Structural Control in Weighted Voting Games*

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ABSTRACT

Inspired by the study of control scenarios in elections and complementing manipulation and bribery settings in cooperative games with transferable utility, we introduce the notion of structural control in weighted voting games. We model two types of influence, adding players to and deleting players from a game, with goals such as increasing a given player's Shapley–Shubik power index in relation to the original game. We study the complexity of the problems of whether such structural changes can achieve the desired effect.

Keywords

Game theory; computational social choice; computational complexity; structural control

1. INTRODUCTION

A major task in computational social choice is the complexity analysis of the question of whether a certain form of influence (such as manipulation, bribery, and control) is possible in an election under some voting rule (see, e.g., [11, 2]). Whenever successful manipulative actions are generally possible, a high computational complexity may provide some protection against them. Similar ideas have been adapted to other fields, such as judgment aggregation (again, see, e.g., [11, 2]). In algorithmic game theory, the question of influencing the outcome of a game has also been studied extensively. For example, the complexity of manipulation by merging, splitting, and annexation has been studied in weighted voting games (WVGs) [1, 6, 9], as well as manipulation of the quota [13]. In dynamic WVGs [5], the quota changes dynamically over time. Bribery has been studied for path-disruption games [10].

Inspired by the notion of control in elections, we consider control scenarios in WVGs. We define the problems of whether it is possible to change the structure of a game by either *adding* or *deleting* players so as to achieve certain goals. One could, for instance, think of a committee able to decide upon an issue with a certain quota of votes. In order to increase the significance of some participant, an organizer might invite further participants or choose a meeting schedule to make sure that others are excluded. Structural changes could also be viewed as a change of participation over time without malicious intentions. Goals include *increasing* and *decreasing* a distinguished player's power in relation to the original game. Moreover, if an *exact* number of players is to be added, it might be desirable to *maintain* an original player's power index. All defined control types are possible in WVGs, we therefore analyze the complexity of whether structural control can be successful in a given game. The complexity depends on the control type, the goal, and on whether the number of players that can be added or deleted is fixed or given in the problem instance.

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2. PRELIMINARIES

A simple game $\mathscr{G} = (N, v)$ can be compactly represented as a *weighted voting game* $\mathscr{G} = (w_1, \dots, w_n; q)$, where w_i is player i's *weight*, q is a *quota*, and a coalition $C \subseteq N$ *wins* if $\sum_{i \in C} w_i \geq q$ and otherwise it *loses*. To measure player i's significance in $\mathscr{G}$, power indices such as the *probabilistic Penrose–Banzhaf index (PBI)* [4] and the *Shapley–Shubik index (SSI)* [12] can be used:

$$\mathrm{PBI}(\mathscr{G}, i) = 2^{1-n} \sum_{C \subseteq N \smallsetminus \{i\}} (v(C \cup \{i\}) - v(C)) \text{ and}$$

$$\mathrm{SSI}(\mathscr{G}, i) = \frac{1}{n!} \sum_{C \subseteq N \smallsetminus \{i\}} \|C\|!(n-1-\|C\|)(v(C \cup \{i\}) - v(C)).$$

For more background on cooperative game theory, see, e.g., [3]. We use the standard notions of *hardness* and *completeness* for a complexity class (e.g., NP or coNP) with respect to *many-one polynomial-time reducibility*. #P is the class of all functions that give the number of solutions of a problem in NP. Probabilistic polynomial time, PP, is the class of all decision problems X for which there exist a function $f \in$ #P and a polynomial p such that for all instances x, $x \in X \iff f(x) \geq 2^{p(|x|)-1}$. It is considered to be a rather large complexity class, since it is known that $\mathrm{PH} \subseteq \mathrm{P}^{\mathrm{PP}}$.

3. CONTROL TYPES AND GOALS

We define control by adding and by deleting players in WVGs. For each control type, we consider goals, such as increasing or decreasing a distinguished player's power, in relation to the original game. We first define how adding and deleting a player affects the coalitional function for WVGs: For control by adding players, from a given WVG $\mathscr{G} = (w_1, \dots, w_n; q)$ with $N = \{1, \dots, n\}$ and a set $M = \{n+1, \dots, n+m\}$ of m unregistered players with weights $w_{n+1}, \dots, w_{n+m}$, we obtain a new game $\mathscr{G}_{\cup M} = (w_1, \dots, w_{n+m}; q)$. For example, for control by adding players to increase a power index PI we ask: Given a WVG $\mathscr{G}$, a set M of unregistered players with weights $w_{n+1}, \dots, w_{n+m}$, a distinguished player p, $1 \leq p \leq n$, and a positive integer k, can at most k players $M' \subseteq M$ be added to $\mathscr{G}$ such that for the new game $\mathscr{G}_{\cup M'}$ it holds that $\mathrm{PI}(\mathscr{G}_{\cup M'}, p) > \mathrm{PI}(\mathscr{G}, p)$? Analogously, we can ask whether the game can be controlled so as to decrease a certain player's index; or whether it is possible to add players to a game without changing the distribution of power among the original players. Deleting a subset

