Attachment Centrality: An Axiomatic Approach to Connectivity in Networks

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ABSTRACT

Centrality indices aim to quantify the importance of nodes or edges in a network. A number of new centrality indices have recently been proposed to try and capture the role of nodes in connecting the network. While these indices seem to deliver new insights, to date not enough is known about their theoretical properties. To address this issue, we propose an axiomatic approach. Specifically, we prove that there exists a unique centrality index satisfying some intuitive properties related to network connectivity. This new index, which we call *Attachment Centrality*, is equivalent to the Myerson value of a particular graph-restricted coalitional game. Building upon our theoretical analysis, we show that our Attachment Centrality has certain computational properties that are more attractive than the Myerson value for an arbitrary game.

General Terms

Measurement, Theory

Keywords

network centrality; axiomatization

1. INTRODUCTION

In many social networks, certain nodes play more important roles than others. For example, popular individuals with frequent social contacts are more likely to spread a disease in the event of an epidemic [8]; airport hubs such as Heathrow or Schiphol concentrate intercontinental passenger traffic [1]; and certain parts of the brain's neural network may be indispensable for breathing or to perform other vital activities [11]. Consequently, the concept of *centrality index*, which aims at identifying the key nodes in the network, has been extensively studied in the literature [12, 7]. Arguably, the most well-known such indices are: *Degree*, *Closeness*, and *Betweenness* centralities [9]. In particular, Degree Centrality quantifies the power of a node by the number of its incident edges. In contrast, Closeness Centrality promotes nodes that are close to all other nodes in the network. Betweenness Centrality counts the shortest paths between any two nodes in the network, and ranks nodes according to the number of the shortest paths they belong to.

More recently, a number of new centrality indices have been proposed in an attempt to reflect the following fundamental property of nodes: *the role they each play in connecting the network* [10, 3, 4]. Typically, such measures are built upon the well-known coalitional-game model of Myerson [16], where cooperation is restricted to *connected coalitions* (i.e., connected subgraphs of the network). Within this general framework, different centrality indices are obtained by manipulating the function that assigns the payoff from cooperation in each connected coalition.

Unfortunately, although these recent indices offered new insight, they were only evaluated empirically. The lack of a theoretical underpinning not only makes it difficult to choose among them, but also hinders the development of efficient algorihtms to compute them, especially given the inherent computational difficulty of Myerson's model [22].

To address this issue, we propose an axiomatic approach built around five basic requirements. The first two, namely (1) *Locality* and (2) *Normalization*, seem to be reasonable requirements for any centrality index (indeed, all three standard centrality indices satisfy them). Other requirements that we focus on are (3) *Monotonicity* and (4) *Gain-loss*—two somewhat-opposing views of how adding an edge affects other nodes (i.e., nodes that are not part of the edge). In a nutshell, when an edge is added, Monotonicity requires that the indices of other nodes *do not decrease*, whereas Gain-loss implies that they *do not increase*. More precisely, Gain-loss requires that the gain of a node always results in the loss of another. The final requirement is (5) *Fairness*, requiring that the addition of an edge equally affects the two nodes connected by it. Arguably, this requirement seems reasonable when an edge is considered to be equally owned by both nodes. That is to say, both nodes have an equal say in whether the edge is formed or disbanded (e.g., both sides of a relationship must consent to, and can break, the relation). From this point of view, it seems "fair" that both nodes equally benefit from their edge, since they both equally own it.[1]

Building upon these axioms, we prove that Degree Centrality is the unique index satisfying Normalization, Locality, Fairness and Monotonicity. Furthermore, we prove that replacing Monotonicity with Gain-loss leads to yet another unique index, which we call: *Attachment Centrality*; this is the first axiomatized centrality index focusing on connectivity in the literature. Finally, to compute our index, we propose a dedicated algorithm and test it on a relatively large, real-life social network.

[1] Of course one can think of many alternative ways to interpret "fairness". We do not argue that ours is necessarily *the best way*, but rather that it seems to be *a reasonable way*.

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2. PRELIMINARIES

Since our work falls at the interface of graph theory and coalitional game theory, this section provides the necessary background and notation from both sides.

Graph theory: A graph, or a network, is a pair, $G = (V, E)$, where V is the set of $n = |V|$ nodes, and E is the set of edges. We sometimes write $|G|$ instead of $|V|$ to denote the number of nodes in G. An edge $\{v, u\} \in E$ is said to be *incident* to nodes v and u. The degree of a node v, denoted by $degree_G(v)$, is the number of edges incident to v, i.e.:

$$degree_G(v) = \big|\{\{v, u\} \in E : u \in V\}\big|.$$

When there is no risk of confusion, we will often omit the G from the notation, and simply write $degree(v)$ instead of $degree_G(v)$. Nodes $u, v \in V$ are said to be *neighbors* if they are connected by an edge. If $degree(v) = 0$, we say that v is *isolated*.

A path, $p = (v_1, \dots, v_k)$, is a sequence of nodes in which every two consecutive nodes are connected by an edge, i.e., $\{v_i, v_{i+1}\} \in E,\ \forall i \in \{1, \dots, k-1\}$. The length of a path is the number of edges in it (i.e., the number of nodes in it minus 1). We write $v \in p$ if v is one of the nodes in p. The *distance* between any two nodes, $v, u \in V$, is denoted by $dist(v, u)$, and is defined as the length of a shortest path between them. If there exists no path between u and v, we assume that $dist(v, u) = \infty$.

Next, we introduce the concept of a *minimal path*. In particular, a path, p, between v and u is said to be *minimal* if there exists no shorter path between v and u that can be obtained by removing nodes from p (see Figure 1 for an example). The set of all *shortest* paths between v and u is denoted by $\Pi_s(v, u)$, and the set of all *minimal* paths between v and u is denoted by $\Pi_m(v, u)$. Note that every *shortest* path between u and v is a *minimal* path between u and v, i.e., $\Pi_s(v, u) \subseteq \Pi_m(v, u)$.

Nodes $v, u \in V$ are said to be *connected* if there exists a path between them. A graph G is said to be connected if every two nodes in it are connected. For any subset of nodes, $S \subseteq V$, the *subgraph induced by* S is denoted by $G[S]$ and is defined as a graph whose nodes are S and whose edges are those in G that connect some node in S with some other node in S. Formally:

$$G[S] = (S, \{\{v, u\} \in E : v, u \in S\}).$$

Any subset of nodes, $S \subseteq V$, is said to be connected if the subgraph induced by S is connected. If G is not connected, we denote by $K(G)$ the partition of V into disjoint sets of nodes that each induce a *maximal connected subgraph* in G.

