

1. INTRODUCTION

The runs-and-systems approach [8] and, later, dynamic epistemic logics [19] have been used to specify multi-agent systems dynamic at a high level of systems architecture. Such modal logical approaches have some advantages, e.g., that they are typically decidable, the model checking complexity is fairly low (even linear for the base modal logic), and satisfiability may also be tamed quite a bit (public announcement logic and the base modal logic are NP-complete in the single-agent case and PSPACE-complete in the multi-agent case [12]). Unfortunately, when combining epistemic (or more basic modal) logics with logical dynamics, it is highly unpredictable whether the resulting logic is decidable even when the base logics are. A well-known result achieved by very minimal (linguistic) means is [13]. Another such result, described below, was reported in [9]. There are several open problems concerning decidability of dynamic epistemic logics and in this paper we answer such a question negatively: Group Announcement Logic is undecidable. For the development of proof tools for such logics, and given the frequent claim that such logics are applicable, we consider this result relevant to report.

Group Announcement Logic (GAL) [3, 2] is an extension of public announcement logic [15] that includes an operator to quantify over what a group of agents can achieve by publicly announcing information that one or more agents in the group know to be true. It is related to Arbitrary Public Announcement Logic (APAL) [4] which has no restrictions on what may be announced.

In Public Announcement Logic (PAL) [15], the so-called 'truthful public announcements' are assumed to be made by an outside observer ('truthful' here simply means 'true'). The outside observer is not modelled as an agent and does not appear in the logical language. If we wish to formalize a truthful public announcement of a formula ϕ made by an agent a that is modelled in the system, it is common to see this as the announcement of a formula $K_a\phi$, for 'the agent knows ϕ'. Now, of course, there is a difference between true and truthful. Truthful means that the agent believes what it announces. Group Announcement Logic models what can be achieved by simultaneous truthful announcements by a subset of the set of all agents. It can also be used to reason about finite sequences of announcements – communication protocols – where agents take turns in saying something. We illustrate group announcement by a simple example.

Given are two agents a, b such that a knows whether p and b knows whether q, and this is common knowledge (Figure 1, left). Anne (a) can achieve that Bill knows whether p (namely by informing him of the value of p) (Figure 1, right). In other words, there is an announcement that a can make and that a knows to be true (namely $K_a p$ when p is true, and $K_a \neg p$ when p is false) such that after that announcement, b knows whether p. This is formalized as $\langle a \rangle (K_b p \vee K_b \neg p)$. Bill ($b$) can achieve that Anne knows whether q (in a formula, $\langle b \rangle (K_a q \vee K_a \neg q)$) (Figure 2, left). But *neither* agent can achieve both outcomes at the same time. However, together they can achieve that: $\langle ab \rangle (K_a(p \wedge q) \wedge K_b(p \wedge q))$ (we even have common knowledge) (Figure 2, right). Not anything goes with group announcements. For example, one cannot get the submodel with domain $\{00, 10, 11\}$ (where we use obvious names for states). This can be used in an expressivity argument to show that APAL is able to express properties that are inexpressible in GAL. But it is unknown if the reverse is true (see below).

With GAL one can formalize communication protocols, such as security protocols. Let Alice be a sender a, Bob a receiver b, and Eve a spy / eavesdropper e. Let ϕ be some *information goal*. For example, suppose Alice wishes to inform Bob of the latest transatlantic scandal p, then an information goal could be that Alice, Bob, and Eve commonly know that either Alice and Bob share knowledge of p

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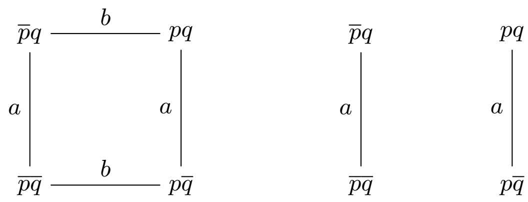

Figure 1: An announcement by a.

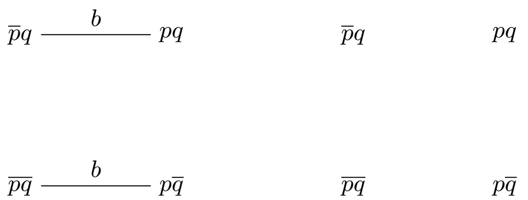

Figure 2: An announcement by b, and by a and b.

or Alice and Bob share knowledge of $\neg p$. (This can be more succinctly formalized with common knowledge operators but we bypass these in this paper.) The requirement that not only Alice and Bob but also they and Eve commonly know this, is common in a security setting. It formalizes that the protocol is known to have terminated: we may assume that everything is public about the protocol except the message (and private keys). There is also a *security goal* ψ that needs to be preserved throughout protocol execution, e.g., Alice, Bob, and Eve commonly know that Eve is ignorant about p (or some more involved aspect of p, such as the identity of those involved in the scandal). A finite protocol for a and b to learn the secret safely should observe

$$\psi \to \langle ab \rangle(\phi \wedge \psi).$$

It is not known if the existence of a finite two-agent protocol specification as above is formalizable in APAL.

The logic GAL shares various properties with APAL, e.g., the axiomatization is similar, and the model checking complexity is PSPACE-complete [2].

A variation on GAL is coalition announcement logic (CAL). In group announcement logic we investigate the consequences of the simultaneous announcement (joint public event) by G. The agents not in G do not take part in the action. In CAL we quantify over what the agents in G can achieve by their joint announcement, no matter what the other agents simultaneously announce. Thus, we get into the domain of coalitions logics [14], see also the recent [16]. CAL is further discussed in Section 4.

Arbitrary Public Announcement Logic, which has no restrictions on what may be announced (except that the formulas may not contain arbitrary public announcement operators), was shown to be undecidable in [9]. This has been an open question for GAL. However, the complexity (and, more specifically, the decidability) of GAL should be of great interest. GAL allows us to formulate more interesting and practical properties than APAL, since we have the requirement that the publicly announced information must be known to the agents in the group. Genuine collaboration and information sharing is required to progress the knowledge of the group. Furthermore the undecidability proof given for APAL does not directly apply to GAL.

In this paper we address the open problem of the decidability of GAL, showing that its satisfiability problem, like that of APAL, is not recursively enumerable. Based on the complete axiomatization of GAL, so that GAL validity is recursively enumerable, we may conclude that satisfiability for GAL is co-RE complete.

The remainder of the paper is organised as follows. In the next section we review the syntax and semantics of GAL. The undecidability proof is described in Section 3, before future work is discussed in Section 4.

2. SYNTAX AND SEMANTICS

We focus our attention on a scenario where a finite set of agents, A, are aware of a set of atomic (Boolean) propositions, P. They consider different worlds possible where different sets of propositions may be true, and where different agents may have different states of knowledge. Our interest is in providing a formal language for reasoning about what agents know about the propositions, what agents know about what other agents know, and what agents can find out through informative events (such as public announcements).

The base language we work with is epistemic logic $\mathcal{L}_{el}$:

$$\phi ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_a\phi$$

where $p \in P$ and $a \in A$.