$M \subseteq N$ of m players from a WVG $\mathscr{G} = (w_1, \dots, w_n; q)$ yields a WVG $\mathscr{G}_{\smallsetminus M} = (w_{j_1}, \dots, w_{j_{n-m}}; q)$ with $\{j_1, \dots, j_{n-m}\} = N \smallsetminus M$. We ask whether at most k players $M' \subseteq N \smallsetminus \{p\}$ can be deleted from $\mathscr{G}$ to reach some goal for p. If a player i is deleted from a WVG, any other player j gains the same amount of power that i would gain if j were deleted [7]. After deleting a subset $M \subseteq N \smallsetminus \{i\}$ of size $m \geq 1$ from a WVG $\mathscr{G}$, player i's PBI changes by at most $1 - 2^{-m}$ and by at least $-1 + 2^{-m}$; her SSI changes by at most $1 - {(n-m+1)!}/{2n!}$ and by at least $-1 + {(n-m-1)!}/{2(n-2)!}$.

4. IN- AND DECREASING AN INDEX

Similarly to control by adding or deleting voters or candidates in elections, adding and deleting players are not merely inverse operations, as when adding players all original players are guaranteed to be part of the game before and after the structural change, whereas when deleting players each player except the distinguished one can be removed from the game. Hardness in terms of complexity can be seen as a shield to prevent a game from being controlled to improve or worsen a player's significance.

4.1 Control by Adding Players

We show PP-hardness via techniques inspired by related work [9, 6, 13]: For a #P-parsimonious-complete function F, COMPARE-$F = \{(x,y) \mid F(x) > F(y)\}$ is PP-complete. In this manner, COMPARE-#SUBSETSUM is PP-complete. We reduce from a PP-hard restricted variant of this problem, COMPARE-#SUBSETSUM-RR: Given a set $A = \{1, \dots, n\}$ and a function $a : A \to \mathbb{N} \smallsetminus \{0\}$, $i \mapsto a_i$, with $\alpha = \sum_{i=1}^{n} a_i$, is the number of subsets of A with values that sum up to $(\alpha/2) - 2$ greater than the number of those summing up to $(\alpha/2) - 1$? The results can be adapted to the SSI by means of a transformation from X3C allowing constant solution sizes.

THEOREM 1. *Control by adding a given number of players in order to increase (decrease) a player's PBI or SSI is* PP-*hard.*

An upper bound of $\mathrm{NP}^{\mathrm{PP}}$ can be established whenever the number of players to be added is given. If the number of players to be added is fixed, we even obtain a PP upper bound.

THEOREM 2. *Control by adding a fixed number of players in order to increase (decrease) a player's PBI or SSI is* PP-*complete.*

4.2 Control by Deleting Players

We have the following two initial results.

THEOREM 3.
1. *Control by deleting players to increase a player's SSI is* NP-*hard (even if only one player is deleted).*
2. *Control by deleting players to decrease a player's PBI is* coNP-*hard (even if only one player is deleted).*

NP-hardness holds by a reduction from SUBSETSUM with constant solution sizes. Our coNP-hardness result can be shown by a reduction from the complement of PARTITION.

5. MAINTAINING AN INDEX

In addition to constructive or destructive goals, we now consider situations with the goal of maintaining a player's index when an exact number of players is added. It may, for instance, happen that legislative bodies such as the EU Commission, national parliaments, or the United Nations Security Council, have to be expanded by adding a certain number of new members. Then, an old member may be interested in maintaining power. This goal is not only differently motivated but also different in its algorithmic nature. Note that here we require an *exact* (fixed or given) number of players to be added to or deleted from the game to avoid the trivial case of adding or deleting none of *at most* a number of players. While PP-hardness is also valid for control by adding a given number of players to maintain a player's PBI or SSI, PP-hardness for a fixed number of players to be added cannot immediately be deduced. However, we have a PP upper bound in this case (as PP is closed under complement) and a coNP-hardness lower bound.

THEOREM 4.
1. *Control by adding a fixed number of players to maintain a player's PBI or SSI is* coNP-*hard and in* PP.
2. *Control by deleting players in order to maintain a player's PBI is* coNP-*hard (even if only one player can be deleted).*

6. CONCLUSIONS AND FUTURE WORK

The complexity of some control problems is left open; interesting gaps remain, e.g., between NP-hardness and PP as well as PP-hardness and $\mathrm{NP}^{\mathrm{PP}}$. So far we have studied goals in *relation to the original game*. Alternatively, one might think of a situation where the goal is to increase a player's power in *comparison to the other players*; or where a player is required to exceed a certain *constant index*. We might also study obtaining an exact value. There may be different ways to reasonably model the new game, e.g., similar to weighted majority games, where players do not make an absolute but a relative contribution to the game. Studying a change of players dynamically over time is an interesting task for future work.

There seems to be a close connection to the notion of *synergies* in cooperative games (see, e.g., [8]), and it will be interesting to have a closer look at related results here. Other classes of games might also be affected by control scenarios.

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