The set of all possible graphs with nodes V is denoted by $\mathcal{G}^V$. Two special graphs are *cliques* and *stars*. Specifically, a graph is a *clique* if every two nodes in it are connected by an edge:

$$(V, \{\{v, u\} : v, u \in V\}).$$

On the other hand, a graph is a *star* if there exists a node v (called the *center* of the star) such that every node $u \in V \setminus \{v\}$ is connected to v and not connected to any other node. That is:

$$(V, \{\{v, u\} : u \in V \setminus \{v\}\}).$$

Centrality indices: A *centrality index*, $F : \mathcal{G}^V \to \mathbb{R}^V$, is a function that assigns to every node a number reflecting its importance. Typically, the higher this number, the more *important* or *central* the node. Arguably, the most well-known centrality indices are the following (we will refer to them *the standard centrality indices*):

- *Degree Centrality* (D_v) is simply the degree of a node, i.e.,

$$D_v(G) = degree_G(v);$$

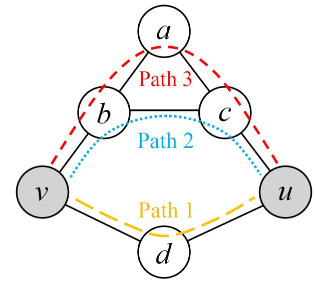

Figure 1: Out of all three paths between v and u, only Path 1 **and** Path 2 **are *minimal* paths. Furthermore, out of those two minimal paths, only** Path 1 **is a *shortest* path between v and u.**

- *Closeness Centrality* (C_v) is the sum of of the inverses of distances to other nodes (under the assumption that $\frac{1}{\infty} = 0$):[2]

$$C_v(G) = \sum_{u \in V \setminus \{v\}} \frac{1}{dist(v, u)};$$

- *Betweenness Centrality* (B_v) is the average percentage of shortest paths between any two other nodes that goes through the node under consideration. More formally, if we denote by $K_v(G)$ the connected component containing v in G, then:

$$B_v(G) = \frac{1}{\big|K_v(G)\big| - 2} \sum_{\substack{s,t \in K_v(G) \\ s \neq v \neq t}} \frac{|\{p \in \Pi_s(s,t) : v \in p\}|}{|\Pi_s(s,t)|}.$$

Note that the above formula of Betweenness Centrality is normalized to ensure that it yields the same range of values as Degree and Closeness centralities, i.e., $[0, n-1]$.

Coalitional game theory: A *game* is a pair (N, f), where N is the set of *players* and $f : 2^N \to \mathbb{R}$ is the *characteristic function*, which assigns a real number to each subset of players (with the only assumption being that $f(\emptyset) = 0$). Any subset of players, $S \subseteq N$, is called a *coalition*, and $f(S)$ is called *the value of coalition* S. In contrast, "*a value of a game*" is a function that assigns a *payoff* to each player $v \in N$, i.e., $\varphi : (2^N \to \mathbb{R}) \to \mathbb{R}^N$. This payoff traditionally represents v's share out of the value of the *grand coalition*, i.e., the coalition of all players. Alternatively, the payoff of v can be interpreted as an assessment of the importance of v in the game. Thus, a *value* of a game plays the same role as a *centrality index* of a network; the former ranks players whereas the latter ranks nodes.

Shapley [21] was the first to propose an axiomatic approach to the problem of payoff division. In particular, Shapley proved that there exists a unique "value" satisfying some intuitive and desirable properties (also known as "*axioms*"). This value—now known as *the Shapley value*—is denoted for player v by $SV_v(f)$, defined as:

$$SV_v(f) = \sum_{S \subseteq N \setminus \{v\}} \beta(S, N)\big(f(S \cup \{v\}) - f(S)\big), \qquad (1)$$

where $\beta(S, N) = |S|!(|N| - |S| - 1)!/|N|!$. Here, the expression $f(S \cup \{v\}) - f(S)$ is known as the *marginal contribution* of player v to coalition S; it is the difference that v makes when joining S.

Myerson [16] considered a model under which the cooperation of players is restricted by a communication graph, G. Specifically in this model, only connected coalitions, i.e., coalitions in which all players can communicate (either directly or indirectly through

[2] In the classical definition [20], the summation sign is in the denominator. However, we consider the well-respected version for graphs which do not have to be strongly connected [5].

intermediaries) can generate value added from cooperation. As for any disconnected coalition, its value equals the sum of the values of its connected components. As such, Myerson's model is defined by a graph, G, and a function, f, that specifies the value of every connected subgraph of G. Over the past decades, this became widely accepted as the canonical model of restricted cooperation. Myerson also proposed a value—now known as *the Myerson value*—which is denoted for player i by $MV_i(f, G)$; this value is simply the Shapley value of the *restricted game* $(N, f/G)$ whose characteristic function is defined as follows:

$$f/G(S) = \sum_{C \in K(G[S])} f(C), \quad \text{for every } S \subseteq N. \tag{2}$$

To put it differently, we have:

$$MV_v(f, G) = SV_v(f/G). \tag{3}$$

3. ATTACHMENT CENTRALITY

Our aim in this section is to define a new centrality index that reflects connectivity, by following an "axiomatic approach". That is, we want to identify a set of requirements, and then prove that there exists exactly one possible centrality index that satisfies all of those requirements, or "*axioms*". This way, the axioms would serve as a theoretical foundation for the centrality index that they uniquely define. Ideally, those axioms should be as intuitive and desirable as possible, to justify the use of the resulting index. In reality, however, any such set of axioms would probably be more suitable and intuitive for some settings, and less so for others. Still, identifying such a set of axioms would serve as a important step towards better understanding how centrality can be tailored to reflect or capture connectivity.

To this end, we propose five requirements, namely *Locality*, *Normalization*, *Fairness*, *Monotonicity*, and *Gain-loss*. Starting with Locality, this requirement is defined as follows, where $K_v(G)$ is the connected component containing v in G.

Locality: *For every graph $G = (V, E)$ and every node $v \in V$, the centrality of v depends solely on $G[K_v(G)]$. That is,*

$$F_v\big(G\big) = F_v\big(G[K_v(G)]\big).$$

As centrality indices are typically defined for connected graphs, the Locality requirement can be interpreted as a natural extension to disconnected graphs. Note that all four standard indices satisfy this requirement.

