Weighted Matching Markets with Budget Constraints

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ABSTRACT

We investigate markets with a set of students on one side and a set of colleges on the other. A student and college can be linked by a weighted contract that defines the student's wage, while a college's budget for hiring students is limited. Stability is a crucial requirement for matching mechanisms to be applied in the real world. A standard stability requirement is coalitional stability, i.e., no pair of a college and group of students has incentive to deviate. We find that a coalitionally stable matching is not guaranteed to exist, verifying the coalitional stability for a given matching is coNP-complete, and the problem to find whether a coalitionally stable matching exists in a given market, is $\mathrm{NP}^{\mathrm{NP}}$-complete (that is Σ_2^P-complete). Given these computational hardness results, we pursue a weaker stability requirement called pairwise stability, i.e., no pair of a college and single student has incentive to deviate. We then design a strategy-proof mechanism that works in polynomial-time for computing a pairwise stable matching in typed markets in which students are partitioned into types that induce their possible wages.

Keywords

Matching, Complexity, Mechanism design

CCS Concepts

•Computing methodologies → Multi-agent systems;

1. INTRODUCTION

Investigation into two-sided matchings began with Gale and Shapley [12], who introduced the college admissions problem. Since then, the theory of two-sided matching and its application to real-life problems have been extensively developed in the literatures of economics, artificial intelligence and multi-agent systems [13, 25]. The problems of matching students to schools [2, 3, 4], doctors to hospitals [27, 28], and military cadets to army branches [30, 31] are important formal settings that have been considered. Central notions are pairwise and coalitional stability of a matching, which should be immune to deviations by a pair or group of agents. Also, a mechanism must be strategy-proof: there should be no incentive for students[1] to misreport their preferences.

The presence of maximum quotas (i.e. capacity limit of a college) is assumed in most standard models. In real-life examples, there are different kinds of distributional constraints other than maximum quotas [7, 18], and in recent years, various types of distributional constraints have been addressed and a series of mechanisms have been introduced to achieve desirable outcomes under such constraints [10, 11, 14, 21, 22]. In this paper, we revisit the standard distributional constraint of the maximum quotas, by assuming that each college has a fixed amount of resource, or budget, that can be distributed among students; and by assuming that students may receive a different amount. The amount may differ among different types of students (e.g., tuition of a state university in the US is lower for local students), a student may be allocated a different amount of resource, depending on the contract she made (e.g., full scholarship or partial scholarship), or both. In our model, we explicitly take into account the total amount of resources of each school and possible amount a student may receive. Therefore, we model a *weighted matching market with budget constraints*.

Although our model is a natural extension of the standard maximum quotas, there have been very few literature that have addressed this issue, possibly due to its intractability: there are two conditions from the literature, substitutability and the law of aggregate demand, that make an analysis tractable [16], but neither is satisfied in our model. The most relevant work [5] studies college admissions with budget constraints, in which a student receives a college-stipend pair, and develops a strategy-proof mechanism that satisfies a weaker notion of stability. The major differences between this model and ours are that we deal with general ordinal (instead of quasi-linear) preferences of students, which becomes possible since we focus on discrete sets of wages (instead of a continuum), and we allow different types of students to be in a market (instead of assuming all students are the same type). Another relevant work [24] shows that the core can be empty in a job market with budge constraints. We cannot apply their result to our model since they assume the utility of each school/firm is quasi-linear, while in our model, each school is indifferent about the amount of money it spends as long as it is below the budget limit. The environment of grouping students into types has also been studied in the literature of school choice problem [1, 4, 9, 20, 34].

We also address computational issues related to verifying/finding a stable matching. As far as we know, we are

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[1]For the sake of presentation, the rest of this paper is described in the context of a college-student matching problem.

the first to address these issues in two-sided matching with budget constraints. We show that coalitional stability in matchings with budget constraints involves a larger complexity class than NP in the polynomial hierarchy [23]. According to a compendium of problems (updated in 2008), there are not many Σ_2^P-complete (that is $\mathrm{NP}^{\mathrm{NP}}$-complete) problems involving numbers [29]. The $\forall\exists$SubsetSum problem that we introduce (as a mid-step in our reduction) is new. This compendium does not reflect the more recent progresses in algorithmic game theory. The complexity of coalitional stability has been studied in several related models, in which checking is often coNP-complete and deciding coalitional stability is also Σ_2^P-complete. For instance, this is the case in additively separable hedonic games [32, 33] or for envy freeness (and Pareto efficiency) [6] and in resource allocation [8]. Furthermore, NP-completeness of the problem of deciding whether there exists a stable outcome has been proved in matching problems with couples [26] and matching problems with minimum quotas [7, 17].

