Facility Location Games with Dual Preference

Shaokun Zou
Department of Computer Science, City University of Hong Kong
Hong Kong
zoushaokun15@gmail.com

Minming Li
Department of Computer Science, City University of Hong Kong
Hong Kong
minming.li@cityu.edu.hk

ABSTRACT

In this paper, we focus on the facility location games with the property of dual preference. Dual preference property indicates that both two preferences of agents, staying close to and staying away from the facility(s), exist in the facility location game. We will explore two types of facility location games with this property, the dual character facility location game and the two-opposite-facility location game with limited distance which model the scenarios in real life. For both of them, we wish to design strategy-proof mechanisms or group strategy-proof mechanisms with the objective of optimizing the social utility. For the dual character facility location game, we propose a strategy-proof optimal mechanism when misreporting is restricted to agents' preferences, and give a $\frac{1}{3}$-approximation deterministic group strategy-proof mechanism when both location and preference are considered as private information. For the two-opposite-facility location game with limited distance, when the number of agents is even (denoted as $2k$), we give a $\frac{1}{k}$-approximation deterministic group strategy-proof mechanism, and when the number of agents is odd (denoted as $2k-1$), we propose a $\frac{1}{2k-1}$-approximation deterministic group strategy-proof mechanism. The approximation ratios for both mechanisms are proved to be the best a deterministic strategy-proof mechanism can achieve.

Categories and Subject Descriptors

F.6.1 [**Theory of computation**]: Algorithmic mechanism design

General Terms

Economics, Theory

Keywords

Algorithmic Mechanism Design, Facility Location, Dual Preferences, Mechanisms without Money

Appears in: *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2015), Bordini, Elkind, Weiss, Yolum (eds.), May 4–8, 2015, Istanbul, Turkey.*

1. INTRODUCTION

In this paper, we study the facility location games with the property of dual preference. Its origin, the facility location game, models the scenario where the government is going to build a facility on a line segment where some self-interested agents who tend to maximize their own utility are situated. The agents are required to report their locations as private information, which will then be mapped to a single facility location by a mechanism, with the purpose of optimizing the social utility. Dual preference property means that both preferences of agents, staying as close as possible to and staying as far away as possible from the facility(s), exist in the facility location game. To the best of our knowledge, this is the first time that the dual preference property is introduced in the facility location game.

We find that dual preference property describes well the fact in real life that apparent individual difference among citizens exists in terms of life styles and individual demands, and thus the emergence of distinct attitudes towards a certain facility is quite natural and common. Consider the case where the government plans to build a farmer's market on a line segment. Some agents may prefer living closer to the farmer's market for easy access to fresh vegetables, while others would like to keep away from it because of the garbage left by vegetable vendors as well as noise and transport inconvenience caused by large amounts of people and vehicles inside and around the market. For this case, we formulate the dual character facility location game.

In addition, dual preference property would also be useful to capture the scenarios where different characteristics of facilities result in different preferences. This scenario is possible to appear when several facilities, related but serving diverse functions, are to be built by the government in order to cooperate for a particular purpose. For example, to maintain the public order in an area, the government is going to build a police station along with a detention house to detain criminals arrested by police. The agents in this area would prefer a shorter distance towards the police station for timely rescue from police in case of emergency. However, they would wish to keep far away from the detention house concerning its potential security risks such as prison break. Besides, to guarantee the quick response and efficient control from police when security incidents happen in the detention house, the distance between two facilities should be limited. Take, for instance, another similar case that on a line segment where some factories are located, the government plans to build a refuse collection point to collect garbage and a waste treatment plant to dispose collected waste.

Naturally, all factories would wish to stay closer to the collection point for less cost in sending garbage, but keep away from the waste treatment plant to alleviate the effect of pollution in the process of waste disposal. Also, in order to save the cost of transportation and enhance garbage disposal efficiency, the government should set a limitation to the distance between the two facilities. The distance limitation mentioned in above examples reflects the relation between two facilities, which will be incorporated as an important element in our second formulated model called two-opposite-facility location game.

In the scenarios mentioned above, assume all agents know the mechanisms that the government will adopt to aggregate agents' information to the final locations of the facilities. An agent may have a chance to improve its utility, i.e., shorten or lengthen the distance to a certain facility according to preferences, by misreporting. Therefore, we emphasize the strategy-proofness of a mechanism, which guarantees that an agent cannot acquire more utility from misreporting. We also try to find group strategy-proof mechanisms, which discourages the simultaneous misreporting of a group of agents. In addition, we need to evaluate the mechanisms in terms of optimization of social utility, usually defined to be the sum of utilities of all agents. The evaluation is mainly conducted by the approximation ratio for the social utility of a mechanism, which is the worst ratio between the social utility of the mechanism output and the optimal social utility value among all possible profiles.

1.1 Related work

The classic facility location game where all agents on a line segment only prefer staying close to the facility to be built was firstly studied by Procaccia and Tennenholtz [17], deriving from the work of single peaked preference problem studied by Moulin [16] and extending its primary result with the objectives of optimizing the social cost and the maximum cost. For the problem where two facilities are to be built, Procaccia and Tennenholtz [17], Lu et al. [15] [14] gave and improved lower and upper bound of approximation ratios for deterministic and randomized strategy-proof mechanisms in succession. For the k-facility location game, Fotakis and Tzamos [9] showed that the addition of winner-imposing constraint can guarantee the strategy-proofness of Proportional Mechanism, and in [11], they extended the study to the cases where concave cost functions between agents and facilities exist. Other extended settings of the classic facility location game are studied in [1] [10] [7] [6] [8] [2] [19] [18].

Mechanism designs for the obnoxious facility location game where all agents on a line segment have the preference of staying as far away as possible from the facility was initiated by Cheng et al. [4] and they gave deterministic and randomized group strategy-proof mechanisms for it. Cheng et al. [5] further studied the scenarios where agents are located on circles and trees. Complete characterization for deterministic (group) strategy-proof mechanisms in line metric is presented in [13] and [12]. [3] further explored the case where a service radius r is assigned to the obnoxious facilities.

2. PRELIMINARIES

In this section, we introduce some notations and definitions used in this paper. Let $N = (1, 2, ..., n)$ be the set of agents. In our setting, all agents are located on a line segment. We denote the length of the line segment as l ($l > 0$), the leftmost point of the line segment as 0 and the rightmost point as l. For two points a, b on the line segment, we use $d(a, b)$ to denote the distance between two points.

We use b_i to denote the information (e.g. position and/or preference) of agent i which alternatively can represent the **bid** from agent i if he tells the truth, and use the set $\mathbf{b} = (b_1, b_2, ..., b_n)$ to indicate the **profile** which contains bids of all agents. A mechanism is a function f which maps the profile to an **output** O containing locations of all facilities to be built, which can be written as $O = f(\mathbf{b})$. Notice that due to the different natures of two games we study in this paper, here we only use general notations for the concepts of bid, profile and output. The specific form of notations for these concepts will vary in the following sections. We use $SU(f, \mathbf{b})$ to indicate the **social utility** for the profile $\mathbf{b}$ under the mechanism f and use $su(O, \mathbf{b})$ to indicate the **social utility** for the profile $\mathbf{b}$ with a given output. For agent i, we use $u(b_i, O)$ to denote its **utility** with respect to output O. In addition, for a profile $\mathbf{b}$, we define the sub profile which contains bids of all agents except b_i as $\mathbf{b}_{-i}$, and we use $<>$ to connect two profile sets $\mathbf{b}_1$ and $\mathbf{b}_2$, i.e. $< \mathbf{b}_1, \mathbf{b}_2 >$, to indicate the new profile set composed of bids in $\mathbf{b}_1$ and $\mathbf{b}_2$. Using this notation, $\mathbf{b}$ can be expressed as $< \mathbf{b}_{-i}, b_i >$.

A mechanism f is strategy-proof if no agent can acquire more utility by misreporting. That is, for any agent $i \in N$, suppose it misreports its information to b'_i, we have $u(b_i, f(< \mathbf{b}_{-i}, b'_i >)) \leq u(b_i, f(\mathbf{b}))$.

A mechanism f is group strategy-proof if for any group of agents, at least one of them cannot acquire more utility if they misreport simultaneously. That is, for any group $G \subseteq N$, suppose they misreport their profiles to $\mathbf{b}'_G$, there exists an agent $i \in G$ such that $u(b_i, f(< \mathbf{b}_{-G}, \mathbf{b}'_G >)) \leq u(b_i, f(\mathbf{b}))$, where $\mathbf{b}_{-G}$ denotes the sub profile containing bids of all agents not in G.