As for *Normalization*, it is inspired by the observation that Degree, Closeness, and normalized Betweenness all return a *minimum* value of 0, and a *maximum* value of $n - 1$. Moreover, they are all *minimized* when the node is isolated, and *maximized* when the node is the center of a star. Normalization generalizes this observation.

Normalization: *For every $G = (V, E)$ and $v \in V$, we have:*

- $F_v(G) \in [0, n - 1]$*;*
- $F_v(G) = 0$ *when v is isolated in G;*
- $F_v(G) = n - 1$ *when G is a star, center of which is v.*

The remaining three requirements are concerned with the impact of adding an edge; *Fairness* focuses on how this addition affects both ends of the edge, whereas *Monotonicity* and *Gain-loss* focus on how this addition affects every node other than the two ends of the edge. Next, we explain each requirement in more detail.

Fairness: *For every $G = (V, E)$ and every $v, u \in V$, adding the edge $e = \{u, v\}$ affects the centrality of u and v equally:*

$$F_v\big((V, E \cup \{e\})\big) - F_v\big((V, E)\big) = F_u\big((V, E \cup \{e\}\big) - F_u\big((V, E)\big)$$

This notion of Fairness was first proposed by Myerson [16]. Arguably, this seems to be a reasonable requirement when the two ends of the edge are considered to be equally responsible for it. This is perhaps more evident in settings where the formation of an edge requires the consent of both ends, and where the edge can be broken at any time by either end, such as friendship relationships for example. Interestingly, Closeness and Betweenness centralities do not satisfy the Fairness requirement. As mentioned earlier, this requirement is clearly not the only possible interpretation of a "fair" centrality index. However, we choose to call it "Fairness" because, at least in some settings, it seems to be reasonably fair.

Moving on to the final two requirements, namely *Monotonicity* and *Gain-loss*; these reflect somewhat-opposing views of how adding an edge affects the indices of the remaining nodes. In a nutshell, *Monotonicity* requires that these indices *do not decrease*, whereas *Gain-loss* implies that they *do not increase*. Next is a formal definition of the two requirements, followed by a discussion of the intuition behind each.

Monotonicity: *For every graph, $G = (V, E)$, adding an edge to E does not decrease the index of any node in V. That is, for every $v, u, w \in V$:*

$$F_v\big((V, E \cup \{u, w\})\big) \geq F_v\big((V, E)\big).$$

Gain-loss: *For every connected graph, $G = (V, E)$, and every pair of nodes, $u, w \in V$, adding the edge $\{u, w\}$ to E does not affect the sum of indices, i.e.:*

$$\sum_{v \in N} F_v\big((V, E \cup \{\{u, w\}\})\big) = \sum_{v \in N} F_v\big((V, E)\big).$$

Arguably, from the connectivity point of view, Gain-loss makes more sense compared to Monotonicity. To see why this is this case, consider a situation in which the removal of a node, v, breaks a connected graph, G, into two components, G_1 and G_2. Here, v obviously plays an important role in terms of connectivity, since its presence is necessary to connect G_1 with G_2. Now suppose that the edge $\{u, w\}$ was added to G, where u belongs to G_1 and w belongs to G_2. With this addition, it seems reasonable to claim that the role played by u and w grows more important, whereas the role played by v diminishes, since its presence is no longer necessary to connect G_1 with G_2. From this perspective, the Gain-loss requirement seems reasonable, whereas Monotonicity seems rather unintuitive, since it assumes that the connectivity role of v remains unchanged, or even grows more important, after the addition of $\{u, w\}$.

Note that the Gain-loss requirement only deals with the addition of an edge to an already-connected graph. As for disconnected graphs, if we add an edge that connects some of its components, then the Gain-loss requirement places no assumptions or restriction on how this would affect the centrality of nodes across the graph.

Having described the necessary five requirements, we are now ready to introduce one of our main results.

THEOREM 1. *There exists a unique centrality index that satisfies Locality, Normalization, Fairness, and Monotonicity; this index is Degree Centrality.*

PROOF. We begin by showing that Degree Centrality satisfies the four requirements listed in the statement of the theorem. To this end, for any graph, $G = (V, E)$, and any node, $v \in V$:

- $degree(v)$ depends solely on the connected component containing v in G, meaning that Locality is met;
- $degree(v) = 0$ when v is isolated, and $degree(v) = n - 1$ when G is a star center of which is v. It also holds that: $0 \leq degree(v) \leq n - 1$. Thus, Normalization is met;

- $degree(v)$ does not decrease by adding an edge $\{u, w\}$ for some $u, w \in V \setminus \{v\}$, meaning that Monotonicity is met;
- Adding an edge, $\{v, u\}$ for any $u \in V \setminus \{v\}$ increases the degree of both u and v by 1. Therefore, Fairness is met.

It remains to prove that Degree Centrality is the only possible centrality index satisfying the above four requirements. To put it differently, assuming that F is a centrality index satisfying those requirements, it remains to prove that $F_v(G) = D_v(G)$ for any graph $G = (V, E)$ and any node $v \in V$. We will do so by first proving that $F_v(G) \geq D_v(G)$ and then proving that $F_v(G) \leq D_v(G)$.

Step 1: In this step, we will prove that:

$$F_v(G) \geq D_v(G), \ \ \forall v \in V. \tag{4}$$

Let us fix a node, $v \in V$, and remove all the edges from G except those connecting v to its neighbors. In so doing, we obtain a new graph, G', such that $K_v(G')$ forms a star center of which is v, and: $|K_v(G')| = D_v(G) + 1$. Since v is the center of the star $G'(v)$, then based on Normalization: $F_v(G'[K_v(G')]) = |K_v(G')| - 1$. Next, we know from Locality that: $F_v(G') = F_v(G'[K_v(G')])$. Finally, we know from Monotonicity that: $F_v(G) \geq F_v(G')$ which concludes the proof of correctness of (4).