We consider extension of this logic with

1. public announcements $[\psi]\phi$.
2. arbitrary public announcements $\Box\phi$
3. arbitrary group announcements $[G]\phi$

where $G \subseteq A$ and ψ and ϕ may be any formula of the (extended) logic.

We let PAL refer to $\mathcal{L}_{el}$ augmented with public announcements, APAL refer to PAL augmented with arbitrary public announcements, and GAL refer to PAL augmented with arbitrary group announcements.

The formulas of these logics are interpreted over structures $M = (S, \sim, V)$, where S is a non-empty set of worlds, $\sim$: $A \longrightarrow \wp(S \times S)$ assigns a reflexive, transitive and symmetric accessibility relation, $\sim_a$, to each agent a, and $V : P \longrightarrow \wp(S)$ maps each proposition to the set of worlds where it is true. For each $a \in A$, the set of worlds $\{t \mid s \sim_a s\}$ is an equivalence class that we will denote $[s]_a$.

Let $M = (S, \sim, V)$ and suppose that $s \in S$. The semantics of $\mathcal{L}_{el}$ and the operators above are given recursively with respect to the *pointed model* M_s:

$$\begin{array}{rcl}
M_s \models p & \text{iff} & s \in V(p) \\
M_s \models \neg\phi & \text{iff} & M_s \not\models \phi \\
M_s \models \phi_1 \wedge \phi_2 & \text{iff} & M_s \models \phi_1 \text{ and } M_s \models \phi_2 \\
M_s \models K_a\phi & \text{iff} & \forall t \in S \text{ where } s \sim_a t,\ M_t \models \phi \\
M_s \models [\psi]\phi & \text{iff} & M_s \models \psi \Longrightarrow M_s^\psi \models \phi \\
M_s \models \Box\phi & \text{iff} & \forall \psi \in \mathcal{L}_{el},\ M_s \models [\psi]\phi \\
M_s \models [G]\phi & \text{iff} & \forall \psi \in \mathcal{L}_{el}^G,\ M_s \models [\psi]\phi
\end{array}$$

where $M^\psi = (S', \sim', V')$ is such that: $S' = \{s \in S \mid M_s \models \psi\}$; for all $a \in A$, $\sim'_a = \sim_a \cap (S' \times S')$; and for all $p \in P$, $V'(p) = V(p) \cap S'$. For the semantics of $[G]$ we define the sublanguage $\mathcal{L}_{el}^G$ to be the set of formulas of the type $\bigwedge_{a\in G} K_a\phi_a$, where for each $a \in G$, $\phi_a \in \mathcal{L}_{el}$.

As usual, we take $K_a\phi$ to mean *agent a knows* ϕ, and let $L_a\phi$ abbreviate $\neg K_a \neg\phi$ (*agent a considers* ϕ *to be possible*).

We say that a formula ϕ is satisfiable if there exists some model $M = (S, \sim, V)$ and some world $s \in S$ such that $M_s \models \phi$, and if $M_s \models \phi$ for all model-world pairs, M, s, we say ϕ is valid.

The formula $[\psi]\phi$ expresses the property *after the true announcement of* ψ, ϕ *will hold*. If ψ is not true (in a given world) then that world is not consistent with the announcement of ψ, so $[\psi]\phi$ is deemed to be vacuously true in such a world. We note that $\langle\psi\rangle\phi$ abbreviates $\neg[\psi]\neg\phi$, which has the same interpretation except that when ψ is not true in a given world, its interpretation defaults to false. It is known that epistemic logic extended with public announcements is expressively equivalent to the base epistemic logic, and as such it is decidable [19].

The formula $\Box\phi$ expresses the statement *"after publicly announcing any true formula of epistemic logic,* ϕ *must be true."* This statement implicitly quantifies over all true formulae of epistemic logic. For example, suppose ϕ were the formula $K_a p \to K_b p$. The formula $\Box\phi$ is true at some world where p is true, if and only if for every b-related world, u, where p is not true, *for every epistemic formula* ψ, there is some a-related world, v, that agrees with u on the interpretation of ψ. This is a very strong property, and in [9] it was shown that such a property could be exploited to encode a recursively enumerable tiling problem.

The formula $[G]\phi$ expresses the property *after a group of agents (simultaneously) announce any statements that they know to be true,* ϕ *will be true.* This statement has an implicit quantifier in it as well, but this time it is only quantifying over formulas that agents know to be true, rather than all formulas (in base epistemic logic). We cannot use the example above again, as the worlds v and u would now need to agree only on formulas that some agent in G knows to be true. Therefore, in order to establish undecidability by a tiling argument, we need a different form of proof.

3. THE UNDECIDABILITY OF GAL

In this section we give a result showing that Group Announcement Logic is undecidable. The proof follows a path similar to the proof for APAL. We reduce the problem of tiling the plane to the satisfiability problem for GAL.

3.1 Tilings and undecidability

The undecidability of a logic may be shown by encoding an undecidable tiling problem into the logic. The tiling problem we will use is as follows.

Definition Let C be a finite set of *colours* and define a C-tile γ to be a four-tuple over C, $\gamma = (\gamma^t, \gamma^r, \gamma^f, \gamma^\ell)$, where the elements are referred to as, respectively, *top, right, floor* and *left*. The tiling problem is, for any given finite set of C-tiles, Γ, determine if there is a function $\lambda : \mathbb{Z} \times \mathbb{Z} \longrightarrow \Gamma$ such that for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$:

1. $\lambda(i, j)^t = \lambda(i, j + 1)^f$
2. $\lambda(i, j)^r = \lambda(i + 1, j)^\ell$.

The tiling problem is known to be co-RE complete [11].

In [9] it was shown that given a set of tiles Γ, we could define a formula of APAL that was satisfiable if and only if Γ could tile $\mathbb{N} \times \mathbb{N}$ (which is also an undecidable problem [11]).

In this paper we take the same approach. The main steps are:

1. enforcing the structure of a satisfying model to have a grid-like structure;
2. defining a formula to represent common knowledge;
3. using propositional atoms to represent tiles, express the formula "it is common knowledge that adjacent tiles on the grid have matching sides".

The proof presented in [9] was complicated because the grid-like structure could only be enforced up to n bisimilarity (for an arbitrary n). For GAL the restriction to arbitrary group announcements makes it harder to capture the notion of n-bisimilarity, which via the correspondence theory of [6] is equivalent to two worlds agreeing on the interpretation of all formulas up to a given modal depth. As GAL does not quantify over all formulas, but rather only the formulas that are known by the group of agents, this approach is more challenging.

The key concept of n-bisimilarity is defined as follows.

Definition Fix a finite set of atoms, Π. Given a model, $M = (S, \sim, V)$, an n-Π-bisimulation over M is a relation $R_n \subseteq S \times S$ defined recursively as:

1. sR_0t if and only if for all $p \in \Pi$, $s \in V(p)$ if and only if $t \in V(p)$.
2. $sR_{m+1}t$ if and only if sR_mt and:
   (a) for all $i \in A$, for all u where $s \sim_i u$, there is some v where $t \sim_i v$ and uR_mv;
   (b) for all $i \in A$, for all v where $t \sim_i v$, there is some u where $s \sim_i u$ and vR_mu.