The contributions of this paper are twofold. First, we investigate the computational issues regarding the coalitional stability of a matching. We find negative results that (1) there may not exist a coalitionally stable outcome, (2) checking whether a given matching is coalitionally stable is coNP-complete, and (3) it is $\mathrm{NP}^{\mathrm{NP}}$-complete to decide whether there exists a coalitionally stable matching. Therefore, coalitional stability is a notion that is very difficult to obtain. Secondly, following the above results, we focus on pairwise stability, a weaker notion that involves only a pair of a student and a college. For a student, finding a profitable deviation as a group involving other students would be difficult. Thus, we assume eliminating such a deviation is important in practice. In the presence of budget constraint, substitutability, a sufficient condition for the existence of a stable matching [15], is not guaranteed to hold. However, in a typed weighted market, in which students are grouped into several types, we show that there always exists a pairwise stable matching by developing a strategy-proof mechanism that finds such a matching in polynomial-time.

2. MODELS

Here, we present our two-sided weighted matchings models, the most general being *weighted markets*. *Simply weighted markets* and *typed weighted markets* will be particular cases.

Definition 1. A *weighted market* is formally defined by a tuple $\pi = (S, C, W, X, b_C, \succ_S, \tilde{\succsim}_C)$, where:

- $S = \{s_1, \ldots, s_n\}$ is a set of *students*.
- $C = \{c_1, \ldots, c_m\}$ is a set of *colleges*.
- $W = \{w_1, \ldots, w_p\}$ are non-negative integer *wages*.
- $X \subseteq \{(s, c, w) \mid s \in S, c \in C, w \in W\}$ is a set of possible *contracts* where contract $x = (s, c, w)$ means that student s is assigned to college c with wage w.
- $b_C = (b_c \in \mathbb{N}_+)_{c \in C}$ is a profile of colleges' *budgets*.
- $\succ_S = (\succ_s)_{s \in S}$ is a profile of student preferences $\succ_s$ over college-wage couples $C \times W$ and an additional couple $(c_\emptyset, 0)$ which means that she stays home with no wage[2]. We assume that $w > w'$ implies $(c, w) \succ_s (c, w')$.
- $\tilde{\succsim}_C = (\tilde{\succsim}_c)_{c \in C}$ is a profile of college weak preferences over sets $S' \subseteq 2^S$ of students. Each college weak preference $\tilde{\succsim}_c$ is based on a weak preference $\succsim_c$ over students and null student $s_\emptyset$. A weak preference $\succsim$ partitions into asymmetric part $\succ$ and symmetric part $\sim$. Here, $s \succ_c s'$ means college c strictly prefers s over s' and $s \sim_c s'$ means c is indifferent between s and s'. We assume college preferences satisfy *responsiveness*; that is, for every pair of students $s, s' \in S$ and subset of students $S' \subseteq S \setminus \{s, s'\}$, it holds that:
$$s \succsim_c s' \quad \Leftrightarrow \quad S' \cup \{s\} \;\tilde{\succsim}_c\; S' \cup \{s'\}$$
which also implies $s \succ_c s' \Leftrightarrow S' \cup \{s\} \;\tilde{\succ}_c\; S' \cup \{s'\}$, since $\succ_c$ and $\tilde{\succ}_c$ are asymmetric parts[3]. Also, for every subset of students $S' \subsetneq S$ and student $s \in S \setminus S'$,
$$s \succsim_c s_\emptyset \quad \Leftrightarrow \quad S' \cup \{s\} \tilde{\succsim}_c S'$$
holds, which similarly implies $s \succ_c s_\emptyset \Leftrightarrow S' \cup \{s\} \tilde{\succ}_c S'$.

We say a market is *simply weighted* if, between every student college couple, there exists at most one possible wage. In this simpler case, the set of contracts can be represented by a bipartite graph between students on one side and colleges on the other side, while each possible student-college edge is weighted by the corresponding wage. Therefore, in simply weighted markets, notation w can be abused in a functional manner $w : S \times C \to W$ where $w(s, c) \in W$ is the wage that student s receives for going to college c and function w is only defined on couples (s, c) for which there is a contract.

Furthermore, in the general setting, the market could be represented by a bipartite multigraph between students and colleges and possibly multiple edges between each student-college pair, corresponding to their possible contracts. The functional abuse for wages would be $w : S \times C \to 2^W$.

Weighted markets also admit *typed weighted markets* as a particular case in which students are partitioned into types.