Step 2: In this step, we will prove that:

$$F_v(G) \leq D_v(G), \ \ \forall v \in V. \tag{5}$$

To this end, for any set of nodes, V, and any $v \in V$, let $\mathcal{G}_v^\dagger(G)$ denote the set of all possible graphs whose set of nodes is V, and whose set of edges, E, includes every pair in $V \setminus \{v\}$, meaning that all nodes in $V \setminus \{v\}$ form a clique. We will prove that the following holds for all $G \in \mathcal{G}_v^\dagger(G)$:

$$F_v(G) = D_v(G). \tag{6}$$

We will prove this by induction over the degree of v. Specifically, for any graph, $G \in \mathcal{G}_v^\dagger(G)$, if $degree_G(v) = 0$ then v is isolated, implying that (6) holds based on Normalization. Now, assuming that (6) holds for all $G \in \mathcal{G}_v^\dagger(G) : degree_G(v) \leq d$ for some positive integer $d < n - 1$, we will prove that (6) also holds for every graph $G \in \mathcal{G}_v^\dagger(G) : degree_G(v) = d + 1$. Let G' be one such graph, i.e., a graph in $\mathcal{G}_v^\dagger(G)$ such that the degree of v in G' equals $d + 1$. Furthermore, let $u \in V$ be a neighbor of v in G'. Then, if we remove the edge $\{u, v\}$ from G', we obtain a new graph, $G'' \in \mathcal{G}_v^\dagger(G)$, in which the degree of v equals d. This means that $F_v(G'') = D_v(G'')$ based on the aforementioned assumption. Now observe that:

- $D_u(G') = n-1$. This is because u is connected to v in G' (since u and v are neighbors) and connected to every other node in G' (since $V \setminus \{v\}$ forms a clique). Thus, based on Normalization and (4), it holds that: $F_u(G') = n - 1$;
- $D_u(G'') = n - 2$. This is because $D_u(G') = n - 1$, and G'' is identical to G' except that u and v are no longer neighbors. Thus, from Normalization and (4): $n - 2 \leq F_u(G'') \leq n - 1$.

The above two observations imply that, with the addition of $\{v, u\}$ to G'', the centrality of u increases by *at most* 1. That is,

$$F_u(G') - F_u(G'') \leq 1. \tag{7}$$

Now let us turn our attention back to node v. So far, we know that $F_v(G'') = D_v(G'')$ and that $D_v(G') = D_v(G'') + 1$. We also know from (4) that $F_v(G') \geq D_v(G')$. Thus:

$$F_v(G') - F_v(G'') \geq 1. \tag{8}$$

Based on (7) and (8), as well as the Fairness requirement, we conclude that: $F_v(G') - F_v(G'') = 1$, implying that $F_v(G') = D_v(G')$. This concludes our proof of the correctness of (6).

Let us now move back to our original goal in **Step 2**—proving the correctness of (5). Here, we have two possibilities:

- If $V \setminus \{v\}$ happened to form a clique in G, then we know from (6) that $F_v(G) = D_v(G)$, implying that (5) holds;
- If $V \setminus \{v\}$ did not form a clique in G, then we can add every missing edge between pairs of nodes in $V \setminus \{v\}$. In so doing, we end up with a new graph, G^*, in which $V \setminus \{v\}$ forms a clique. Thus, based on (6), we have:

$$F_v(G^*) = D_v(G^*). \tag{9}$$

Furthermore, based on Monotonicity, we know that:

$$F_v(G) \leq F_v(G^*). \tag{10}$$

Finally, since none of the added edges involved v:

$$D_v(G^*) = D_v(G). \tag{11}$$

Taken together, (9), (10) and (11) imply that (5) holds.

Thus, we have proved that (5) always holds, which is what we want to show in **Step 2**. This concludes our proof of Theorem 1. $\square$

Next, we will introduce our *Attachment Centrality*, and then prove that it is the unique index satisfying Locality, Normalization, Fairness and Gain-Loss.

DEFINITION 1. ***Attachment Centrality*** *is the centrality index defined for every graph, $G = (V, E)$, and every node, $v \in V$, as:*

$$A_v(G) = \sum_{S \subseteq V \setminus \{v\}} 2\beta(S, V)(|K(G[S])| - |K(G[S \cup \{v\}])| + 1), \tag{12}$$

where $\beta(S, V) = |S|!(|V| - |S| - 1)!/|V|!$, and $K(G[S])$ denotes the partition of S into disjoint sets of nodes that each induce a maximal connected subgraph in $G[S]$.

In the above definition, for any $S \subseteq V$ (connected or otherwise), the expression $|K(G[S])| - |K(G[S \cup \{v\}])| + 1$ equals the number of components in $G[S]$ that node v connects.

The intuition behind the Attachment Centrality is as follows. If we were to remove nodes from the graph one by one in a random order, then the Attachment Centrality of $v \in V$ would be the expected number of components created from the removal of v, multiplied by 2 for normalization purposes.

By comparing the definition of the Attachment Centrality with Equation (1)—the definition of the Shapley value—we find that:

$$A_v(G) = SV_v(f_G^*), \tag{13}$$

where

$$f_G^*(S) = 2(|S| - |K(G[S])|). \tag{14}$$

To put it in words, the Attachment Centrality is equivalent to the Shapley value of the game (V, f_G^*). Importantly, if we now write $f_G^*(S)$ differently as follows:

$$f_G^*(S) = \sum_{C \in K(G[S])} 2(|C| - 1),$$

we obtain an equation similar to (2)—the equation defining f/G—except that $f(C)$ is now replaced with $2(|C| - 1)$. This observation, together with (3) and (13), imply that:

$$A_v(G) = MV_v(f^*, G), \tag{15}$$

where f^* is computed for every connected coalition, S, as:

$$f^*(S) = 2(|S| - 1). \tag{16}$$

To put it in words, the Attachment Centrality is equivalent to the Myerson value under the model of restricted cooperation defined by the communication graph G and the evaluation function f^*.

THEOREM 2. *There exists a unique centrality index that satisfies Locality, Normalization, Fairness, and Gain-loss; this index is the Attachment Centrality.*

PROOF. We begin by recalling a crucial result of Myerson [16], who proved that for an arbitrary game (V, f), there exists a unique function $\varphi : \mathcal{G}^V \to \mathbb{R}^V$ satisfying the following two properties: (1) *Fairness*—defined exactly as in our paper—and (2) *Component Efficiency*—a property defined for game (V, f) as follows:

Component Efficiency (for game (V, f)***):*** *For every graph, G, and every connected component $C \in K(G)$ of a graph:*

$$\sum_{v \in C} \varphi_v(G) = f(C).$$

Myerson proved that this unique function is actually the Myerson value under the model of restricted cooperation defined by the communication graph G and the evaluation function f.