Given such a model M, and some $s \in S$, we let $[s]_n \subseteq S$ be the set of worlds n-Π-bisimilar to s. (We omit the Π when it is clear from context).

It is clear that $[s]_n$ is an equivalence class for all $s \in S$.

An important property of n-bisimilarity is that any two worlds s and t that are not n-bisimilar must have a witnessing formula $\phi \in \mathcal{L}_{el}$, such that $M_s \models \phi$ and $M_t \not\models \phi$.

Lemma 3.1. *Let* Π *be a finite set of propositional atoms. Suppose that* $M = (S, \sim, V)$ *and for some* $s, t \in S$*, for some* n*, we have* $t \notin [s]_n$*. Then there is some* $\mathcal{L}_{el}$ *formula,* ϕ*, such that* $M_s \models \phi$ *and* $M_t \not\models \phi$*.*

Proof. We show this by induction over n, where the induction hypothesis is that for all n, there are a finite set of formulas that are sufficient to distinguish all states that are not n-Π-bisimilar. For the base case, it is clear that if $t \notin [s]_0$, then t and s must disagree on the interpretation of some atom p. If $s \in V(p)$, we set ϕ to p, and otherwise, we set ϕ to $\neg p$. It is clear that there are a finite number of such formulas.

Now suppose given m, for all $u, v \in S$, if $u \notin [v]_m$ there is some formula ψ such that $M_v \models \psi$ and $M_u \not\models \psi$. Now if $t \notin [s]_{m+1}$, there are three possible scenarios:

1. $t \notin [s]_m$, in which case there is some $\mathcal{L}_{el}$ formula ψ such that $M_s \models \psi$ and $M_t \not\models \psi$. In this case we let $\phi = \psi$.

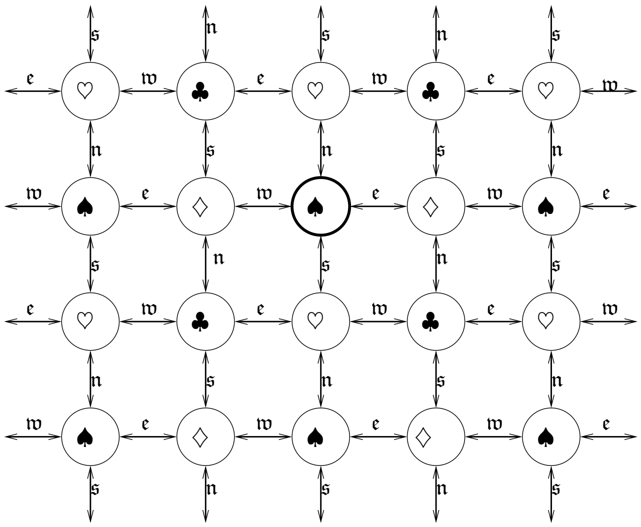

Figure 3: The basic structure of a grid on which to base the proof of undecidability for GAL. For readability, the directions are given as away from the dark suits (spades and clubs), and towards the light suits (hearts and diamonds). We have omitted the common knowledge relations, $\mathfrak{t}$, which is understood to be the universal relation over all points in the grid.

2. there is some $i \in A$, and some u where $s \sim_i u$, but for all v where $t \sim_i v$, $v \notin [u]_m$. Therefore, for every such v there is some formula ψ_v where $M_u \models \psi_v$ and $M_v \not\models \psi_v$. As these formulas are taken from a finite set, there is a finite formula $\phi = L_i(\bigwedge_{t \sim_i v} \psi_v)$ such that $M_s \models \phi$ and $M_t \not\models \phi$.

3. there is some $i \in A$ and some v where $t \sim_i v$, but for all u where $s \sim_i u$, $u \notin [v]_m$. A construction similar to that of the previous case may be given.

Thus we are able to define a $\mathcal{L}_{el}$ formula ϕ that distinguishes all worlds that are not n-bisimilar. It is also clear from construction that for each n, there are a finite number of such formula, so the induction hypothesis will hold. Given such a finite set of formulas Φ, the lemma follows by setting ϕ to be $\bigwedge\{\psi \in \Phi | M_s \models \psi\} \wedge \bigwedge\{\neg\psi \in \Phi | M_s \not\models \psi\}$. $\square$

3.2 The grid-like structure

The model we aim to build represents a grid. It consists of five agents: two "horizontal" agents, *East* ($\mathfrak{e}$) and *West* ($\mathfrak{w}$); two "vertical" agents *North* ($\mathfrak{n}$) and *South* ($\mathfrak{s}$); and one agent for the common knowledge of all agents ($\mathfrak{t}$). We also use the "card-suit" propositions $\heartsuit$, $\clubsuit$, $\diamondsuit$and $\spadesuit$(respectively hearts, clubs, diamonds and spades) to mark the different states of the model. The five agents and four propositions allow us to encode the grid-like structure depicted in Figure 3.

There is a clear pattern in this grid, where there is a period of two in any direction (with respect to both the card suits and the directions). The grid is infinite all directions (so the points may be thought of as elements of $\mathbb{Z} \times \mathbb{Z}$). It is clear that determining the existence of a tiling of $\mathbb{Z} \times \mathbb{Z}$ is equivalent to determining the existence of an tiling of $\mathbb{N} \times \mathbb{N}$.

The point with a darker boundary, containing $\spadesuit$, has been emphasized as we will refer to that point when discussing the construction.

Given a model with such a structure it is not hard to give a formula with satisfiability that is equivalent to the existence of a Γ tiling of the plane. Let $\Gamma = \{\gamma_1, ...\gamma_n\}$ be a set of tiles. For each $\gamma \in \Gamma$, let p_γ be a unique propositional atom (distinct from those we have already introduced). We define:

$$up_\Gamma = \bigwedge_{\gamma \in \Gamma} \left(p_\gamma \to \bigvee_{\substack{\delta \in \Gamma \\ \gamma^t = \delta^f}} \bigwedge \begin{bmatrix} \heartsuit \to K_\mathfrak{s}(\spadesuit \to p_\delta) \\ \clubsuit \to K_\mathfrak{n}(\diamondsuit \to p_\delta) \\ \diamondsuit \to K_\mathfrak{s}(\clubsuit \to p_\delta) \\ \spadesuit \to K_\mathfrak{n}(\heartsuit \to p_\delta) \end{bmatrix} \right)$$

$$down_\Gamma = \bigwedge_{\gamma \in \Gamma} \left(p_\gamma \to \bigvee_{\substack{\delta \in \Gamma \\ \gamma^f = \delta^t}} \bigwedge \begin{bmatrix} \heartsuit \to K_\mathfrak{n}(\spadesuit \to p_\delta) \\ \clubsuit \to K_\mathfrak{s}(\diamondsuit \to p_\delta) \\ \diamondsuit \to K_\mathfrak{n}(\clubsuit \to p_\delta) \\ \spadesuit \to K_\mathfrak{s}(\heartsuit \to p_\delta) \end{bmatrix} \right)$$