3. DUAL CHARACTER FACILITY LOCATION GAME

In the dual character facility location game, all agents are situated on a line segment with length l. Each agent reports its location and preference, and the location of the facility planned to be built on the same line segment will be determined by the complete profile of all agents. Different agents may have different preference values which indicate whether the agents want to stay close to the facility (1) or not (0). Let $N = (1, ..., n)$ be a set of agents, in which each agent i has its location x_i, preference value p_i and together $c_i = (x_i, p_i)$. We use set $\mathbf{x} = (x_1, ..., x_n)$ as the location profile, the set $\mathbf{p} = (p_1, ..., p_n)$ as the preference profile, and the collection $\mathbf{c} = (x_1, p_1, ..., x_n, p_n)$ as the profile of all agents. Assume the facility is built at y, then for an agent i with preference value $p_i = 0$, its utility $u(c_i, y)$ is defined as the distance between the agent and the facility, $d(x_i, y)$; if $p_i = 1$, its utility is defined as the length of the line segment minus the distance between the agent and the facility, i.e. $u(c_i, y) = l - d(x_i, y)$. Both types of agents tend to maximize their utilities by misreporting.

The social utility when the facility is located at location y is equal to the sum of utility values of all agents, i.e.

$su(y, \mathbf{c}) = \sum_{i=1}^{n} u(c_i, y)$. Denoting the optimal social utility for profile $\mathbf{c}$ as $OPT(\mathbf{c})$, we have the following fact.

FACT 1. *For any profile* $\mathbf{c}$*,* $OPT(\mathbf{c}) > 0$*.*

PROOF. As for any agent the utility is not negative, the social utility is not negative and $OPT(\mathbf{c}) \geq 0$.

Assume there exists a profile $\mathbf{c}$ with $OPT(\mathbf{c}) = 0$. Consider a location y such that $su(y, \mathbf{c}) = OPT(\mathbf{c})$. Obviously, $\forall i \in [1, n]$, $u(x_i, y) = 0$. As for every agent i, there can be at most one location y such that $u(x_i, y) = 0$, then consider another location $y' \neq y$, we have $\forall i \in [1, n]$, $u(x_i, y') > 0$, which implies $su(y', \mathbf{x}) > 0 = OPT(\mathbf{c})$, causing a contradiction. Hence, we have $OPT(\mathbf{c}) > 0$. □

For a mechanism f, if there exists a number β such that for any $\mathbf{c}$, the output from f satisfies $\frac{SU(f,\mathbf{c})}{OPT(\mathbf{c})} \geq \beta$, then we say the approximation ratio for the social utility of f is β.

In real life, the utility for an agent defined above can be a good way for the manager to measure the degree of satisfaction of an agent in terms of the location for the facility. In both groups, namely the group composed of all agents with preference value 0 and the group containing all other agents, the agents with higher utility values tend to be more satisfactory in practical significance. Specially, by the different expressions for two types of agents, the utilities for both of them are in the range of $[0, l]$, which makes it possible to compare and deal with utility values for agents of different types in real life.

Next, we will divide the problem into two scenarios according to the extent to which agents can misreport.

3.1 Misreporting Only the Preference or Location

In the first scenario, we assume that the explicit location information for every agent in a profile has been acquired, thus the only possible way for an agent to achieve a better utility is to misreport its preference value.

Because of the given expression of social utility, the optimal value only occurs at two end points of the line segment or the point where an agent stands. Given a profile $\mathbf{c}$ of n agents, define the positions of two end points as x_0 (i.e. 0) and x_{n+1} (i.e. l) separately. Then, the mechanism to achieve the optimal social utility value can be defined as follows:

MECHANISM 1. *Locate the facility at the leftmost point* j *such that* $su(x_j, \mathbf{c}) = max_{i \in [0,n+1]} su(x_i, \mathbf{c})$*.*

Next we will prove that Mechanism 1 is a strategy-proof mechanism for the dual character facility location game when misreporting is limited to the preference value.

THEOREM 2. *Mechanism 1 is strategy-proof.*

PROOF. For a given profile $\mathbf{c}$, assume agent i with $c_i = (x_i, p_i)$ misreports its preference to be $c'_i = (x_i, 1 - p_i)$. Denote the profile after misreporting as $\mathbf{c}'$ and suppose that the output for $\mathbf{c}$ and $\mathbf{c}'$ are y and y'. We distinguish two cases.

Case 1. $p_i = 0$. Define $g(y, \mathbf{c}, i) = \sum_{j \in [1,n] \& j \neq i} u(c_j, y)$, then $su(y, \mathbf{c}) = u(c_i, y) + g(y, \mathbf{c}, i) = g(y, \mathbf{c}, i) + d(x_i, y)$. Similarly, $su(y, \mathbf{c}') = u(c'_i, y) + g(y, \mathbf{c}', i) = g(y, \mathbf{c}', i) + l - d(x_i, y)$. Define $df(y) = su(y, \mathbf{c}') - su(y, \mathbf{c})$, as for every agent j such that $j \neq i$, the bids are the same in $\mathbf{c}$ and $\mathbf{c}'$, $g(y, \mathbf{c}, i) = g(y, \mathbf{c}', i)$ and $df(y) = l - 2 * d(x_i, y)$. Similarly, $df(y') = l - 2 * d(x_i, y')$. By Mechanism 1, $su(y, \mathbf{c}) = max_{i \in [0,n+1]} su(x_i, \mathbf{c})$, so $su(y, \mathbf{c}) \geq su(y', \mathbf{c})$, similarly, $su(y', \mathbf{c}') \geq su(y, \mathbf{c}')$. Hence $su(y', \mathbf{c}) + df(y') \geq su(y, \mathbf{c}) + df(y)$, which implies $df(y') - df(y) \geq su(y, \mathbf{c}) - su(y', \mathbf{c}) \geq 0$. Hence $2 * d(x_i, y) - 2 * d(x_i, y') = df(y') - df(y) \geq 0$. Because $p_i = 0$, agent i cannot gain more utility from the misreporting.

Case 2. $p_i = 1$. The proof for this case is similar.

Intuitively, one can interpret the proof in the following way. If the agent prefers to stay close to the facility, then lying to dislike the facility cannot move the facility towards him. On the other hand, if the agent prefers to stay away from the facility, then lying to like the facility cannot push the facility further away from him. □

If the misreporting is limited to the location value, a special case of this is the obnoxious facility location game (where all the agents prefer to stay as far away from the facility as possible) where the best approximation ratio for strategy-proof mechanisms is $\frac{1}{3}$ [13][4]. We will prove in the next subsection that even if the manipulation is on both the location and the preference, we can provide a strategy-proof mechanism with an approximation ratio of $\frac{1}{3}$. Therefore, we do not elaborate this case.

3.2 Misreporting Both Preference and Location

In this scenario, every agent on the line segment can misreport both its preference value and its location. We can find that Mechanism 1 is not strategy-proof in the following profile of this setting.

Assume $l = 2$ and consider a profile with $n = 4$. The agents profiles of four agents are $x_1 = 0$, $p_1 = 1$, $x_2 = \frac{1}{4}$, $p_2 = 0$, $x_3 = \frac{2}{3}$, $p_3 = 0$, $x_4 = 1$, $p_4 = 1$, the output location of the facility by Mechanism 1 should be 1 and $u(c_3, 1) = \frac{1}{3}$. However, if agent 3 misreports its location to $x'_3 = 1$, then the output location should be 0 and $u(c_3, 0) = \frac{2}{3}$. Hence agent 3 gains larger utility from its misreporting.

We propose another deterministic mechanism which is strategy-proof in this case with approximation ratio $\frac{1}{3}$. Before presenting the details of the mechanism, we will introduce a new attribute **transformed location** x_i^* for every agent i in a profile. For an agent i, if $p_i = 0$, $x_i^* = x_i$; if $p_i = 1$, $x_i^* = l - x_i$. Obviously, for an agent i with $p_i = 1$, x_i^* and x_i are symmetric about the middle point of the line segment.

MECHANISM 2. *For a profile* $\mathbf{c}$*, denote* n_l *as the number of agents with transformed locations in* $[0, \frac{l}{2})$*, and* n_r *as the number of other agents. If* $n_l \leq n_r$*, build the facility at 0, otherwise, build the facility at* l*.*

THEOREM 3. *Mechanism 2 is group strategy-proof.*

PROOF. Consider a profile $\mathbf{c}$ with output y from Mechanism 2. Assume the agents in a group $G \subseteq N$ misreport their profiles to $\mathbf{c}'_G$. For agent $i \in G$, assume the bid after misreporting is $c'_i = (x'_i, p'_i)$. For the new profile $< \mathbf{c}_{-G}, \mathbf{c}'_G >$ with output y', denote the number of agents with transformed locations in $[0, \frac{l}{2})$ as n'_l, and the number of other agents as n'_r. The discussion can be divided into two cases.