Having recalled this result of Myerson, we are now ready to prove that the Attachment Centrality satisfies all four requirements listed in the statement of Theorem 2. In particular, Myerson's result, as well as (15) and (16), imply that the Attachment Centrality satisfies Fairness, and that for every graph G and every connected component $C \in K(G)$ we have:

$$\sum_{v \in C} A_v(G) = 2(|C| - 1). \tag{17}$$

This implies that adding an edge between two nodes in a connected component $C \in K(G)$ does not affect the sum of the Attachment Centrality of every node in that component. This, in turn, implies that the Attachment Centrality satisfies the Gain-loss requirement.

Moving on to Normalization; we need to show that the Attachment Centrality satisfies the three conditions outlined in the definition of Normalization. To this end, for every $v \in V$:

- If v is isolated in G, then v forms a connected component by itself, and (17) implies that $A_v(G) = 0$.
- If G is a star center of which is v, then we need to show that $A_v(G) = n - 1$. To this end, for every $S \subseteq V \setminus \{v\}$, the induced subgraph $G[S]$ consist of $|S|$ connected components, whereas the subgraph $G[S \cup \{v\}]$ consists of a single connected component. Thus,

  $$|K(G[S])| - |K(G[S \cup \{v\}])| + 1 = |S|.$$

  This fact, together with (12), imply that:

  $$A_v(G) = \sum_{S \subseteq V \setminus \{v\}} 2\beta(S, V)|S| = \sum_{s=0}^{n-1} \frac{2s}{n} = \sum_{s=0}^{n-1} = n - 1. \tag{18}$$

- Finally, we need to prove that $A_v(G) \in [0, n-1]$. Note that the expression $|K(G[S])| - |K(G[S \cup \{v\}])| + 1$ is always between 0 and $|S|$ (simply because the number of components in $G[S]$ that are connected by v is between 0 and $|S|$). Furthermore, if this expression equals $|S|$ then $A_v(G) = n-1$; see (18). In contrast, if this expression equals 0, then $A_v(G) = 0$. This implies that $A_v(G) \in [0, n-1]$.

The above three points imply that the Attachment Centrality satisfies the Normalization requirement.

Next, we prove that the Attachment Centrality satisfies Locality. In other words, given a graph, $G = (V, E)$, and a node, $v \in V$, we prove that $A_v(G) = A_v(G[K_v(G)])$, where $K_v(G)$ is the connected component containing v in G. To this end, let us denote by $mc^*_v(S)$ the marginal contribution of v to S in the (V, f^*_G), i.e.,

$$mc^*_v(S) = 2(|K(G[S])| - |K(G[S \cup \{v\}])| + 1).$$

Observe that $mc^*_v(S)$ is basically the number of components from S that node v connects, multiplied by 2. This number is not influenced by any of the nodes lying outside $K_v(G)$—the connected component containing v in G. Thus, for every $S \subseteq V \setminus \{v\}$: $mc^*_v(S) = mc^*_v(S \cap K_v(G))$. Next, we rewrite (12) as follows:

$$A_v(G) = \sum_{P \subseteq V \setminus K_v(G)} \sum_{S \subseteq K_v(G) \setminus \{v\}} \beta(S \cup P, V) mc^*_v(S).$$

Simple calculations show that for every $S \subseteq K_v(G)$ we have:

$$\sum_{P \subseteq V \setminus K_v(G)} \beta(S \cup P, V) = \beta(S, K_v(G)).$$

Thus, $A_v(G) = A_v(G[K_v(G)])$, i.e., the Attachment Centrality satisfies Locality.

It remains to prove that the Attachment Centrality is the only possible centrality index satisfying all four requirements listed in Theorem 2. To put it differently, given a centrality index, F, that satisfies those requirements, it remains to prove that $F_v(G) = A_v(G)$ for any graph $G = (V, E)$ and any node $v \in V$. We will do so by showing that the sum of the F index of every node belonging to the same connected component S equals: $f^*(S) = 2(|S| - 1)$. In so doing, we show that F satisfies *Component Efficiency* for game (V, f^*). Since F also satisfies *Fairness*, then based on Myerson's result, this index is unique.

Let $G = (V, E)$ be a star with node v being the center, and let u be an arbitrary node in $V \setminus \{v\}$. Normalization implies that:

$$F_v(G) = n - 1.$$

Next, we show that $F_u(G) = 1$. To this end, consider the graph $G' = (V, E \setminus \{\{v, u\}\})$ obtained from G by removing the edge $\{v, u\}$. Since u is now isolated in G', Normalization implies that:

$$F_u(G') = 0,$$

and from Locality, we know that:

$$F_v(G') = F_v(G'[V \setminus \{u\}]) = n - 2.$$

Now since the Fairness requirement implies that the removal of $\{v, u\}$ affects the centrality indices of both v and u equally, then:

$$F_u(G) = F_u(G') + (F_v(G) - F_v(G')) = 1.$$

As node u was chosen arbitrarily from the set $V \setminus \{v\}$, we conclude that every node other than the center of the star has a centrality index of 1. Thus, the sum of indices in a star equals $2(n-1)$. As every connected graph can be obtained from a star by adding and/or removing edges, the Gain-loss requirement implies that the sum of indices in any connected graph with n nodes equals $2(n-1)$. Finally, Locality implies that the sum of indices in any connected component S equals $2(|S| - 1)$. $\square$

Note that Theorems 1 and 2 imply that there exist no value that satisfies all five axioms: Locality, Normalization, Fairness, Monotonicity, and Gain-loss.

4. PROPERTIES

In this section, we discuss some key properties of Attachment Centrality, and show that it is closely related to the notion of *minimal paths* between nodes. The following theorem constitutes the cornerstone of our analysis.