$$left_\Gamma = \bigwedge_{\gamma \in \Gamma} \left(p_\gamma \to \bigvee_{\substack{\delta \in \Gamma \\ \gamma^\ell = \delta^r}} \bigwedge \begin{bmatrix} \heartsuit \to K_\mathfrak{w}(\clubsuit \to p_\delta) \\ \clubsuit \to K_\mathfrak{e}(\heartsuit \to p_\delta) \\ \diamondsuit \to K_\mathfrak{w}(\spadesuit \to p_\delta) \\ \spadesuit \to K_\mathfrak{e}(\diamondsuit \to p_\delta) \end{bmatrix} \right)$$

$$right_\Gamma = \bigwedge_{\gamma \in \Gamma} \left(p_\gamma \to \bigvee_{\substack{\delta \in \Gamma \\ \gamma^r = \delta^\ell}} \bigwedge \begin{bmatrix} \heartsuit \to K_\mathfrak{e}(\clubsuit \to p_\delta) \\ \clubsuit \to K_\mathfrak{w}(\heartsuit \to p_\delta) \\ \diamondsuit \to K_\mathfrak{e}(\spadesuit \to p_\delta) \\ \spadesuit \to K_\mathfrak{w}(\diamondsuit \to p_\delta) \end{bmatrix} \right)$$

$$unit_\Gamma = \bigvee_{\gamma \in \Gamma} p_\gamma \wedge \bigwedge_{\gamma \in \Gamma} \left(p_\gamma \to \bigwedge_{\delta \in \Gamma \setminus \{\gamma\}} \neg p_\delta \right)$$

$$Tile_\Gamma = K_\mathfrak{t}(up_\Gamma \wedge down_\Gamma \wedge left_\Gamma \wedge right_\Gamma \wedge unit_\Gamma)$$

LEMMA 3.2. *A grid-like model M, as depicted in Figure 3, satisfies the formula $Tile_\Gamma$ if and only if Γ can tile the integer plane.*

PROOF. (Sketch) This is straightforward to see. For example, the relation $\sim_\mathfrak{n}$ always relates a $\spadesuit$to a $\heartsuit$or a $\clubsuit$to a $\diamondsuit$, and the $\heartsuit$is always above the $\spadesuit$and the $\diamondsuit$is always above the $\clubsuit$. This relationship is used in the subformulas up_Γ and $down_\Gamma$ to make sure that the tiles' tops and bottoms match in adjacent points. The subformula $unit_\Gamma$ ensures that precisely one tile marks each world. The formula $Tile_\Gamma$ uses the common knowledge operator $K_\mathfrak{t}$ to ensure that these properties are maintained at every point in the model. From this it is clear to see that if the model M satisfies $Tile_\Gamma$ then we can extract a tiling of the integer plane. Conversely, given a tiling of the plane we can extract an assignment of the the atoms p_γ to the points in the model M such that $Tile_\Gamma$ will be satisfied. $\square$

We note that $Tile_\Gamma$ does not contain any arbitrary group announcement operators, and is in fact a formula of $\mathcal{L}_{el}$. It is also clear that the formula $Tile_\Gamma$ is satisfiable by some models that do not have the grid-like structure. The real power, and complexity, of this construction is in using the arbitrary group announcement operators to enforce the grid like structure. The critical aspect of enforcing this grid structure is being able to show that two states in the model are n-bisimilar for any n.

We start by defining the basic properties of the grid. We notice that

- No agent ever knows what suit is true in a world.
- Agents $\mathfrak{n}$ and $\mathfrak{s}$ always know that either $\spadesuit$ or $\heartsuit$ are true, or that either $\clubsuit$ or $\diamondsuit$ are true.
- Agents $\mathfrak{e}$ and $\mathfrak{w}$ always know that either $\spadesuit$ or $\diamondsuit$ are true, or that either $\clubsuit$ or $\heartsuit$ are true.
- Agent $\mathfrak{t}$ always considers it possible that any of $\heartsuit$, $\clubsuit$, $\diamondsuit$ or $\spadesuit$ are true.

The constraints are captured by the following edge formulas:

$$\begin{aligned}
E_{\mathfrak{n}} &= \begin{pmatrix} L_{\mathfrak{n}}\spadesuit \wedge L_{\mathfrak{n}}\heartsuit \wedge K_{\mathfrak{n}}(\spadesuit \vee \heartsuit) \\ \vee \\ L_{\mathfrak{n}}\clubsuit \wedge L_{\mathfrak{n}}\diamondsuit \wedge K_{\mathfrak{n}}(\clubsuit \vee \diamondsuit) \end{pmatrix} \\
E_{\mathfrak{s}} &= \begin{pmatrix} L_{\mathfrak{s}}\spadesuit \wedge L_{\mathfrak{s}}\heartsuit \wedge K_{\mathfrak{s}}(\spadesuit \vee \heartsuit) \\ \vee \\ L_{\mathfrak{s}}\clubsuit \wedge L_{\mathfrak{s}}\diamondsuit \wedge K_{\mathfrak{s}}(\clubsuit \vee \diamondsuit) \end{pmatrix} \\
E_{\mathfrak{e}} &= \begin{pmatrix} L_{\mathfrak{e}}\spadesuit \wedge L_{\mathfrak{e}}\diamondsuit \wedge K_{\mathfrak{e}}(\spadesuit \vee \diamondsuit) \\ \vee \\ L_{\mathfrak{e}}\clubsuit \wedge L_{\mathfrak{e}}\heartsuit \wedge K_{\mathfrak{e}}(\clubsuit \vee \heartsuit) \end{pmatrix} \\
E_{\mathfrak{w}} &= \begin{pmatrix} L_{\mathfrak{w}}\spadesuit \wedge L_{\mathfrak{w}}\diamondsuit \wedge K_{\mathfrak{w}}(\spadesuit \vee \diamondsuit) \\ \vee \\ L_{\mathfrak{w}}\clubsuit \wedge L_{\mathfrak{w}}\heartsuit \wedge K_{\mathfrak{w}}(\clubsuit \vee \heartsuit) \end{pmatrix} \\
E_{\mathfrak{t}} &= \begin{pmatrix} L_{\mathfrak{t}}\heartsuit \wedge L_{\mathfrak{t}}\clubsuit \wedge L_{\mathfrak{t}}\diamondsuit \wedge L_{\mathfrak{t}}\spadesuit \\ \wedge \\ (\heartsuit \vee \clubsuit \vee \diamondsuit \vee \spadesuit) \end{pmatrix} \\
Edge &= K_{\mathfrak{t}}(E_{\mathfrak{n}} \wedge E_{\mathfrak{s}} \wedge E_{\mathfrak{e}} \wedge E_{\mathfrak{w}} \wedge E_{\mathfrak{t}})
\end{aligned}$$

These formulas are enough to capture local properties of the grid, but global properties are more complex to capture. We use the group announcement operator to capture the notion of equivalence between two states. The central claim is this: Suppose that at world w, the agent $\mathfrak{n}$ (acting as a group with just one member) made any group announcement (i.e. announced something that he knew). Suppose also that $M_w \models \phi$ and both $\mathfrak{n}$ and $\mathfrak{s}$ did not know whether ϕ was true at w. If, no matter what the announcement was, after making that announcement $\mathfrak{s}$ still does not know whether ϕ is true, it must be the case that for all n, there is some $u \in [w]_{\mathfrak{n}}$ and some $v \in [w]_{\mathfrak{s}}$ such that M_u and M_v are n-bisimilar.

