Coping with Hardness of Welfare Maximization by Introducing Useful Complexity Measures

(Doctoral Consortium)

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ABSTRACT

We introduce complexity measures for set functions and apply them to the welfare maximization problem.

1. INTRODUCTION

Consider the following fundamental problem. We have a set of players and a set of indivisible items. Each player has a valuation set function, giving a value to every possible subset of the items. Our aim is to allocate the items to the players, while maximizing the sum of values of the items the players get, by their personal valuation functions. This problem is called THE WELFARE MAXIMIZATION PROBLEM (also known as "combinatorial auctions") and it has been vastly researched. Unfortunately, it is $\mathcal{NP}$-hard. Moreover, there are lower bounds excluding the possibility of having reasonable approximation guarantees for this problem. One possible approach to cope with this hardness is to restrict the input (e.g. to submodular valuation functions). For some restrictions, the welfare maximization problem is known to admit constant approximation guarantees. However, this approach has an obvious disadvantage; the problem is not promised to be solved with any guarantee (or at least not with an acceptable one) when it does not obey the restriction. It might be most frustrating if an instance seems to be really close to obey the restriction, but however, slightly disobeys it. Another possible approach is to find approximation algorithms, without restricting the problem, but instead, to have approximation guarantees that are proportional to some *complexity measure* of the instances. Roughly speaking, this means to try having some "good" approximation guarantee in some restricted case, approximation guarantee slightly worse for instances that are close to belong to this restricted case, and generally, approximation guarantees with decreasing quality for increasing complexity of instances. The latter approach is the one we study. In order to formalize it, the following fundamental questions should be answered: "What does it mean that an instance is "close" to another instance?" "What does it mean that an instance is "more complex" than another?" and more generally: "How can we measure the "complexity" of instances?" The latter (general) question is formalized by the notion of *complexity measures* of instances of optimization problems. Specifically, a complexity measure for an optimization problem P with a set of possible instances $\mathcal{I}(P)$ is a function $\mathcal{C} : \mathcal{I}(P) \to \mathbb{N}$. Indeed, there are typically infinitely many such functions (since there are typically infinitely many instances of an optimization problem), and it seems to not be necessarily true that each of the measures is meaningful for any optimization problem. But, we aim to find complexity measures that are:

Natural: One can typically intuitively understand what is the meaning of a value given to an instance of P.

Useful: There exists an algorithm with approximation guarantees proportional to the value of the measure for each instance of P, which improves at least some of the currently known guarantees.

The research we have already done includes introducing new complexity measures and designing specific algorithms

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for them for the welfare maximization problem. Our measures apply to any valuation function, and are not specific for the welfare maximization problem.

2. OUR COMPLEXITY MEASURES

The first complexity measure we introduced is the supermodular degree [5]. The supermodular degree measures the distance of a valuation function from being submodular. We briefly describe it. Recall that submodular functions have non-increasing marginal values. That is, a valuation function $f : 2^M \to \mathbb{R}^+ \cup \{0\}$ is submodular, if for every item $j \in M$ and subsets of items $S \subseteq T \subseteq M$, we have that $f(j \mid S) \geq f(j \mid T)$, where $f(j \mid X)$ is the marginal value of j given X: $f(j \mid X) \stackrel{\text{def}}{=} f(\{j\} \cup X) - f(X)$. On the other hand, general valuation functions can admit increasing marginals. That is: $f(j \mid S) < f(j \mid S \cup \{j'\})$. Moreover, the latter is realistic in a welfare maximization setting (as an example, one can think of a battery charger of a phone, with respect to the phone). We see the latter phenomenon as synergy between items, and define the supermodular degree as the maximum number of items that a single item may have synergy with. In particular, this means that submodular valuation functions have supermodular degree of 0, and generally, a valuation function over a set of items M can have supermodular degree of up to $|M| - 1$. Our applications for the supermodular degree include an approximation algorithm for the welfare maximization problem [5], an approximation algorithm for a generalization of the welfare maximization problem [6], an algorithm (and a new model) for a secretary like problem that captures an online setting of welfare maximization [7], and a voting rule that is based on the supermodular degree [8]. Another work in progress studies an adversarial online welfare maximization problem [9]. Our approximation guarantees deteriorate linearly with the supermodular degree in the offline settings and polynomially in the online settings.

Another complexity measure that we introduced is MPH (Maximum over Positive Hypergraphs) [4]. The definition of this measure relies on representing a valuation function by a hypergraph; see [1, 2, 3]. Given a hypergraph with a set of vertices V and a set of weighted hyperedges E, we can see it as a valuation function on the set of items (vertices) V, where the value of a subset $S \subseteq V$ is the sum of weights of hyperedges in the subgraph induced by S. A positive hypergraph valuation function is a valuation function with a hypergraph representation with only non-negative hyperedges. A k-positive hypergraph valuation function is a positive hypergraph valuation function with positive edges of rank at most k. We say that a valuation function f is in $MPH-k$, if there exists a set of k-positive hypergraph valuation functions $\mathcal{F}$, such that for every subset of items S, $f(S) = \max_{f' \in \mathcal{F}} f'(S)$. $MPH-1$ is actually the well known XOS class, and any set function over a set of items M is in $MPH-|M|$. Our applications include an approximation algorithm for the welfare maximization problem with approximation guarantee that deteriorates linearly with MPH.

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