THEOREM 3. *Adding an edge $\{v, u\}$ to a graph G affects only the Attachment Centrality of nodes lying on a minimal path between v and u.*

PROOF. Recall that the value of a coalition, S, in the game (V, f^*_G) depends solely on the number of the nodes in S and the number of connected components in S (see Equation (14)). Now, let $G = (V, E)$ be an arbitrary, incomplete graph, and let $v, u \in V$ be two nodes such that $\{v, u\} \notin E$. Finally, let G' be the graph that results from adding $\{v, u\}$ to G, i.e., $G' = (V, E \cup \{\{v, u\}\})$. Next, we analyse how the marginal contribution of some node $w \in V \setminus \{v, u\}$ to a coalition S differs between G and G':

- Suppose that $\{v, u\} \not\subseteq S$, or that $\{v, u\} \subseteq S$ and both v and u belong to the same component in $G[S]$. Either way, edge $\{v, u\}$ does not affect the value of S and $S \cup \{w\}$: $f^*_G(S) = f^*_{G'}(S)$ and $f^*_G(S \cup \{w\}) = f^*_{G'}(S \cup \{w\})$. So, the marginal contribution of w in G is the same as in G';
- On the other hand, suppose that $\{v, u\} \subseteq S$ and that v and u belong to different component in $G[S]$, namely C_v and C_u. In this case, we have two possibilities: Either w is connected to both C_v and C_u or not. If it is not connected to both, then:

$$\begin{aligned} f^*_{G'}(S \cup \{w\}) - f^*_{G'}(S) &= (f^*_G(S \cup \{w\}) + 2) - (f^*_G(S) + 2) \\ &= f^*_G(S \cup \{w\}) - f^*_G(S), \end{aligned}$$

  meaning that the marginal contribution of w in G is the same as in G'. In contrast, if w was connected to both C_v and C_u, then w would unite the two in $G[S \cup \{w\}]$ but not in $G'[S \cup \{w\}]$, because in G' the nodes in S are sufficient to unite the two components; they no longer need w to do that.

To summarize, we have shown that the marginal contribution of w in G can be different than in G' only when all of the following three conditions are met: (1) $\{v, u\} \subseteq S$; (2) v and u belong to different components in $G[S]$; (3) w is connected to both C_v and C_u. Importantly, however, if those three conditions are met, then w must be on some minimal path between v and u. To see why this is case, consider a minimal path p_1 between v and w in C_v, and another minimal path p_2 between u and w in C_u. Since C_v and C_u are not connected in G, merging p_1 and p_2 results in a minimal path between v and u that goes through w.

So far, we have shown that the marginal contribution of w in G can be different than in G' only when w is on some minimal path between v and u. Finally, from (1) and (13), we know that the Attachment Centrality of w is a weighted average of the marginal contributions of w in the game (V, f^*_G). Thus, by adding $\{v, u\}$, the Attachment Centrality of w may change only when w is on some minimal path between v and u. This concludes the proof. □

The above theorem leads to a series of corollaries that provide additional insights on how the Attachment Centrality measures the role of a node in connecting the network. We divide our analysis into two parts: the first focuses on nodes with high connectivity (such as *cut vertices*), whereas the second part focuses on nodes with low connectivity (such as leafs).

High connectivity: Given a *cut vertex*, $v \in V$ (i.e., a node that connects disjoint parts of the graph), we will show that the Attachment centralities of the nodes in each part are not influenced by the other parts. This, in turn, implies that the Attachment Centrality of v is simply the sum of its Attachment Centrality computed for each part separately.

THEOREM 4. *Let G be a connected graph, and let v be a node removal of which breaks G into k disjoint components consisting of the following sets of nodes: $C_1, \ldots, C_k$. Then,*

$$A_v(G) = \sum_{i \in \{1,\ldots,k\}} A_v(G[C_i \cup \{v\}]).$$

Furthermore, for every $i \in \{1, \ldots, k\}$, and every $u \in C_i$:

$$A_u(G) = A_u(G[C_i \cup \{v\}]).$$

PROOF. First, let us focus on an arbitrary node $u \in C_i$ for some $i \in \{1, \ldots, k\}$. Now let us remove from G an edge $\{w, w'\} \in E$ such that $w, w' \in C_j \cup \{v\}$ for some $j \neq i$. Note that a minimal path between w and w' cannot contain nodes from C_i. Based on this, Theorem 3 implies that the removal of $\{w, w'\}$ from G does not affect the Attachment Centrality of node u. By removing every such edge one by one, we eventually end up removing from G every edge outside $G[C_i \cup \{v\}]$ without affecting the Attachment Centrality of node u. Based on this, as well as Locality, we have that $A_u(G) = A_u(G[C_i \cup \{v\}])$.

Now, let us turn our attention to node v, and let us start by computing the Attachment centrality of v in each of the following subgraphs separately: $G[C_i \cup \{v\}] : i \in \{1, 2, \ldots, k\}$; we find that:

$$A_v(G[C_i \cup \{v\}]) = 2|C_i| - \sum_{u \in C_i} A_u(G[C_i \cup \{v\}]).$$

Since we already proved that $A_u(G[C_i \cup \{v\}]) = A_u(G)$ for every $u \in C_i$, we get:

$$\sum_{i \in \{1,\ldots,k\}} A_v(G[C_i \cup \{v\}]) = 2(|C| - 1) - \sum_{u \in C \setminus \{v\}} A_u(G) = A_v(G)$$

which concludes the proof. □

The next corollary concerns bridges, i.e., edges the removal of which increases the number of connected components in the graph. The corollary follows from Theorem 4 and it is based on the observation that both ends of a bridge are cut vertices.

COROLLARY 1. *Removing a bridge decreases the Attachment Centrality of both its ends by 1, and does not affect the Attachment Centrality of other nodes.*

Interestingly, according to the above corollary, the connectivity role played by a bridge is attributed solely to is two ends. Furthermore, the fact that a node v is an end of a bridge does not influence in any way the Attachment centralities of the nodes connecting v to the rest of the network.

Whereas Corollary 1 focuses on *edges* whose removal increases the number of connected components, Theorem 5 focuses on *cliques* whose removal increases the number of connected components (such cliques are known as *cut cliques*).

THEOREM 5. *Let G be a connected graph. If a set of nodes $K \subseteq G$ forms a clique in G, and the removal of K breaks G into k disjoint components consisting of the sets of nodes: $C_1, \ldots, C_k$, then for every $v \in K$:*

$$A_v(G) = \sum_{i \in \{1,\ldots,k\}} A_v(G[C_i \cup K]) - (k-1)A_v(G[K]).$$

Furthermore, for every $i \in \{1, \ldots, k\}$ and every $u \in C_i$:

$$A_u(G) = A_u(G[C_i \cup K]). \tag{19}$$

PROOF. First, let us focus on an arbitrary node $u \in C_i$ for some $i \in \{1, \ldots, k\}$. Analogously to the proof of Theorem 4, we argue that any edge between two nodes in $C_j \cup K$ for some $j \neq i$ does not affect the Attachment Centrality of u. This, in turn, implies the correctness of (19).