The formula capturing this concept is:

$$\phi \wedge \neg K_{\mathfrak{n}}\phi \wedge \neg K_{\mathfrak{s}}\phi \wedge [\{\mathfrak{n}\}](\neg K_{\mathfrak{n}}\phi \rightarrow \neg K_{\mathfrak{s}}\phi).$$

We sketch the basic argument. There is some $u \in [w]_{\mathfrak{n}}$ where $M_u \models \neg\phi$ (since $\mathfrak{n}$ does not know ϕ). Also, we must have that no matter what announcement is made, there is some $v \in [w]_{\mathfrak{s}}$, such that $M_v \models \neg\phi$. If for every $v \in [w]_{\mathfrak{s}}$, for every $u \in [w]_{\mathfrak{n}}$ there was some formula δ_u^v such that $M_u \models \delta_u^v$, and $M_v \not\models \delta_u^v$, then $\mathfrak{n}$ could have announced $\Delta^v = \bigvee_{u \in [w]_{\mathfrak{n}}} \delta_u^v$ (since it must be known) and afterwards, agent $\mathfrak{s}$ would know that v is not possible. If we took the conjunction of Δ^v for all $v \in [w]_{\mathfrak{s}}$ where $M_v \models \neg\phi$, after this announcement, agent $\mathfrak{s}$ would know that ϕ is true, contradicting our initial assumption. Therefore there must be some world $v \in [w]_{\mathfrak{s}}$ that cannot be distinguished from u by a finite formula, which is enough to show u and v are n-bisimilar for all n. Of course, bisimilarity and language equivalence only coincide for image-finite models [7] in which case Δ^v would be a finite formula. However, we can generalise this argument to show that for all n, the formulas δ_u^v up to modal depth n, consisting only of atoms taken from finite set are inadequate to distinguish the worlds $u \in [w]_{\mathfrak{s}}$ where $M_u \not\models \phi$ from the successors $v \in [w]_{\mathfrak{n}}$. This is sufficient to show that for all n, there is some state $v \in [w]_{\mathfrak{s}}$ such that v is n-bisimilar to some world in $[u]_{\mathfrak{n}}$ with respect to a finite set of atoms. If the finite set of atoms are the atoms corresponding to the tiling and the board, this is sufficient to enforce the integrity of the tiling.

The formula above is a simplification. It shows how we might assign to worlds as being n-bisimilar, but in order to capture a grid-like arrangement, we need a much larger formula which is described below. Consider the emphasized state in Figure 3, and suppose that we are interested in agent $\mathfrak{n}$. Agent $\mathfrak{n}$ cannot distinguish the $\spadesuit$ state from the $\heartsuit$ state above it, so any group announcement the agent makes cannot distinguish those states. We exploit this to define the grid-like structure as follows:

$$\begin{aligned}
G_{\spadesuit}^{\mathfrak{n}} &= [\{\mathfrak{n}, \mathfrak{t}\}] \bigwedge \begin{bmatrix} K_{\mathfrak{e}}(\diamondsuit \rightarrow (K_{\mathfrak{s}}\clubsuit \rightarrow L_{\mathfrak{w}}\heartsuit)) \\ K_{\mathfrak{w}}(\diamondsuit \rightarrow (K_{\mathfrak{s}}\clubsuit \rightarrow L_{\mathfrak{e}}\heartsuit)) \end{bmatrix} \\
G_{\spadesuit}^{\mathfrak{s}} &= [\{\mathfrak{s}, \mathfrak{t}\}] \bigwedge \begin{bmatrix} K_{\mathfrak{e}}(\diamondsuit \rightarrow (K_{\mathfrak{n}}\clubsuit \rightarrow L_{\mathfrak{w}}\heartsuit)) \\ K_{\mathfrak{w}}(\diamondsuit \rightarrow (K_{\mathfrak{n}}\clubsuit \rightarrow L_{\mathfrak{e}}\heartsuit)) \end{bmatrix} \\
G_{\spadesuit}^{\mathfrak{e}} &= [\{\mathfrak{e}, \mathfrak{t}\}] \bigwedge \begin{bmatrix} K_{\mathfrak{n}}(\heartsuit \rightarrow (K_{\mathfrak{w}}\clubsuit \rightarrow L_{\mathfrak{s}}\diamondsuit)) \\ K_{\mathfrak{s}}(\heartsuit \rightarrow (K_{\mathfrak{w}}\clubsuit \rightarrow L_{\mathfrak{n}}\heartsuit)) \end{bmatrix} \\
G_{\spadesuit}^{\mathfrak{w}} &= [\{\mathfrak{w}, \mathfrak{t}\}] \bigwedge \begin{bmatrix} K_{\mathfrak{n}}(\heartsuit \rightarrow (K_{\mathfrak{e}}\clubsuit \rightarrow L_{\mathfrak{s}}\diamondsuit)) \\ K_{\mathfrak{s}}(\heartsuit \rightarrow (K_{\mathfrak{e}}\clubsuit \rightarrow L_{\mathfrak{n}}\diamondsuit)) \end{bmatrix} \\
G_{\spadesuit} &= \spadesuit \rightarrow G_{\spadesuit}^{\mathfrak{n}} \wedge G_{\spadesuit}^{\mathfrak{s}} \wedge G_{\spadesuit}^{\mathfrak{e}} \wedge G_{\spadesuit}^{\mathfrak{w}}
\end{aligned}$$

Similar formulas may be given for $G_{\heartsuit}$, $G_{\diamondsuit}$ and $G_{\clubsuit}$, which are effectively just require a permutation of propositional atoms in the formula $G_{\spadesuit}$. If we were to write the formula $G_{\spadesuit}$ as $G(\heartsuit, \clubsuit, \diamondsuit, \spadesuit)$, then the formulas in question are:

$$\begin{aligned}
G_{\heartsuit} &= G(\spadesuit, \diamondsuit, \clubsuit, \heartsuit) \\
G_{\clubsuit} &= G(\diamondsuit, \spadesuit, \heartsuit, \clubsuit) \\
G_{\diamondsuit} &= G(\clubsuit, \heartsuit, \spadesuit, \diamondsuit) \\
Grid &= K_{\mathfrak{t}}(G_{\heartsuit} \wedge G_{\clubsuit} \wedge G_{\diamondsuit} \wedge G_{\spadesuit})
\end{aligned}$$

We must also constrain the knowledge relation for the agent $\mathfrak{t}$. As this agent is meant to represent the common knowledge of all agents, it should have the weakest knowledge set. This means that any group announcement coming just from this agent should be unable to distinguish between any worlds the the other agents consider possible.