Cases 1: $y = 0$, which indicates $n_l \leq n_r$.

If there exists an agent $i \in G$ such that $x_i^* \geq \frac{l}{2}$, then $p_i = 0$ and $x_i \geq \frac{l}{2}$ or $p_i = 1$ and $x_i \leq \frac{l}{2}$. In both conditions, agent i has achieved the maximum utility from Mechanism 2, and cannot gain more utility by misreporting.

If $\forall i \in G$, $x_i^* < \frac{l}{2}$, then every agent in G has been counted in n_l, we have $n_l' \leq n_l$, and $n_r' \geq n_r$. Hence, $n_l' \leq n_r'$, and $y' = 0 = y$. Therefore the output cannot be changed and agents in G cannot have more utility.

Case 2: $y = l$. The proof is similar. □

In the following discussion, we will prove that the approximation ratio for the social utility of Mechanism 2 is $\frac{1}{3}$.

LEMMA 4. *For any point y and two points a, b on the line segment, if a and b are symmetric about the middle point of the line segment, then* $l - d(y,a) \geq d(y,b)$.

PROOF. If $a = b$, then a, b must be the middle point of the line segment and $\forall y \in [0,l]$, $d(y,a) = d(y,b) \leq \frac{l}{2}$. We can get $l - d(y,a) \geq \frac{l}{2} \geq d(y,b)$.

If $a < b$, we have $a < \frac{l}{2}$. The proof can be divided into three subcases.

Case 1. $y \geq a$ and $y \leq b$. In this case, we can get $l - d(y,a) = d(y,b) + 2*a \geq d(y,b)$.

Case 2. $y < a$. In this case, we define $y' = l - y$. We can get $d(y,b) = d(y',a)$ and $l - d(y,a) = d(y',a) + 2*y = d(y,b) + 2*y \geq d(y,b)$.

Case 3. $y > b$. The proof is similar to that for Case 2.

If $a > b$, the proof is similar to that when $a < b$. □

LEMMA 5. *Under Mechanism 2, if a profile* $\mathbf{c}$ *with output y satisfies* $\forall i \in [1,n]$, $p_i = 1$, *then* $\frac{SU(f,c)}{OPT(c)} = \frac{su(y,c)}{OPT(c)} \geq \frac{1}{3}$.

PROOF. It is obvious that $OPT(\mathbf{c})$ occurs when the facility is built at the location of the middle agent (if n is odd) or any location between two middle agents (if n is even). Denote the agents as i_1, i_2,..., i_n from left to right. Given y_o such that $su(y_o,\mathbf{c}) = OPT(\mathbf{c})$, for any integer k satisfying $k \leq \lfloor \frac{n}{2} \rfloor$, we have $x_{i_k} \leq y_o \leq x_{i_{n+1-k}}$, with $d(x_{i_k}, y_o) + d(x_{i_{n+1-k}}, y_o) = d(x_{i_k}, x_{i_{n+1-k}})$. Hence $u(c_{i_k}, y_o) + u(c_{i_{n+1-k}}, y_o) = l - d(y_o, x_{i_k}) + l - d(y_o, x_{i_{n+1-k}}) = 2l - d(x_{i_k}, x_{i_{n+1-k}})$.

The proof can be further divided into two cases.

Case 1. $y = 0$, which indicates $n_l \leq n_r$. We define the number of agents in $(\frac{l}{2}, l]$ as m_r, and the number of other agents as m_l. As $\forall i \in [1,n]$, $x_i^* = l - x_i$, we have $m_r = n_l$ and $m_l = n_r$, which implies $m_l \geq m_r$. Hence, for any integer k such that $k \leq \lfloor \frac{n}{2} \rfloor$, $x_{i_k} \leq \frac{l}{2}$.

If n is even, we define $ur(k) = \frac{u(c_{i_k},y)+u(c_{i_{n+1-k}},y)}{u(c_{i_k},y_o)+u(c_{i_{n+1-k}},y_o)} = \frac{2l-2x_{i_k}-d(x_{i_k},x_{i_{n+1-k}})}{2l-d(x_{i_k},x_{i_{n+1-k}})}$. Obviously, $ur(k)$ decreases as $d(x_{i_k}, x_{i_{n+1-k}})$ increases, and because $d(x_{i_k}, x_{i_{n+1-k}}) + x_{i_k} \leq l$, $ur(k) \geq \frac{2l-2x_{i_k}-(l-x_{i_k})}{2l-(l-x_{i_k})} = \frac{l-x_{i_k}}{l+x_{i_k}}$. As $x_{i_k} \leq \frac{l}{2}$, we can get $ur(k) \geq \frac{l-x_{i_k}}{l+x_{i_k}} \geq \frac{l-\frac{l}{2}}{l+\frac{l}{2}} = \frac{1}{3}$. Hence, we have $u(c_{i_k}, y) + u(c_{i_{n+1-k}}, y) \geq \frac{1}{3} * (u(c_{i_k}, y_o) + u(c_{i_{n+1-k}}, y_o))$. As $su(y,\mathbf{c}) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k}, y) + u(c_{i_{n+1-k}}, y))$ and $su(y_o,\mathbf{c}) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k}, y_o) + u(c_{i_{n+1-k}}, y_o))$, we have $su(y,\mathbf{c}) \geq \frac{1}{3} * su(y_o,\mathbf{c}) = \frac{1}{3} * OPT(\mathbf{c})$, which implies $\frac{SU(f,\mathbf{c})}{OPT(\mathbf{c})} = \frac{su(y,\mathbf{c})}{OPT(\mathbf{c})} \geq \frac{1}{3}$.

If n is odd, as $m_l \geq m_r$, the location of the middle point $x_{i_{(n+1)/2}} \leq \frac{l}{2}$. Also, we have $y_o = x_{i_{(n+1)/2}}$, which gives $u(c_{i_{(n+1)/2}}, y_o) = l - d(y_o, x_{i_{(n+1)/2}}) = l$ and $\frac{u(c_{i_{(n+1)/2}},y)}{u(c_{i_{(n+1)/2}},y_o)} = \frac{l-x_{i_{(n+1)/2}}}{l} \geq \frac{l/2}{l} = \frac{1}{2} > \frac{1}{3}$. Similar to n is even case, we can get $\frac{\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k},y)+u(c_{i_{n+1-k}},y))}{\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k},y_o)+u(c_{i_{n+1-k}},y_o))} \geq \frac{1}{3}$. As $\frac{u(c_{i_{(n+1)/2}},y)}{u(c_{i_{(n+1)/2}},y_o)} > \frac{1}{3}$, $\frac{SU(f,\mathbf{c})}{OPT(\mathbf{c})} = \frac{\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k},y)+u(c_{i_{n+1-k}},y))+u(c_{i_{(n+1)/2}},y)}{\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(u(c_{i_k},y_o)+u(c_{i_{n+1-k}},y_o))+u(c_{i_{(n+1)/2}},y_o)} > \frac{1}{3}$.

Case 2. $y = l$. The proof is similar. □

THEOREM 6. *The approximation ratio for the social utility of Mechanism 2 is* $\frac{1}{3}$.

PROOF. Consider two profiles $\mathbf{c}_0$ and $\mathbf{c}_1$. To distinguish these two profiles, we use $c_{(0)i} = (x_{(0)i}, p_{(0)i})$ and $c_{(1)i} = (x_{(1)i}, p_{(1)i})$ to indicate the bids for agent i in $\mathbf{c}_0$ and $\mathbf{c}_1$. $\mathbf{c}_0$ and $\mathbf{c}_1$ satisfy the following conditions. Both of these two profiles have n agents. For any integer $i \in [1,n]$, $p_{(1)i} = 1$. If $p_{(0)i} = 1$, $x_{(1)i} = x_{(0)i}$; if $p_{(0)i} = 0$, $x_{(1)i} = l - x_{(0)i}$. Notice that, under the above conditions, $\forall i \in [1,n]$, $x_{(0)i}^* = x_{(1)i}^*$, so the output for the two profiles will be the same. Denote the common output as y.

Given an integer $i \in [1,n]$, if $p_{(0)i} = 1$, then $x_{(0)i} = x_{(1)i}$, $p_{(0)i} = p_{(1)i}$, so $u(c_{(0)i}, y) = u(c_{(1)i}, y)$; if $p_{(0)i} = 0$, then $x_{(1)i} = l - x_{(0)i}$, $p_{(1)i} = 1$, as y can only be 0 or l, we have $d(y, x_{(1)i}) = l - d(y, l - x_{(1)i}) = l - d(y, x_{(0)i})$. Hence $u(c_{(1)i}, y) = l - d(y, x_{(1)i}) = l - (l - d(y, x_{(0)i})) = u(c_{(0)i}, y)$. As $\forall i \in [1,n]$, $u(c_{(1)i}, y) = u(c_{(0)i}, y)$, we have $su(y, \mathbf{c}_0) = su(y, \mathbf{c}_1)$.