Now let us turn our attention to an arbitrary node $v \in K$, and let us analyse the marginal contribution of this node to an arbitrary coalition $S \subseteq V \setminus \{v\}$. Without loss of generality, let $K(G[S]) = \{C_1, \ldots, C_l\}$ be the components of S, and assume that v is connected to the first m components, where $1 \leq m \leq l$. Following the definition of f^*_G (i.e., Equation (14)) the marginal contribution of v to S equals $2m$. Every connected component C_i for $i \leq m$ either contains all elements $K \cap S$ or is a subset C_j for some j. Thus, whenever S contains at least one element of K, then node v gets 2 for this component k times instead of 1. S contains at least one element of K with the probability $1 - \frac{1}{|K|}$. With the same probability node v has non-zero (and equal 2) marginal contribution in a clique of nodes K. This concludes the proof. □

Low connectivity: This part focuses on nodes with almost no connectivity role. The first corollary concerns *leafs*.

COROLLARY 2. *The Attachment Centrality of a leaf equals* 1. *Furthermore, removing a leaf decreases the Attachment Centrality of its neighbor by 1, and does not affect the Attachment Centrality of any other node in the graph.*

PROOF. Let v be a leaf, and let u be its only neighbor. Furthermore, let $S \subseteq V$ be the set of nodes comprising the component that contains both v and u. The presence of the edge $\{v, u\}$ increases the "*profit*" of $S \cup \{v\}$ by 2, since $f^*(S \cup \{v\}) - f^*(S) - f^*(v) = 2(|S|) - 2(|S| - 1) - 0 = 2$. Furthermore, since we know from the proof of Theorem 2 that the Attachment Centrality satisfies the *Component Efficiency* requirement, we know that the profit of 2 must be divided wholly among the nodes of the graph. However, according to Theorem 3, the edge $\{v, u\}$ only affects the Attachment Centrality of nodes lying on a minimal path between v and u, and since v is a leaf, the edge $\{v, u\}$ only affects the Attachment Centrality of v and u. In other words, the profit must be divided *wholly* between v and u. Finally, according to Fairness, this profit must be divided *equally* between v and u. This implies the correctness of the theorem and concludes the proof. □

So far, we discussed a type of nodes that plays a relatively-small connectivity role, namely a leaf. The following theorem focuses on yet another such type—a node whose set of neighbors forms a clique. The reason such a node, v, plays a relatively-small connectivity role is that v does not appear on any minimal path between any two nodes $u, w \in V \setminus \{v\}$.

THEOREM 6. *Given a node, v, whose set of neighbors, K, forms a clique, the Attachment Centrality of v equals $\frac{2|K|}{|K|+1}$. Furthermore, removing v decreases the Attachment Centrality of each of its neighbors by $\frac{2}{|K|(|K|+1)}$, and does not affect the Attachment Centrality of any other node in the graph.*

PROOF. Let $G = (V, E)$ be a graph in which a node, $v \in V$, has a set of neighbors, K, that forms a clique. It suffices to prove:

$$A_u(G) = \begin{cases} \frac{2|K|}{|K|+1} & \text{if } u = v, \\ A_u(G[V \setminus \{v\}]) + \frac{2}{|K|(|K|+1)} & \text{if } u \in K, \\ A_u(G[V \setminus \{v\}]) & \text{otherwise.} \end{cases}$$

Node v has non-zero marginal contribution to coalition S, equal 2, if and only if S contains at least one of its neighbors. Based on the permutation interpretation of the Shapley value, this happens with the probability $\frac{|K|}{|K|+1} \to 1$. All his neighbors benefit in a marginal contribution from v (have a greater marginal contribution to coalition with v, than without him) only if S does not contain any other neighbor. Such marginal contribution happens with the probability $\frac{1}{|K|(|K|+1)}$. Since other nodes do not appear on the minimal path between v and his neighbors in G', all edges of v can be removed without the change in their value. □

5. ALGORITHM AND APPLICATION

As an application, we focus on the identification of key terrorists in covert organisations. In particular, we analyse of the terrorist network responsible for the 2004 attacks on Madrid trains. The reasons behind our choice of the application and the network are twofold. Firstly, it has been recently argued that connectivity plays a crucial role in identifying the key members of terrorist networks [13, 15]. Secondly, the Madrid network is relatively big and, thus far, has never been analysed with a centrality index of this kind.

The *Madrid network* consists of 70 nodes and 243 edges. The size of the network makes it impractical to compute the existing connectivity-based centrality indices. In more detail, the computation involves enumerating all induced connected subgraphs of the network. Unfortunately, even the state-of-the-art algorithm for this purpose [22] takes over 100 seconds to compute the Myerson value for sparse network with only 36 nodes. Furthermore, the running time grows exponentially with the size of the network; every additional node nearly doubles it. To address this challenge, we use techniques introduced in the previous section to narrow down the set of nodes for which the Myerson value has to be calculated.

The original Madrid network [18] contains 6 isolated nodes. From Normalization, we know that the Attachment Centrality of each of these nodes is 0. We also know from Locality that those 6 nodes can be removed without affecting the Attachment Centrality of others. Furthermore, we observe that the Madrid network contains 8 leaf nodes. From Corollary 2, we know that every such node has an Attachment Centrality of 1, and can easily be removed from the network (since the corollary specifies the impact of this removal). Moreover, from Theorem 6 we know that every node whose set of neighbors, K, forms a clique has an Attachment Centrality of $\frac{2|K|}{|K|+1}$, and can easily be removed from the network (since the theorem specifies the impact of this removal on the Attachment Centrality of other nodes). Note that removing nodes results in a chain reaction, meaning that the above rules can be applied repeatedly (e.g., by removing a leaf, some other node might become a leaf, which can then be removed, and so on).

Our algorithm carries out the above process systematically, by finding the *cut-clique decomposition* of a graph. In a nutshell, the cut-clique decomposition of a graph G is a binary tree in which every node t is labeled with a subset $S \subseteq V$. If subgraph $G[S]$ has a cut-clique, then for a (possibly one of many) cut-clique K both his children are labeled according to the decomposition of $G[S]$ using K. Specifically, children l and r are labeled with two subsets $L, R \subseteq S$ such that $L \cap R = K$, and there exists no edge between $L \setminus K$ and $R \setminus K$. Theorem 5 allows for considering those subgraphs independently. By using the algorithm proposed by Tarjan [23], which utilizes the *simple elimination ordering* [19], we know that only one child can be further decomposed. This simplifies our algorithm (see Algorithm 1 for the pseudocode).