$$\begin{aligned}
CK_{\spadesuit} &= \spadesuit \rightarrow [\{\mathfrak{t}\}](L_{\mathfrak{n}}\heartsuit \wedge L_{\mathfrak{s}}\heartsuit \wedge L_{\mathfrak{e}}\diamondsuit \wedge L_{\mathfrak{w}}\diamondsuit) \\
CK_{\heartsuit} &= \heartsuit \rightarrow [\{\mathfrak{t}\}](L_{\mathfrak{n}}\spadesuit \wedge L_{\mathfrak{s}}\spadesuit \wedge L_{\mathfrak{e}}\clubsuit \wedge L_{\mathfrak{w}}\clubsuit) \\
CK_{\clubsuit} &= \clubsuit \rightarrow [\{\mathfrak{t}\}](L_{\mathfrak{n}}\diamondsuit \wedge L_{\mathfrak{s}}\diamondsuit \wedge L_{\mathfrak{e}}\heartsuit \wedge L_{\mathfrak{w}}\heartsuit) \\
CK_{\diamondsuit} &= \diamondsuit \rightarrow [\{\mathfrak{t}\}](L_{\mathfrak{n}}\clubsuit \wedge L_{\mathfrak{s}}\clubsuit \wedge L_{\mathfrak{e}}\spadesuit \wedge L_{\mathfrak{w}}\spadesuit) \\
CK &= K_{\mathfrak{t}}(CK_{\spadesuit} \wedge CK_{\heartsuit} \wedge CK_{\clubsuit} \wedge CK_{\diamondsuit})
\end{aligned}$$

The formula CK states that in every world that agent $\mathfrak{t}$ considers possible, there is no announcement that $\mathfrak{t}$ can knowledgeably make that would cause any other agent to be able to deduce what the actual suit was. Suppose that $\spadesuit$ marks the current world. If a set of worlds accessible via some agent's (say $\mathfrak{n}$'s) relation were distinct from the set of worlds that agent $\mathfrak{t}$ considered possible, then agent $\mathfrak{t}$ would be able to announce that those worlds are not possible, and $\mathfrak{n}$ would be able to deduce that $\spadesuit$ were true. As this can't happen we can deduce that at least one $\heartsuit$-world the agent $\mathfrak{n}$ considers possible is equivalent (up to epistemic formula) to a world that $\mathfrak{t}$ considers possible. This is a weaker property than common-knowledge (that guarantees that $\mathfrak{t}$ considers all such worlds possible), but it is sufficient for our purposes.

The final property we require is that there may only be one suit true at each world. Let $D = \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$:

$$Sep = K_{\mathfrak{t}}(\bigvee_{d \in D}(d \wedge \bigwedge_{f \in D \setminus d} \neg f)). \quad (1)$$

3.3 Putting it all together

To construct the complete formula we put the separate parts together:

$$T(\Gamma) = Grid \wedge Edge \wedge CK \wedge Tile_{\Gamma} \wedge Sep \quad (2)$$

LEMMA 3.3. *The formula $T(\Gamma)$ is satisfiable if and only if Γ can tile the plane.*

One direction of the proof is much easier than the other. Given a tiling, it is straightforward to build a model that satisfies $T(\Gamma)$. However, given an arbitrary model that satisfies $T(\Gamma)$ it is a non-trivial exercise to extract a tiling of the plane. We will do this by induction over the structure of the model.

PROOF. ($\Longleftarrow$) Suppose that Γ can tile the plane, so that there is a mapping $\tau : \mathbb{Z} \times \mathbb{Z} \longrightarrow \Gamma$ such that adjacent tiles have matching coloured sides.
We build a model $M^{\tau} = (S, \sim, V)$ as follows:

1. $S = \mathbb{Z} \times \mathbb{Z}$;
2. $\sim$ is defined such that:
   (a) $(a, b) \sim_{\mathfrak{n}} (c, d)$ iff $a = c$, $|b - d| \leq 1$ and either $b = d$ or $a + b + c + d = 1 \mod 4$;
   (b) $(a, b) \sim_{\mathfrak{s}} (c, d)$ iff $a = c$, $|b - d| \leq 1$ and either $b = d$ or $a + b + c + d = 3 \mod 4$;
   (c) $(a, b) \sim_{\mathfrak{e}} (c, d)$ iff $b = d$, $|a - c| \leq 1$ and either $a = c$ or $a + b + c + d = 1 \mod 4$;
   (d) $(a, b) \sim_{\mathfrak{w}} (c, d)$ iff $b = d$, $|a - c| \leq 1$ and either $a = c$ or $a + b + c + d = 3 \mod 4$;
   (e) $(a, b) \sim_{\mathfrak{t}} (c, d)$ for all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$.
3. V is defined such that:
   (a) $V(\heartsuit) = \{(a, b) \mid a \text{ is even, and } b \text{ is odd}\}$;
   (b) $V(\clubsuit) = \{(a, b) \mid a \text{ is odd, and } b \text{ is odd}\}$;
   (c) $V(\diamondsuit) = \{(a, b) \mid a \text{ is odd, and } b \text{ is even}\}$;
   (d) $V(\spadesuit) = \{(a, b) \mid a \text{ is even, and } b \text{ is even}\}$;
   (e) $V(p_{\gamma}) = \{(a, b) \mid \tau(a, b) = \gamma\}$

It is easy to see that M^{τ} is a valid $S5$ model. We claim that $T(\Gamma)$ holds at all points of the model, M^{τ}. The formulas $Edge$, $Tile_{\Gamma}$ and Sep are easy to verify, and are left as an exercise for the reader. For CK it is clear that as $\sim_{\mathfrak{t}}$ is the universal relation, there is nothing that $\mathfrak{t}$ can knowledgeably announce that would change the model in anyway. As all subformulas in the scope of the group announcements operators are implied by $Edge$, the formula CK will hold. Finally, we consider $G^{\mathfrak{n}}_{\spadesuit}$ at the world $(0, 0)$. Whatever announcements $\mathfrak{n}$ and $\mathfrak{t}$ may make must preserve the worlds $(0, 0)$ and $(0, 1)$. Now suppose that after these announcements $K_{\mathfrak{e}}(\diamondsuit \wedge K_{\mathfrak{s}}\clubsuit)$ holds at world $(0, 0)$. The only way this be true is if whatever was announced did not contradict what was true at world $(1, 0)$ and world $(1, 1)$. Therefore all worlds $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ must remain, and hence all relations between these worlds remain. Therefore we have

$$M^{\tau}_{(0,0)} \models [\{\mathfrak{n}, \mathfrak{t}\}]K_{\mathfrak{e}}(\diamondsuit \rightarrow K_{\mathfrak{s}}(\clubsuit \rightarrow L_{\mathfrak{w}}\heartsuit))$$

and the other subformulas may be shown in a similar way. Consequently we can see that M satisfies $T(\Gamma)$ at all worlds in the model.

($\Longrightarrow$) Suppose now that $M_s = (S, \sim, V, s)$ is some arbitrary pointed model such that $M_s \models T(\Gamma)$. We fix a finite set of atoms $\Pi = \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit, p_{\gamma} \mid \gamma \in \Gamma\}$. We will assume, without loss of generality, that $M_s \models \spadesuit$. In this proof we will build a map $\tau_n : S \longrightarrow \mathbb{Z} \times \mathbb{Z}$ such that if $\tau_n(u) = \tau_n(v)$, then $v \in [u]_n$. We define $\tau_n(s) = (0, 0)$ for all n, and continue the map as follows.