Given an arbitrary point y_0 on the line segment, for any integer $i \in [0,n]$, if $p_{(0)i} = 1$, it is easy to see that $u(c_{(0)i}, y_0) = u(c_{(1)i}, y_0)$; if $p_{(0)i} = 0$, $u(c_{(0)i}, y_0) = d(x_{(0)i}, y_0)$ and $u(c_{(1)i}, y_0) = l - d(y_0, x_{(1)i})$. Notice that $x_{(1)i}$ and $x_{(0)i}$ are symmetric about the middle point of the line segment, by Lemma 4, $l - d(y_0, x_{(1)i}) \geq d(y_0, x_{(0)i})$, which implies $u(c_{(1)i}, y_0) \geq u(c_{(0)i}, y_0)$. Therefore for the social utility, we can also get $su(y_0, \mathbf{c}_1) \geq su(y_0, \mathbf{c}_0)$, which implies $OPT(\mathbf{c}_1) \geq OPT(\mathbf{c}_0)$. Because in $\mathbf{c}_1$, $\forall i \in [1,n]$, $p_{(1)i} = 1$, by Lemma 5, $\frac{su(y,\mathbf{c}_1)}{OPT(\mathbf{c}_1)} \geq \frac{1}{3}$. Hence, in $\mathbf{c}_0$, $\frac{SU(f,\mathbf{c}_0)}{OPT(\mathbf{c}_0)} = \frac{su(y,\mathbf{c}_0)}{OPT(\mathbf{c}_0)} \geq \frac{su(y,\mathbf{c}_0)}{OPT(\mathbf{c}_1)} = \frac{su(y,\mathbf{c}_1)}{OPT(\mathbf{c}_1)} \geq \frac{1}{3}$.

For any $\mathbf{c}_0$, we can find $\mathbf{c}_1$ satisfying the requirements defined at the beginning, which then completes the proof. □

The tight case for this approximation ratio occurs when $n = 2k$, $k \in N^+$, where k agents with preference value 1 are located at $\frac{l}{2}$ and k agents with preference value 0 are located at l. For this profile, Mechanism 2 will output the rightmost point with social utility $\frac{1}{2} * kl$, but the optimal social utility could be $\frac{3}{2} * kl$ when the facility is built at the leftmost point.

Specially, the problem in this section has some relationships with the obnoxious facility location game studied in [13] and [4]. For the profiles studied in the obnoxious facility location game, all agents tend to stay away from the facility, which is actually one special kind of profiles in the dual character facility location game. As implied by the main results in [13], any deterministic strategy-proof mechanism cannot achieve an approximation ratio better than $\frac{1}{3}$ for the obnoxious facility location game. Hence $\frac{1}{3}$ is also the best any deterministic mechanism can achieve for our problem. In addition, for the obnoxious facility location game, Cheng et al. [4] gives a strategy-proof mechanism with approximation ratio $\frac{1}{3}$. The length of the line segment is set to 2 in the mechanism, and the mechanism is as follows:

MECHANISM 3. *Let n_1 be the number of agents in $[0,1)$, and n_2 be the number of agents in $[1,2]$. The mechanism outputs 0 if $n_2 \geq n_1$, and 2 otherwise.*

We can see that Mechanism 3 can be regarded as a special version of Mechanism 2 when $\mathbf{c}$ satisfies that $\forall i \in [1,n]$, $p_i = 0$. Because under this condition, $\forall i \in [1,n]$, $x_i = x_i^*$, these two mechanisms will have the same output. The conclusion about the approximation ratio for Mechanism 3 proposed by Cheng et al. [4] can be rewritten as the following lemma:

LEMMA 7. *Under Mechanism 2, if a profile $\mathbf{c}$ satisfies that $\forall i \in N$, $p_i = 0$, then $\frac{SU(f,\mathbf{c})}{OPT(\mathbf{c})} \geq \frac{1}{3}$.*

The reason why we use Lemma 5 instead of Lemma 7 in the proof of Theorem 6 is as follows. In the proof of Theorem 6, if we use Lemma 7, we can set preferences of all agents in $\mathbf{c}_1$ to be 0 and change the relationship between $\mathbf{c}_0$ and $\mathbf{c}_1$ to be if $p_{(0)i} = 0$, then $x_{(1)i} = x_{(0)i}$; otherwise, $x_{(1)i} = l - x_{(0)i}$. Under this condition, similarly, $\mathbf{c}_0$ and $\mathbf{c}_1$ will have the same output from Mechanism 2. Assume the common output is 0, then we can get similar conclusion that $su(0,\mathbf{c}_0) = su(0,\mathbf{c}_1)$. Hence by Lemma 7, $\frac{su(0,\mathbf{c}_0)}{OPT(\mathbf{c}_1)} = \frac{su(0,\mathbf{c}_1)}{OPT(\mathbf{c}_1)} \geq \frac{1}{3}$. However, in this case, we have $OPT(\mathbf{c}_1) \leq OPT(\mathbf{c}_0)$ and $\frac{SU(f,\mathbf{c}_0)}{OPT(\mathbf{c}_0)} = \frac{su(0,\mathbf{c}_0)}{OPT(\mathbf{c}_0)} \leq \frac{su(0,\mathbf{c}_0)}{OPT(\mathbf{c}_1)}$, which is not sufficient to obtain a explicit relationship between $\frac{SU(f,\mathbf{c}_0)}{OPT(\mathbf{c}_0)}$ and $\frac{1}{3}$.

4. TWO-OPPOSITE-FACILITY LOCATION GAME WITH LIMITED DISTANCE

In the two-opposite-facility location game with limited distance, all agents are located on a line segment with length l. Two facilities need to be built on the line segment based on the location information reported by each agent. Let $N = (1,...,n)$ be a set of agents. We use x_i to indicate the location of each agent i and the set $\mathbf{x} = (x_1,...,x_n)$ to represent the location profile of all agents. The two facilities are of opposite characteristics for agents, which means all agents want to stay as close as possible to one facility (denoted as f_1) and stay as far away as possible from the other one (denoted as f_0). Another important constraint for the construction of two facilities is that the distance between them cannot exceed a certain value C with $0 < C < l$. In a building scheme $S = (y_0, y_1)$, we use y_1 and y_0 to indicate the locations of f_1 and f_0 respectively. We define the **length** of S as the distance between two facilities (i.e. $|y_0 - y_1|$) and it can be denoted as $|S|$. For a certain location profile with building scheme S used, the utility of agent i can be defined as the difference between its distances towards f_0 and f_1, i.e., $u(x_i,S) = u(x_i,y_0,y_1) = d(x_i,y_0) - d(x_i,y_1)$. In this game, each agent tends to maximize its utility value by misreporting its location information.

The social utility of this game is defined as the sum of the utilities of all agents, i.e. $su(S,\mathbf{x}) = su(y_0,y_1,\mathbf{x}) = \sum_{i=1}^{n} u(x_i,y_0,y_1) = \sum_{i=1}^{n}(d(x_i,y_0) - d(x_i,y_1))$. In this game, we try to find a strategy-proof mechanism with the objective of maximizing the social utility. Given a location profile $\mathbf{x}$, denote the optimal social utility for $\mathbf{x}$ as $OPT(\mathbf{x})$. For a mechanism f, if there exists a number β such that for any location profile $\mathbf{x}$ with $OPT(\mathbf{x}) \neq 0$, the building scheme for $\mathbf{x}$ from f satisfies $\frac{SU(f,\mathbf{x})}{OPT(\mathbf{x})} \geq \beta$, then we say the approximation ratio for the social utility of f is β. The following is an important fact about applicability of the approximation ratio for the social utility.

FACT 8. *Given a location profile $\boldsymbol{x}$ with n agents, if $n = 2k$, $k \in N^+$, k agents are located at 0 and other k agents are located at l, then $OPT(\boldsymbol{x}) = 0$ and the approximation ratio is not applicable; otherwise, $OPT(\boldsymbol{x}) > 0$.*

PROOF. When $n = 2k$, k agents are located at 0 and other k agents are located at l. Suppose the building scheme for $\mathbf{x}$ is $S = (y_0, y_1)$. If $y_0 \leq y_1$, for any agent i with $x_i = 0$, $u(x_i,S) = -|S|$; for any agent i with $x_i = l$, $u(x_i,S) = |S|$. Therefore $su(S,\mathbf{x}) = \sum_{i=1}^{n} u(x_i,S) = k*(-|S|)+k*|S| = 0$. If $y_0 > y_1$, similarly, we also have $su(S,\mathbf{x}) = 0$. Hence, $OPT(\mathbf{x}) = 0$.