The results of our analysis are summarized in Table 1. As can be seen, the Attachment Centrality significantly differs from the standard centrality indices. For instance, let us consider two nodes with the highest number of edges – 1 and 3 – which are positioned

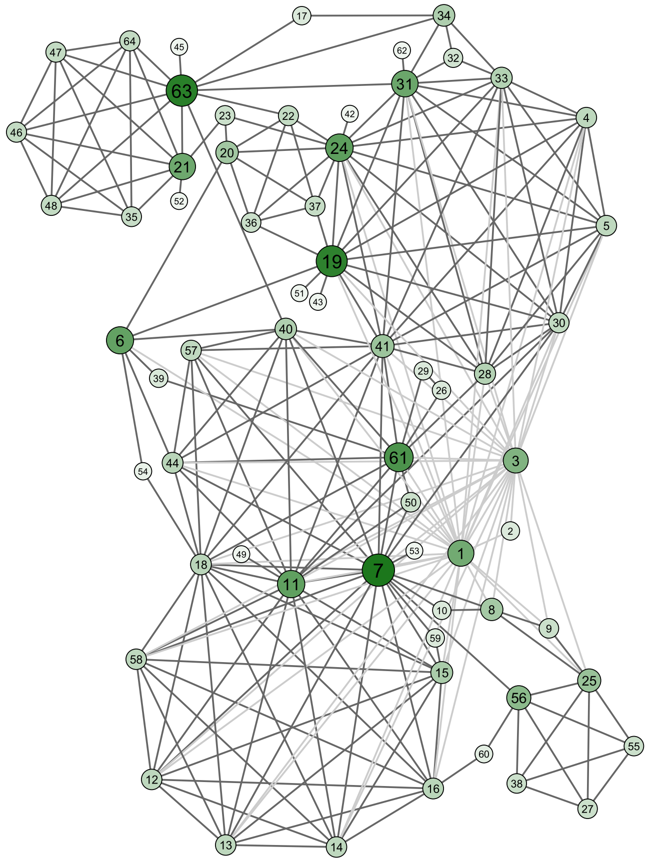

Figure 2: *Madrid network*. The node size reflects the Attachment Centrality (the larger the node the greater its centrality). To highlight the differences even further, the node color is set to reflect the node size (the larger the node the darker the color).

top by Degree and Closeness Centralities. Interestingly, Betweenness Centrality also gives them very high (the second and the third) positions while Attachment Centrality ranks node 1 as the tenth and node 3 even lower. Such a substantial difference between Betweenness and Attachment centralities is quite surprising, given that nodes that connect different parts of the network are more likely to belong to shortest paths than other nodes. However, nodes 1 and 3 have so many neighbors that they are very often parts of the shortest paths in the network, more often than nodes important from the connectivity perspective.

The running time of our algorithm depends on the topology of the graph. In the case of the Madrid network, the largest subgraph for which the Myerson value had to be calculated in line (11) was a subgraph consisting of 26 nodes. The running time on the entire network was 15.01 seconds on a standard desktop PC.

Our analysis revealed a previously unknown aspect of the Madrid network. In particular, we identified several overlapping parts of the network that are almost fully connected, i.e., each part is almost a clique, except for very few missing edges. This new insight confirms the existing belief, that *terrorist networks consist of rather sparsely-connected, highly-dense parts* [24].

6. RELATED WORK

A number of game-theoretic centrality indices have been recently proposed in the literature. In particular, Suri and Narahari [17] proposed an extension of degree centrality defined as the Shapley value of the game $f(S) = |\bigcup_{v \in S} neighbors(v)|$. Michalak et al.

Algorithm 1: Algorithm for the Attachment Centrality

Input: Graph $G = (V, E)$
Output: Attachment Centrality A_v for every $v \in V$

```
1  find the simple elimination ordering π;
2  create a cut-clique decomposition T of graph G from π;
3  f ← function defined as f(C) = 2(|C| − 1);
4  t ← root of T;
5  while t has children do
6      (l, r) ← children of t (left one without children);
7      (L, R) ← labels of l, r;
8      K ← L ∩ R;
9      foreach v ∈ L do
10         calculate MV_v(f, G[L]);
11         A_v ← A_v + MV_v(f, G[L]);
12         if v ∈ K then A_v ← A_v − 2 + 2/|K|;
13     t ← r;
14 return A_v for every v ∈ V;
```

Rank	A_v		B_v	C_v	D_v
1^{st}	7	(Imad Eddin Barakat)	63	1	1
2^{nd}	63	(Semaan Gaby Eid)	1	3	3
3^{rd}	19	(Abderrahim Zbakh)	3	41	7
4^{th}	61	(Mohamed El Egipcio)	40	7	11
5^{th}	24	(Naima Oulad Akcha)	7	31	41
6^{th}	11	(Amer Azizi)	31	40	18
7^{th}	6	(Mohamed Chedadi)	24	24	24

Table 1: The seven highest ranked nodes in the Madrid network, according to different centrality indices.

[14] considered a number of generalizations of this game. All these measures do not satisfy Fairness and Normalization. Also, if we consider their normalized version then they do not satisfy Locality.

To expose the connectivity role of a node, several authors proposed indices that are based on the Myerson value. Skibski et al. [22] considered several characteristic functions (e.g., $f(S) = |S|^2$ or $f(S) =$ number of edges in $G[S]$) combined with the graph-restrictions from Myerson's model. Depending on the function used, the resulting centrality measures do not satisfy Normalization nor Fairness (note that when the function f used in the Myerson value is based on the graph, Fairness may be violated).

A slightly different model (compared to Myerson's) was proposed by Amer and Gimenez [2], whereby the centrality of a node is the Shapley value of the following function: $f(S) = 1$ if $G[S]$ is connected and $f(S) = 0$ otherwise. This was later expanded by Lindelauf et al. [13] to an arbitrary $f(S)$ when $G[S]$ is connected. The resulting centrality measure does not satisfy Locality, since all centrality indices equal zero in a network with two disjoint parts.

7. CONCLUSIONS

While there were some attempts in the literature to provide theoretical foundations to the standard centrality indices [20, 12, 6], our analysis is the first in the literature that proposes an axiomatization of an index focusing on connectivity.

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