Since $M_s \models CK$, so we have that

$$M_s \models [\{\mathfrak{t}\}](L_{\mathfrak{n}}\heartsuit \wedge L_{\mathfrak{s}}\heartsuit \wedge L_{\mathfrak{e}}\diamondsuit \wedge L_{\mathfrak{w}}\diamondsuit).$$

Therefore, no matter what announcement that $\mathfrak{t}$ makes, the agents $\mathfrak{n}$, $\mathfrak{s}$, $\mathfrak{e}$ and $\mathfrak{w}$ are unable to deduce that the current world is $\spadesuit$. Now, let us fix some positive n and examine the set of worlds $\{w \in V(\heartsuit) \mid s \sim_{\mathfrak{n}} w\}$. Suppose, for contradiction, that all of these worlds were not n-bisimilar to any world u where $s \sim_{\mathfrak{t}} u$. From Lemma 3.1 there will be a finite set of formulas Ψ such that $M_s \models K_{\mathfrak{t}}(\heartsuit \rightarrow \bigvee_{\psi \in \Psi} \psi)$. The agent $\mathfrak{t}$ would be able to announce $\heartsuit \rightarrow \bigvee_{\psi \in \Psi} \psi$, after which agent $\mathfrak{n}$ would know that $\heartsuit$ cannot be true in the current world. But this contradicts the implication of CK, so it must be that at least one world u where $s \sim_{\mathfrak{t}} u$ and one world v where $s \sim_{\mathfrak{n}} v$ such that $v \in [u]_n$. Now since $u \sim_{\mathfrak{t}} s$ and $M_s \models CK$, it must also be the case that $M_u \models CK$. We may set $\tau_n(u) = \tau_n(v) = (0, 1)$, and proceed. By reasoning similar to the above, if $u \sim_{\mathfrak{e}} w$, then there is some $x \sim_{\mathfrak{t}} u$ such that $x \in [w]_n$. Thus we can let $\tau_{n-1}(w) = \tau_{n-1}(x) = (1, 1)$, and since $u \sim_{\mathfrak{t}} s$ we have via transitivity, $x \sim_{\mathfrak{t}} s$. We will define $\tau_n : S \longrightarrow \mathbb{Z} \times \mathbb{Z}$ such that:

- for all n, $\tau_n(s) = (0, 0)$.
- for all $u, v \in S$, for all $n \geq 0$, $\tau_n(u) = \tau_n(v)$ implies $u \in [v]_n$.
- for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, for all n, there is some $x \in S$ such that $\tau_n(x) = (a, b)$ and $s \sim_{\mathfrak{t}} x$.
- if $\tau_n(u) = (a, b)$, and $M_u \models \mathfrak{B}$ then there exists
  1. $v_1 \sim_{\mathfrak{n}} u$ where $M_{v_1} \models \mathfrak{R}$ and $\tau_{n-1}(v_1) = (a, b+1)$
  2. $v_2 \sim_{\mathfrak{s}} u$ where $M_{v_2} \models \mathfrak{R}$ and $\tau_{n-1}(v_2) = (a, b-1)$
  3. $v_3 \sim_{\mathfrak{e}} u$ where $M_{v_3} \models \mathfrak{R}$ and $\tau_{n-1}(v_3) = (a+1, b)$
  4. $v_4 \sim_{\mathfrak{w}} u$ where $M_{v_4} \models \mathfrak{R}$ and $\tau_{n-1}(v_4) = (a-1, b)$
- if $\tau_n(u) = (a, b)$, and $M_u \models \mathfrak{R}$ then there exists
  1. $v_1 \sim_{\mathfrak{n}} u$ where $M_{v_1} \models \mathfrak{B}$ and $\tau_{n-1}(v_1) = (a, b-1)$
  2. $v_2 \sim_{\mathfrak{s}} u$ where $M_{v_2} \models \mathfrak{B}$ and $\tau_{n-1}(v_2) = (a, b+1)$
  3. $v_3 \sim_{\mathfrak{e}} u$ where $M_{v_3} \models \mathfrak{B}$ and $\tau_{n-1}(v_3) = (a-1, b)$
  4. $v_4 \sim_{\mathfrak{w}} u$ where $M_{v_4} \models \mathfrak{B}$ and $\tau_{n-1}(v_4) = (a+1, b)$

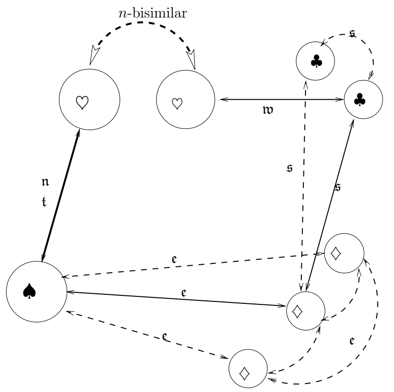

Figure 4: This figure represents the key construction of the grid from the formula $G^{\mathfrak{n}}_{\spadesuit}$. The solid lines indicate the relations that form the edges in the grid.

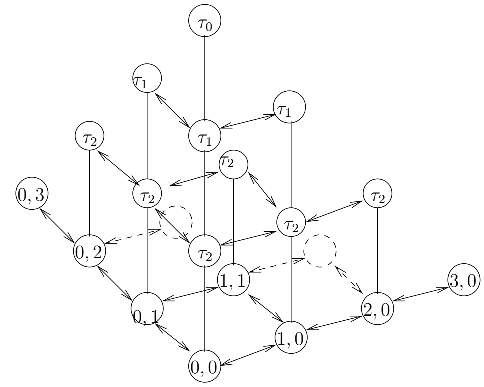

Figure 5: This figure represents the functions τ_n for $n = 0, 1, 2, 3$, and their relationship to the integer plane.

where $\mathfrak{B} = \clubsuit \vee \spadesuit$ and $\mathfrak{R} = \heartsuit \vee \diamondsuit$. The last two clauses are the most difficult to show, and they depend on the formula *Grid*. We will show how the formula $G^{\mathfrak{n}}_{\spadesuit}$ helps enforce this property, and the remaining 15 subformulas of *Grid* may be handled similarly. So suppose that for some $u \in S$, we have $u \sim_{\mathfrak{t}} s$, $\tau_n(u) = (a, b)$, and $M_u \models \spadesuit$. From the formula *Grid*, it must be that $M_u \models G^{\mathfrak{n}}_{\spadesuit}$. We also know $M_u \models Sep$, so each world must satisfy a unique suit ($\heartsuit$, $\clubsuit$,$\diamondsuit$,$\spadesuit$). Now, we have

$$M_u \models [\{\mathfrak{n}, \mathfrak{t}\}] \bigwedge \left[\begin{array}{l} K_{\mathfrak{e}}(\diamondsuit \rightarrow (K_{\mathfrak{s}}\clubsuit \rightarrow L_{\mathfrak{w}}\heartsuit)) \\ K_{\mathfrak{w}}(\diamondsuit \rightarrow (K_{\mathfrak{s}}\clubsuit \rightarrow L_{\mathfrak{e}}\heartsuit)) \end{array} \right]$$

so no matter what announcements $\mathfrak{n}$ and $\mathfrak{t}$ might make, every $\diamondsuit$-world indistinguishable to $\mathfrak{e}$, has a $\clubsuit$-world indistinguishable to $\mathfrak{s}$ such that there is some $\heartsuit$-world, v, indistinguishable to $\mathfrak{w}$. It is possible that $\mathfrak{n}$ could make an announcement such as $\heartsuit \vee \spadesuit$ so the $\diamondsuit$-world and the $\clubsuit$-world would be removed. However, any announcement made by $\mathfrak{n}$ and $\mathfrak{t}$ that preserves the two intermediate worlds must preserve a $\heartsuit$-world. From Lemma 3.1 there cannot be a finite set of formulas, Ψ known by either of $\mathfrak{n}$ and $\mathfrak{t}$ that could rule out every $\heartsuit$ world reachable in such a way. The only way this may happen is if there is at least one world $x \sim_{\mathfrak{n}} u$ and at least one world $y \sim_{\mathfrak{t}} u$ such that $v \in [x]_{n-1} \cap [y]_{n-1}$. As $[x]_n$ and $[y]_n$ are equivalence classes it follows that v, x and y are all $n - 1$ bisimilar. This construction is depicted in Figure 4.