When the above condition is not satisfied, if we can find a building scheme S such that $su(S,\mathbf{x}) > 0$, as S is one possible building scheme, we have $OPT(\mathbf{x}) \geq su(S,\mathbf{x}) > 0$. We should consider the following two cases.

Case 1. $n = 2k-1$, $k \in N^+$. Denote the location of the middle agent in $\mathbf{x}$ as x_m.

If $x_m < l$, consider the building scheme $S = (a, x_m)$ with $|S| > 0$ where $a = min(x_m + C, l)$. Define G to be the set of all agents i with $x_i \leq x_m$. Assume the number of agents in G is n_G. Obviously, $n_G \geq k$ and $n_G > n - n_G$. $\forall i \in G$, $u(x_i,S) = |S|$, and $\forall i \notin G$, $u(x_i,S) \geq -|S|$. Therefore, $su(S,\mathbf{x}) = \sum_{i\in G} u(x_i,S) + \sum_{i\notin G} u(x_i,S) \geq n_G * |S| - (n - n_G) * |S| > 0$.

If $x_m = l$, consider the building scheme $S = (l - C, l)$. Define G to be the set of all agents i with $x_i = l$. Assume the number of agents in G is n_G. We have $n_G > n - n_G$, and similarly, $su(S,\mathbf{x}) \geq n_G * |S| - (n - n_G) * |S| > 0$.

Case 2. $n = 2k$, $k \in N^+$. Denote the locations of the left and right middle agents in $\mathbf{x}$ as x_{m_1} and x_{m_2}.

If there exists an agent in the first k agents not located at 0, we have $x_{m_1} \neq 0$. Consider the building scheme $S = (a, x_{m_1})$ where $a = max(x_{m_1} - C, 0)$. Define G to be the set of all agents i with $x_i \geq x_{m_1}$. Assume the number of agents in G is n_G. We have $n_G \geq k+1 > n - n_G$, and similarly, $su(S,\mathbf{x}) \geq n_G * |S| - (n - n_G) * |S| > 0$.

If there exists an agent in the last k agents not located at l, we have $x_{m_2} \neq l$. Consider the building scheme $S = (a, x_{m_2})$ where $a = min(x_{m_2} + C, l)$. Define G to be the set of all agents i with $x_i \leq x_{m_2}$. Assume the number of agents in G is n_G. We have $n_G \geq k+1 > n - n_G$, and similarly, $su(S,\mathbf{x}) \geq n_G * |S| - (n - n_G) * |S| > 0$. □

We will continue discussion in these two cases.

4.1 n is Even

In this subsection, we consider the case when the total number of agents n is even and we define $n = 2k$. We give a deterministic strategy-proof mechanism with approximation ratio $\frac{1}{k}$, which will also be proved to be the best approximation ratio a deterministic strategy-proof mechanism can achieve for any C and l. For a location profile $\mathbf{x}$, arranging agents from left to right, denote the location of the left and right middle agents as x_{m_1} and x_{m_2}, then the mechanism can be described as follows.

MECHANISM 4. *Define $k_l = min(x_{m_1}, C)$ and $k_r = min(l - x_{m_2}, C)$. If $k_l \geq k_r$, the output will be $(0, k_l)$; otherwise, the output will be $(l, l - k_r)$.*

Before proving that Mechanism 4 is strategy-proof, we need to give some definitions and a lemma. For a location profile $\mathbf{x}$, arranging agents from left to right, we define the first k agents as **left agents** and the other agents as **right agents**. We use **left set** to indicate the set of all left agents and **right set** to indicate that of all right agents. Denote the left set and the right set of $\mathbf{x}$ as N_L and N_R, obviously, $\forall i \in N_L$, $x_i \leq x_{m_1}$ and $\forall i \in N_R$, $x_i \geq x_{m_2}$. For an agent i, if it satisfies $i \in N_L$ and $d(0, x_i) = x_i > C$, or $i \in N_R$ and $d(l, x_i) = l - x_i > C$, we define it as a **free agent**.

LEMMA 9. *Consider a location profile $\boldsymbol{x}$ with building scheme S from Mechanism 4 and a group $G \subseteq N$. Suppose agents in G misreport locations to $\boldsymbol{x}'_G$, and the location of agent i after misreporting is x'_i. Denote the location profile after misreporting as $\boldsymbol{x}_t =< \boldsymbol{x}_{-G}, \boldsymbol{x}'_G >$ with building scheme S_t from Mechanism 4. If $S = (0, k_l)$ and $\forall i \in G$, $x_i < k_l$, or $S = (l, l - k_r)$ and $\forall i \in G$, $x_i > l - k_r$, then for any agent $e \in G$, $u(x_e, S) \geq u(x_e, S_t)$.*

PROOF. If $S = (0, k_l)$ and $\forall i \in G$, $x_i < k_l$, select an arbitrary agent $i_0 \in G$ (i_0 can be e or not) and denote $\mathbf{x}' =< \mathbf{x}_{-i_0}, x'_{i_0} >$ with building scheme S'. Denote the locations of the left and right middle agents in $\mathbf{x}'$ as x'_{m_1} and x'_{m_2}, $k'_l = min(x'_{m_1}, C)$ and $k'_r = min(l - x'_{m_2}, C)$.

If $x'_{i_0} < x_{i_0}$, then $x_{m_1} = x'_{m_1}$, $x_{m_2} = x'_{m_2}$, $k_l = k'_l$, $k_r = k'_r$, and output will not change after misreporting, which implies $u(x_e, S') = u(x_e, S)$. If $x'_{i_0} > x_{i_0}$, then $x'_{m_1} \geq x_{m_1}$, $x'_{m_2} \geq x_{m_2}$, so $k'_l \geq k_l \geq k_r \geq k'_r$ and $S' = (0, k'_l)$. As $u(x_e, S) = x_e + x_e - k_l$, $u(x_e, S') = x_e + x_e - k'_l$, we have $u(x_e, S) \geq u(x_e, S')$.

Since $k'_l \geq k_l$, for any agent $i \in G$, $x_i < k'_l$. Select another agent i_1 which has not been moved yet and repeat the previous procedure. Then we can get the building scheme S'' for $< (\mathbf{x}')_{-i_1}, x'_{i_1} >$ satisfying $u(x_e, S'') \leq u(x_e, S')$, implying $u(x_e, S'') \leq u(x_e, S)$. Continue moving the agents staying at the original location in G until all agents have been moved to misreported locations. Then we get $u(x_e, S) \geq u(x_e, S_t)$.

If $S = (l, l - k_r)$ and $\forall i \in G$, $x_i > l - k_r$, the proof is similar. $\square$

THEOREM 10. *Mechanism 4 is group strategy-proof.*

PROOF. Consider a building scheme S from Mechanism 4 for an arbitrary location profile $\mathbf{x}$. Assume the agents in a group $G \subseteq N$ misreport their locations $\mathbf{x}'_G$ and for agent $i \in G$, the location after misreporting is x'_i. The building scheme for the new profile $\mathbf{x}' =< \mathbf{x}_{-G}, \mathbf{x}'_G >$ is $S' = (y'_0, y'_1)$ and the locations of the left and right middle agents in $\mathbf{x}'$ are x'_{m_1} and x'_{m_2}. Also, $k'_l = min(x'_{m_1}, C)$ and $k'_r = min(l - x'_{m_2}, C)$. The proof requires analysis for the following two scenarios.

Scenario 1. There are no free agents in $\mathbf{x}$. In this scenario, $x_{m_1} \leq C$ and $k_l = min(x_{m_1}, C) = x_{m_1}$; similarly, $k_r = min(l - x_{m_2}, C) = l - x_{m_2}$. Hence the output of the mechanism can only be $(0, x_{m_1})$ or (l, x_{m_2}).

If $S = (0, x_{m_1})$, we have $x_{m_1} \geq l - x_{m_2}$. Under this condition, we need to discuss three cases.

Case 1: Every agent in G is a left agent. In this case, if there exists an agent $i \in G$, such that $x_i = k_l$, then $u(x_i, S') \leq u(x_i, S)$. Brief proof is as follows:

If $y'_0 = 0$, obviously, $u(x_i, S') = u(x_{m_1}, S') \leq x_{m_1} = u(x_i, S)$. If $y'_0 = l$, as the locations of all right agents remain the same, $x'_{m2} \geq x_{m2} \geq x_i$, so $u(x_i, S') = l - x'_{m2} \leq l - x_{m2} \leq x_{m_1} = u(x_i, S)$.

If $\forall i \in G$, $x_i < k_l$, by Lemma 9, for any agent $i \in G$, $u(x_i, S') \leq u(x_i, S)$.

Case 2: Every agent in G is a right agent. In this case, consider an arbitrary agent i in G. If $y'_0 = l$, then $u(x_i, S') \leq l - x_i \leq l - x_{m_2} \leq x_{m_1}$. As $u(x_i, S) = x_{m_1}$, agent i cannot get more utility. If $y'_0 = 0$, as the locations of all left agents remain the same, $x'_{m_1} \leq x_{m_1}$, which implies $u(x_i, S') = x'_{m_1} \leq x_{m_1} = u(x_i, S)$.

Case 3: Left and right agents coexist in G. Consider a right agent i and a left agent j in G. Similar to the analysis in Case 2, if $u(x_i, S') > x_{m_1} = u(x_i, S)$, then S' must satisfy that $y'_0 = 0$ and $y'_1 > k_l$. However, if $y'_0 = 0$ and $y'_1 > k_l$, then for left agent j, $u(x_j, S') < u(x_j, S)$. Therefore agent i and agent j cannot get more utility at the same time.

If $S = (l, x_{m_2})$, the analysis is similar.

Scenario 2. There exist free agents in $\mathbf{x}$. In this scenario, at least one of the values of k_r and k_l is C.

If $S = (0, k_l)$, then k_l must be C. We should discuss the following two cases.

Case 1: $\forall i \in G$, $x_i < k_l$, by Lemma 9, for any agent i, $u(x_i, S') \leq u(x_i, S)$.

Case 2: There exists agent $i \in G$ such that $x_i \geq k_l$. For agent i, $u(x_i, S) = C$ which is the largest utility it can achieve and agent i cannot get more utility by misreporting.

If $S = (l, l - k_r)$, we should consider two cases, the case where $\forall i \in G$, $x_i > l - k_r$ and the case where there exists agent $i \in G$ such that $x_i \leq l - k_r$. The proof is similar. $\square$

Define a function $g(y, \mathbf{x}) = \sum_{i=1}^{n} d(x_i, y)$ for a location profile $\mathbf{x}$ and a point $y \in [0, l]$. Assume the building scheme for $\mathbf{x}$ is $S = (y_0, y_1)$, then $su(y_0, y_1, \mathbf{x}) = \sum_{i=1}^{n}(d(x_i, y_0) - d(x_i, y_1)) = g(y_0, \mathbf{x}) - g(y_1, \mathbf{x})$. Considering the sample graph of $g(y, \mathbf{x})$ below, the optimal social utility must occur when the building scheme is $(0, k_l)$ or $(l, l - k_r)$.

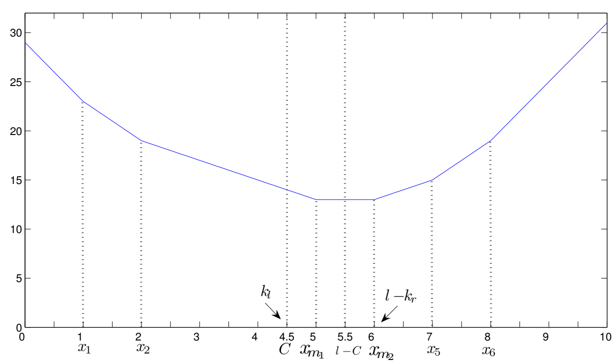

Figure 1: g(y,x)=|y-1|+|y-2|+|y-5|+|y-6|+|y-7|+|y-8| with C=4.5

THEOREM 11. *Mechanism 4 has approximation ratio $\frac{1}{k}$.*

PROOF. Consider an output building scheme S from Mechanism 4 for a location profile $\mathbf{x}$ with $OPT(\mathbf{x}) \neq 0$.

If $S = (0, k_l)$, which indicates $k_l \geq k_r$, but the optimal social utility occurs when $(l, l - k_r)$ is used. In this situation, $SU(f, \mathbf{x}) = su(S, \mathbf{x}) \geq 2 * k_l$, and $2 * k_l$ occurs when $k - 1$ left agents are located at 0 and only one is located at k_l. Also, $OPT(\mathbf{x}) = su(l, l - k_r, \mathbf{x}) \leq 2 * k * k_r$ and $2 * k * k_r$ occurs when all right agents are located at k_r. As $OPT(\mathbf{x}) \neq 0$, it is easy to have $k_r > 0$ and $k_l \geq k_r > 0$. So $\frac{SU(f, \mathbf{x})}{OPT(\mathbf{x})} \geq \frac{2 * k_l}{2 * k * k_r} = \frac{k_l}{k * k_r} \geq \frac{1}{k}$.

If $S = (l, l-k_r)$ but the optimal social utility occurs when $(0, k_l)$ is used, the proof is similar. □

With respect to a given profile $\mathbf{x}$, we define $S = (y_0, y_1)$ as **left pattern** if $y_0 \in [0, x_{m_1})$ and $y_1 > y_0$, or we define it as **right pattern** if $y_0 \in (x_{m_2}, l]$ and $y_1 < y_0$. From $su(y_0, y_1, \mathbf{x}) = g(y_0, \mathbf{x}) - g(y_1, \mathbf{x})$ and the graph of $g(y, \mathbf{x})$, we can easily get that when $y_0 \in [0, x_{m_1})$ and $y_1 \leq y_0$, or $y_0 \in (x_{m_2}, l]$ and $y_1 > y_0$, or $y_0 \in [x_{m_1}, x_{m_2}]$, $su(y_0, y_1, \mathbf{x}) \leq 0$. Because the optimal social utility for any location profile cannot be negative by Fact 8, we have the following lemma.

LEMMA 12. *Given a location profile $\boldsymbol{x}$ with $OPT(\boldsymbol{x}) \neq 0$, if a building scheme S for $\boldsymbol{x}$ satisfies $\frac{su(S,\boldsymbol{x})}{OPT(\boldsymbol{x})} > 0$, then S must be left pattern or right pattern.*

LEMMA 13. *Assume a deterministic strategy-proof mechanism with positive approximation ratio is adopted. In a location profile $\boldsymbol{x}_0$ with $OPT(\boldsymbol{x}_0) \neq 0$ and building scheme S_0, a right agent i with $d(x_{m_1}, x_i) > C$ exists. If the building scheme S_1 for location profile $\boldsymbol{x}_1 =< (\boldsymbol{x}_0)_{-i}, l >$ is left pattern and $d(x_i, l) < |S_1|$, then S_0 should be left pattern and $|S_0| = |S_1|$.*

PROOF. As a positive approximation ratio is guaranteed, by Lemma 12, S_0 must be left pattern or right pattern. Denote the left pattern scheme $S_1 = (y_0, y_1)$.

Consider agent i in $\mathbf{x}_0$. Because agent i is a right agent and $d(x_{m_1}, x_i) = x_i - x_{m_1} > C$, we have $y_0 < x_{m_1} < x_i$ and $y_1 = y_0 + |S_1| < x_{m_1} + C < x_i$. Then $x_i > y_1 > y_0$, which implies $u(x_i, S_1) = |S_1|$. Assume S_0 for $\mathbf{x}_0$ is right pattern. As $d(l, x_i) < |S_1|$, $u(x_i, S_0) \leq d(l, x_i) < |S_1| = u(x_i, S_1)$, implying that agent i will misreport its location to l to gain larger utility, which contradicts the strategy-proofness. Therefore the assumption is false and S_0 must be left pattern. As agent i is a right agent and $d(x_{m_1}, x_i) > C$, $u(x_i, S_0) = |S_0|$. For strategy-proofness, $|S_0| = u(x_i, S_0) \geq u(x_i, S_1) = |S_1|$.

Consider agent i in $\mathbf{x}_1$ with location l. Denoting $x'_i = l$, we have $u(x'_i, S_1) = |S_1|$ and $u(x'_i, S_0) = |S_0|$. If agent i misreports its location to x_i, then S_0 will be used, for strategy-proofness, $|S_1| = u(x'_i, S_1) \geq u(x'_i, S_0) = |S_0|$. Combining with $|S_0| \geq |S_1|$, we have $|S_0| = |S_1|$. □

To represent the location profile effectively, we use another notation in the form of $(d_1 * n_1, d_2 * n_2, d_3 * n_3, ..., d_m * n_m | d_{m+1} * n_{m+1}, d_{m+2} * n_{m+2}, ..., d_w * n_w)$. The '$|$' symbol separates the location profile for left agents (including distance to left endpoint d_i and occurrence number n_i) and that of right agents (including distance to right endpoint d_i and occurrence number n_i). Specially, in the expression, d_i appears in an ascending order in the left part and in a descending order in the right part. Define function $c_k(x, a)$ for three numbers k, x and a as $c_k(x, a) = \frac{x}{k*a}$. Specially, if k is clear in the context, we simplify the expression of the function to be $c(x, a)$.