We let $\tau_{n-1}(x) = \tau_{n-1}(y) = \tau_{n-1}(v) = (a, b + 1)$ and continue. We can repeat the construction for every pair of directions ($\mathfrak{n}$,$\mathfrak{s}$,$\mathfrak{e}$and $\mathfrak{w}$) and suits ($\heartsuit$,$\clubsuit$, $\diamondsuit$and $\spadesuit$). As the argument universally quantifies over the intermediate worlds (the $\diamondsuit$-world and $\clubsuit$-world above), the definition of τ_n will be consistent. Each time the argument is applied we only require that worlds be $n - 1$ bisimilar to one another, since if a world u' was n-bisimilar to u, worlds reachable from it will be $n - 1$ bisimilar at best. This way, the function τ_n if defined inductively with respect to τ_{n-1}. This layered construction is represented in Figure 5.

Given the set of mappings τ_n, we are now able to construct a tiling. Since $M_s \models Tile_\Gamma$, for every world t where $s \sim_{\mathfrak{t}} t$, there is a unique tile γ such that $t \in V(p_\gamma)$. Since for all $u, v \in S$, if $\tau_n(u) = \tau_n(v)$ then $u \in [v]_n$ so it must be that $u \in V(p_\gamma) = v \in V(p_\gamma)$. Therefore for each n, we can describe a partial map $\pi_n : \mathbb{Z} \times \mathbb{Z} \longrightarrow \Gamma$, by $\pi_n(a, b) = \gamma$ if and only if for any $t \in S$ such that $\tau_n(t) = (a, b)$, we have $t \in V(p_\gamma)$. We are then able to build a complete tiling $\pi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \Gamma$ by enumerating $\mathbb{Z} \times \mathbb{Z}$ as $a_0, a_1, ...$ where $a_0 = (0, 0)$. For each i we let $a^n_i \in \Gamma$ be the tile such that for all v where $\tau_n(v) = a_i$, $v \in V(p_\gamma)$. We give an inductive definition of π where $\pi(0, 0) = \gamma$ where $s \in V(p_\gamma)$ and $\pi(a_{i+1}) = \gamma$ where $a^n_{i+1} = \gamma$ for $n \in N$ where N is an infinite subset of $\{m \mid a^m_i = \pi(a_i)\}$. As $\{m \mid a^m_0 = \pi(a_0)\}$ is the set of all natural numbers, and Γ is finite, it follows that $\{m \mid a^m_i = \pi(a_i)\}$ is an infinite set. Furthermore, it follows that for any i, j we have $\{m \mid a^m_i = \pi(a_i)\} \cap \{m \mid a^m_j = \pi(a_j)\}$ is infinite. Therefore the definition of τ_n, and the fact that $M_s \models Tile_\Gamma$ implies that the function π is a valid tiling (i.e. the sides of the tiles match).

This is sufficient to show that for any finite set of tiles Γ, the existence of a Γ-tiling of the integer plane is equivalent to the satisfiability of a computable formula of GAL. $\square$

THEOREM 3.4. *The satisfiability problem for GAL is co-RE complete.*

PROOF. Lemma 3.3 shows that the satisfiability problem of GAL is not recursively enumerable. Given the existence of a sound and complete axiomatization for GAL [2], the validity problem must be recursively enumerable, and hence the satisfiability problem is co-RE complete. $\square$

4. FUTURE WORK

The previous section gives an essentially negative result: we are unable to determine the satisfiability of formulas in Group Announcement Logic. The careful reader will observe that we have in fact proven a stronger result: in the proof of undecidability we did not use the public announcement operators, which means that the logic without those operators is already undecidable.

The negative result does not make the logic redundant, as validity is recursively enumerable and there is a computable model checking procedure [2]. The core purpose of Group Announcement Logic, to determine what agents can achieve by sharing knowledge is still a question of great interest. The undecidability proof given here does not use an intrinsic property of knowledge sharing, but rather exploits the power of quantifying over language. With this in mind, we consider the question of what logics could successfully reason about knowledge sharing agents. We list some of the possible answers below:

- One possibility that remains open is the satisfiability problem for Coalition Announcement Logic (CAL)[3] mentioned in the introduction. CAL quantifies over the true announcements that a group of agents can make at the same time as the remaining agents make an announcement. CAL, like GAL, quantifies over the set of possible announcements, and we suspect that it too is undecidable. A similar strategy to the one above could probably be applied to create a tiling, although the formulas presented above require some adjustment.

- Another approach could be to consider generalizations of announcements as the medium of informative updates. By replacing the public announcements in Arbitrary Public Announcement Logic with refinements [18] it is possible to define a logic that may be used to reason about the quantification of informative updates. The essential difference between APAL and this logic is that while APAL has an operator that quantifies over all public announcements, this logic has an operator that quantifies over all refinements. Recent work [10] presents a logic that quantifies over event models [5] and could also be an alternative to Group Announcement Logic.

- Restricting announcements to positive knowledge is also interesting. A positive knowledge formula is a formula where all K_a operators are in the scope of an even number of negations. This means agents may announce what they know, and what they know other agents know, and so on. However, they may not announce what they don't know, or what they know other agents don't know. The appealing thing about these announcements is that once they are made, they will always remain true, so they have a monotonic nature. The other important aspect of positive knowledge announcements is that they cripple the strategy that has been used to show the undecidability of GAL and APAL. When a similar argument is applied, we are only able to establish that one state is an n-refinement of another, rather than being able to establish that they are n-bisimilar. This suggests that a logic of arbitrary (or group) positive announcements may be more computationally amenable than GAL.

- Distributed knowledge [8] gives a semantic interpretation of knowledge sharing, so that the distributed knowledge of agent i and j at world w, is the set of worlds $\{s \mid s \sim_i w, s \sim_j w\}$. This logic is decidable. However, it's implementation assumes that agents know a shared labelling of worlds, and through this labelling, they are able to share knowledge. In practise, such labellings do not exist, so our interest is in how we might represent a realisable version of distributed knowledge.

These approaches can contribute to building a computation feasible approach to the question of what agents can achieve by sharing their knowledge, and we aim to investigate these approaches to distributed knowledge in future work.

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