LEMMA 14. *Assume a deterministic mechanism with positive approximation ratio β is adopted. For any positive numbers x and m, if $m > c(x, \beta)$, $x < l - m$ and $x, m < C$, then the building scheme S for the location profile $\boldsymbol{x} = (0 * (k-1), x * 1 | m * k)$ must be right pattern.*

PROOF. Obviously, $0 < \beta \leq 1$ and $m > 0$. By Fact 8, $OPT(\mathbf{x}) \neq 0$. As $\beta > 0$, by Lemma 12, S can only be left pattern or right pattern.

As $m > c(x, \beta)$, $m * 2k * \beta > c(x, \beta) * 2k * \beta = 2x$. For $\mathbf{x}$, because $x, m < C$, we have $k_l = x$, $k_r = m$ and the optimal social utility must occur when building scheme is $S_0 = (0, x)$ or $S_1 = (l, l-m)$. Because $su(S_0, \mathbf{x}) = 2x$, $su(S_1, \mathbf{x}) = m * 2k$, we have $su(S_1, \mathbf{x}) \geq su(S_1, \mathbf{x}) * \beta > su(S_0, \mathbf{x})$. Hence, the optimal social utility $OPT(\mathbf{x})$ should be $m * 2k$. Assume S is left pattern, $SU(f, \mathbf{x}) = su(S, \mathbf{x}) \leq su(S_0, \mathbf{x}) = 2x < m*2k*\beta = OPT(\mathbf{x})*\beta$. As $OPT(\mathbf{x}) \neq 0$, $\frac{SU(f,\mathbf{x})}{OPT(\mathbf{x})} < \beta$, which contradicts the approximation ratio of β. Therefore S must be right pattern building scheme. □

Further, we define a function $t(x, a) = \frac{c(x,a)+x}{2}$. When $a > \frac{1}{k}$ and $x > 0$, $c(x, a) = \frac{x}{k*a} < x$, we have $t(x, a) > c(x, a)$ and $t(x, a) < x$.

Now we define a special number P with respect to l and C as $P = min(\frac{l-C}{2}, C)$. Notice that $P < \frac{l}{2}$, $P \leq C$ and $d(P, l-P) \geq C$. If $a > \frac{1}{k}$, then $t(P, a) < P$ and $l - t(P, a) > l - P > \frac{l}{2} > P$. This inequality is the basis for the location profile $\mathbf{x}_0$ defined in the following lemma. Also, because $P \neq 0$, we can guarantee the applicability of the approximation ratio for the social utility in the following lemma.

LEMMA 15. *Assume a deterministic strategy-proof mechanism with approximation ratio $\beta > \frac{1}{k}$ is adopted. Given a location profile $\boldsymbol{x}_0 = (0*(k-1), P | t(P, \beta)*m, 0*(k-m))(1 \leq m \leq k)$, if the building scheme S_0 for $\boldsymbol{x}_0$ is a right pattern building scheme, then the building scheme S_1 for location profile $\boldsymbol{x}_1 = (0*(k-1), P | t(P, \beta)*(m-1), 0*(k-m+1))$ is right pattern.*

PROOF. Consider a right agent i in $\mathbf{x}_0$ with $x_i = l - t(P, \beta)$. As S_0 is a right pattern building scheme, we have $u(x_i, S_0) \leq d(x_i, l) = t(P, \beta)$.

Assume $S_1 = (y_0, y_1)$ is left pattern. If agent i misreports its location to l, as $< (\mathbf{x}_0)_{-i}, l >= \mathbf{x}_1$, S_1 will be used. Because $t(P, \beta) < P$, $d(x_{m_1}, x_i) = d(P, l-t(P, \beta)) > d(P, l-P) \geq C$, which means $x_{m_1} + C < x_i$. As S_1 is left pattern, we have $y_0 < x_{m_1} = P < l - t(P, \beta) = x_i$ and $y_1 = y_0 + |S_1| < x_{m_1} + C < x_i$. Hence $u(x_i, S_1) = |S_1|$. For strategy-proofness, $|S_1| = u(x_i, S_1) \leq u(x_i, S_0) \leq t(P, \beta)$. Also, $t(|S_1|, \beta) < |S_1| \leq t(P, \beta)$.

Then consider another location profile $\mathbf{x}_2 = (0*(k-1), P | t(P, \beta)*(m-1), t(|S_1|, \beta)*1, 0*(k-m))$. For right agent j in $\mathbf{x}_2$ with $x_j = l - t(|S_1|, \beta)$, $d(x_{m_1}, x_j) = d(P, l - t(|S_1|, \beta)) > d(P, l - t(P, \beta)) > C$. Because $< (\mathbf{x}_2)_{-j}, l >= \mathbf{x}_1$, the building scheme for $< (\mathbf{x}_2)_{-j}, l >$ is left pattern S_1. As $d(l, x_j) = t(|S_1|, \beta) < |S_1|$, then by Lemma 13, the building pattern S_2 for $\mathbf{x}_2$ should be left pattern, and $|S_2| = |S_1|$.

Repeating the procedure in the last paragraph, we can find that the building patterns for $(0*(k-1), P | t(P, \beta)*(m-1), t(|S_1|, \beta)*\mathbf{2}, 0*(k-m-\mathbf{1}))$, $(0*(k-1), P | t(P, \beta)*(m-1), t(|S_1|, \beta)*\mathbf{3}, 0*(k-m-\mathbf{2}))$ until $\mathbf{x}_t = (0*(k-1), P | t(P, \beta)*(m-1), t(|S_1|, \beta)*(k-m+\mathbf{1}))$ are all left patterns and the length of all these building schemes including S_t for $\mathbf{x}_t$ are the same as $|S_1|$. Because $|S_t| = |S_1|$, $su(S_t, \mathbf{x}_t) \leq 2*|S_1|$ and as $|S_1| \leq t(P, \beta) < P$, the maximum for the social utility can occur when $S_t = (P - |S_1|, P)$. However, consider building scheme $S'_t = (l, l - t(|S_1|, \beta))$, $|S'_t| = t(|S_1|, \beta) < |S_1| < C$, and $su(S'_t, \mathbf{x}_t) = 2k * t(|S_1|, \beta) > 2k * c(|S_1|, \beta) = 2k * \frac{|S_1|}{k*\beta} = 2 * \frac{|S_1|}{\beta}$. As S'_t is one possible building scheme, we have $OPT(\mathbf{x}_t) * \beta \geq su(S'_t, \mathbf{x}_t) * \beta > \frac{2*|S_1|}{\beta} * \beta = 2 * |S_1| \geq su(S_t, \mathbf{x}_t)$. Hence $\frac{SU(f,\mathbf{x}_t)}{OPT(\mathbf{x}_t)} = \frac{su(S_t,\mathbf{x}_t)}{OPT(\mathbf{x}_t)} < \beta$ and it contradicts that β is the approximation ratio for the social utility. Therefore S_1 for location profile $\mathbf{x}_1$ cannot be left pattern. As $\beta > 0$ is guaranteed and $OPT(\mathbf{x}_1) \neq 0$, by Lemma 12, S_1 is right pattern. □

THEOREM 16. *When $n = 2 * k (k \in N^*)$, any deterministic strategy-proof mechanism cannot have an approximation ratio for the social utility larger than $\frac{1}{k}$.*

PROOF. Assume there exists a deterministic strategy-proof mechanism with approximation ratio $\beta > \frac{1}{k}$ and it is adopted.

For location profile $\mathbf{x}_0 = (0 * (k-1), P|t(P,\beta) * k) = (0*(k-1), P|t(P,\beta)*(k-0), 0*0)$, we have $t(P,\beta) > c(P,\beta)$, $l - t(P,\beta) > P$ and $P, t(P,\beta) < C$. By Lemma 14, the building scheme S_0 for $\mathbf{x}_0$ should be right pattern. Then by Lemma 15, we can know that the building scheme for $(0*(k-1), P|t(P,\beta)*(k-1), 0*1)$, $(0*(k-1), P|t(P,\beta)*(k-2), 0*2)$ until S_t for $\mathbf{x}_t = (0*(k-1), P|t(P,\beta)*(k-k), 0*k) = (0*(k-1), P|0*k)$ should be right pattern. However, because the right middle point is located at l in x_t, by the definition of right pattern building scheme, S_t cannot be a right pattern, which causes a contradiction. Therefore, any deterministic strategy-proof mechanism cannot have an approximation ratio for the social utility larger than $\frac{1}{k}$. □

4.2 n is Odd

In this subsection, the total number of agents is odd and we denote it as $n = 2 * k - 1$. We will give a deterministic mechanism similar to Mechanism 4 and the approximation ratio for the social utility of the new mechanism is $\frac{1}{n}$. For a location profile $\mathbf{x}$, define the location of the middle agent as x_m. The mechanism can be described as follows.

MECHANISM 5. *Define $k_l = min(x_m, C)$, $k_r = min(l - x_m, C)$. If $k_l \geq k_r$, the output will be $(0, k_l)$, otherwise, the output will be $(l, l - k_r)$.*

In this subsection, a newly constructed function with a special form is introduced. Due to the space constraints, we will list core theorems but only the proof for the second one.

THEOREM 17. *Mechanism 5 is group strategy-proof.*

THEOREM 18. *The approximation ratio of Mechanism 5 is $\frac{1}{2k-1}$.*

PROOF. Consider an output building scheme S from Mechanism 5 for a location profile $\mathbf{x}$. Based on the similar analysis to that in the case when n is even, we can know that the optimal social utility must occur when building scheme is $(0, k_l)$ or $(l, l - k_r)$.

If $S = (0, k_l)$, which indicates $k_l \geq k_r$, but the optimal social utility occurs when $(l, l-k_r)$ is used. In this situation, $SU(f,\mathbf{x}) = su(S,\mathbf{x}) \geq k_l$, and k_l occurs when all first k-1 agents are located at 0 and the middle agent is at k_l. Also, $OPT(\mathbf{x}) = su(l, l-k_r, \mathbf{x}) \leq (2k-1) * k_r$ and $(2k-1) * k_r$ occurs when all last k-1 agents are located at $l - k_r$. As $OPT(\mathbf{x}) > 0$, $k_r > 0$ and $k_l \geq k_r > 0$, we have $\frac{SU(f,\mathbf{x})}{OPT(\mathbf{x})} \geq \frac{k_l}{(2k-1)*k_r} \geq \frac{1}{2k-1}$.

If $S = (l, l - k_r)$, but optimal social utility occurs when $(0, k_l)$ is used, the proof is similar. □

THEOREM 19. *When $n = 2 * k - 1 (k \in N^+)$, any deterministic strategy-proof mechanism cannot have an approximation ratio for social utility larger than $\frac{1}{2k-1}$.*

The difference of the approximation ratios in even and odd cases comes from the different worst location profiles. As shown in the proof of Theorem 11, for the even case, the worst approximation ratio is achieved when $k-1$ agents are at 0, one agent is at k_l, k agents are at $l - k_r$ and $k_l = k_r$. For the odd case, the only difference is that the worst profile, given in the proof of Theorem 18, contains one less agent at $l - k_r$. For both profiles, the optimal social utilities occur with the building scheme $(l, l-k_r)$, and both optimal social utilities are equal to k_l multiplied by the number of all agents ($2k$ and $2k-1$ separately). Also, the building schemes output by our mechanisms for two profiles are both $(0, k_l)$. With this building scheme, the utilities of $k-1$ agents at 0 are k_l, and that of remaining agents (we call them non-zero agents) are k_l. Because for the even case, there is one more non-zero agent than that for the odd case, the social utility of the even case $(2 * k_l)$ is k_l more than that of the odd case (k_l), which finally leads to different approximation ratios.

5. CONCLUSION AND FUTURE WORK

In this paper, we investigate the property of dual preference in facility location games and propose two extended games, the dual character facility location game and the two-opposite-facility location game with limited distance, which are modelled from general scenarios in real life.

For the dual character facility location game, we first consider the case that only preference is regarded as private information for each agent. Under this condition, we prove that the mechanism to build the facility at the optimal location for the social utility is strategy-proof. Then we explore a more general case that both preference and location of each agent are private information and find that the previous mechanism is not strategy-proof anymore. Then we give a deterministic group strategy-proof mechanism and prove its approximation ratio for the social utility is $\frac{1}{3}$. We further study the relationship between this game and the obnoxious facility location game studied in [4], and show that the obnoxious facility location game can be regarded as a special case of the dual character facility location game when only agents tending to keep away from the facility exist. Our mechanism is also a generalization of the mechanism proposed in [4].

For the two-opposite-facility location game with limited distance, we divide it into two cases based on the parity of the number of agents. When the number of agents is even, we give a deterministic group strategy-proof mechanism with approximation ratio $\frac{1}{k}$ where the number of agents on the line segment is $2k$. When the number of agents is odd, another deterministic group strategy-proof mechanism is given and its approximation ratio is proved to be $\frac{1}{2k-1}$, where $2k - 1$ is the number of agents on the line segment. We further prove that the approximation ratios for both mechanisms are the best any deterministic strategy-proof mechanism can achieve in their settings.

As a possible future work, it would be interesting to extend our model on a line segment to more complicated and general metric spaces such as circles and trees.

Acknowledgments

This work was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. CityU 117913].

REFERENCES

[1] N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz. Strategyproof approximation of the minimax on networks. *Mathematics of Operations Research*, 35(3):513–526, 2010.

[2] F.-R. Aris, M. Li, J. Zhang, and Q. Zhang. Facility location with double-peaked preferences. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI)*, 2015. To appear.

[3] Y. Cheng, Q. Han, W. Yu, and G. Zhang. Obnoxious facility game with a bounded service range. In *Theory and Applications of Models of Computation*, pages 272–281, 2013.

[4] Y. Cheng, W. Yu, and G. Zhang. Mechanisms for obnoxious facility game on a path. In *Combinatorial Optimization and Applications*, pages 262–271, 2011.

[5] Y. Cheng, W. Yu, and G. Zhang. Strategy-proof approximation mechanisms for an obnoxious facility game on networks. In *Theoretical Computer Science*, pages 154–163, 2013.

[6] E. Dokow, M. Feldman, R. Meir, and I. Nehama. Mechanism design on discrete lines and cycles. In *Proceedings of the 13th ACM Conference on Electronic Commerce (ACM-EC)*, pages 423–440, 2012.

[7] B. Escoffier, L. Gourvès, N. Kim Thang, F. Pascual, and O. Spanjaard. Strategy-proof mechanisms for facility location games with many facilities. *Algorithmic Decision Theory*, pages 67–81, 2011.

[8] M. Feldman and Y. Wilf. Strategyproof facility location and the least squares objective. In *Proceedings of the 14th ACM conference on Electronic Commerce (ACM-EC)*, 2013.

[9] D. Fotakis and C. Tzamos. Winner-imposing strategyproof mechanisms for multiple facility location games. In *Proceeding of the 5th International Workshop of Internet and Network Economics (WINE)*, pages 234–245, 2010.

[10] D. Fotakis and C. Tzamos. On the power of deterministic mechanisms for facility location games. *arXiv preprint arXiv:1207.0935*, 2012.

[11] D. Fotakis and C. Tzamos. Strategyproof facility location with concave costs. In *ACM SIGecom Exchanges*, pages 46–49, 2013.

[12] Q. Han and D. Du. *Moneyless Strategy-proof Mechanism on Single-sinked Policy Domain: Characterization and Applications*. Faculty of Business Administration, University of New Brunswick, 2011.

[13] K. Ibara and H. Nagamochi. Characterizing mechanisms in obnoxious facility game. In *Combinatorial Optimization and Applications*, pages 301–311, 2012.

[14] P. Lu, X. Sun, Y. Wang, and Z. A. Zhu. Asymptotically optimal strategy-proof mechanisms for two-facility games. In *Proceedings of the 11th ACM Conference on Electronic Commerce (ACM-EC)*, pages 315–324, 2010.

[15] P. Lu, Y. Wang, and Y. Zhou. Tighter bounds for facility games. In *Proceeding of the 5th International Workshop of Internet and Network Economics (WINE)*, pages 137–148, 2009.

[16] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[17] A. D. Procaccia and M. Tennenholtz. Approximate mechanism design without money. In *Proceedings of the 10th ACM Conference on Electronic Commerce (ACM-EC)*, pages 177–186, 2009.

[18] P. Serafino and C. Ventre. Truthful mechanisms without money for non-utilitarian heterogeneous facility location. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI)*, 2015. To appear.

[19] Q. Zhang and M. Li. Strategyproof mechanism design for facility location games with weighted agents on a line. *Journal of Combinatorial Optimiazation*, 28(4):756–773, 2014.