- $\Theta = \{\theta_1, \ldots, \theta_k\}$ is a finite set of *student types*.
- Function $\tau : S \to \Theta$ maps each student to its type. We assume for students $s, s' \in S$ such that $\theta_i = \tau(s)$, $\theta_j = \tau(s')$, $i < j$, if $s \succ_c s_\emptyset$ and $s' \succ_c s_\emptyset$ hold, then $s \succ_c s'$ holds. In words, as long as college c thinks both students s and s' are strictly better than $s_\emptyset$, c always prefers the student with the higher type.
- Set W is represented as $W = \bigcup_{c \in C} \bigcup_{\theta \in \Theta} W_{c,\theta}$, where $W_{c,\theta}$ is the set of wages that college c can give to the students of type θ. Formally, for all $s \in S$, for all $c \in C$ such that $s \succ_c s_\emptyset$, and for all $w \in W$, $(s, c, w) \in X$ holds if and only if $w \in W_{c,\tau(s)}$ holds. We assume types are ordered in the following sense. Given a college c, for every $w \in W_{c,\theta_i}$ and $w' \in W_{c,\theta_{i+1}}$, one has $w > w'$.

Definition 2. A *typed weighted market* is defined by a tuple $\pi = (S, C, \Theta, \tau, (W_{c,\theta})_{c \in C, \theta \in \Theta}, X, b_C, \succ_S, \tilde{\succsim}_C)$.

For instance, one may realistically consider the job market of young researchers in which student types are graduate, young doctorate, experienced doctorate. Each college c proposes a set of possible wages $W_{c,\theta}$ to each type of student θ. It is easy to see that typed weighted markets are a particular case of weighted markets, in which we require additional constraints on possible wages and colleges' preferences. For instance, the typed weighted market in Figure 1 amounts to a weighted market with contracts $X =$

[2]This definition allows for preference $(c, 2) \succ_s (c_\emptyset, 0) \succ_s (c, 1)$; the wage matters for the feasibility of the same college.

[3]The same holds with symmetric parts $\sim_c$ and $\tilde{\sim}_c$.

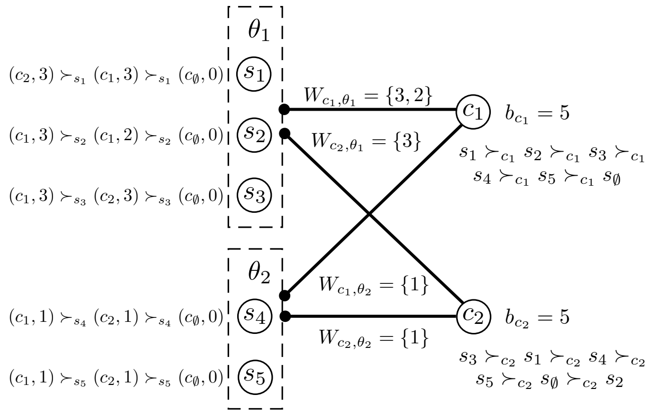


Figure 1: Example of typed-weighted market with two types of students.

$\{(s_1,c_1,3),(s_1,c_1,2),(s_1,c_2,3),(s_2,c_1,3),(s_2,c_1,2),(s_3,c_1,3),(s_3,c_1,2),(s_3,c_2,3),(s_4,c_1,1),(s_4,c_2,1),(s_5,c_1,1),(s_5,c_2,1)\}$.

In our model, to choose an optimal subset of contracts, we need to know $\tilde{\succsim}_c$. For example, assume $s_1 \succ_c s_2 \succ_c s_3$ holds and $b_c = 2$. When choosing an optimal subset within $\{(s_1,c,2),(s_2,c,1),(s_3,c,1)\}$, we cannot tell whether college c prefers $\{(s_1,c,2)\}$ or $\{(s_2,c,1),(s_3,c,1)\}$ without $\tilde{\succsim}_c$. Obtaining $\tilde{\succsim}_c$ is difficult since it is a preference over 2^n sets, unless it can be concisely represented.

For instance, for each college c, weak preference $\tilde{\succsim}_c$ can be represented by an *additively separable utility*; that is, a utility function $u_c : S \to \mathbb{Z}$ that additively extends to sets $S' \subseteq S$ of students by $u_c(S') = \sum_{s \in S'} u_c(s)$. Hence, given two sets of students $S', S'' \in 2^S$, the preference $S' \tilde{\succ}_c S''$ holds if and only if the inequality $u_c(S') > u_c(S'')$ also does, and the indifference $S' \tilde{\sim}_c S''$ holds if and only if one has equality $u_c(S') = u_c(S'')$. This defines a weak preference $\tilde{\succsim}_c = \tilde{\succ}_c \cup \tilde{\sim}_c$. Null-student has utility $u_c(s_\emptyset) = 0$. An additively separable utility satisfies responsiveness. Also, it is an instance of a simply weighted market.

2.1 Matching

Given a contract $x \in X$, let (x_S, x_C, x_W) respectively denote the student, college, and wage that are linked by contract x. Given a subset of contracts $Y \subseteq X$, let us denote the set of contracts of student $s \in S$ as $Y_s = \{x \in Y \mid x_S = s\}$ and the set of contracts of college $c \in C$ as $Y_c = \{x \in Y \mid x_C = c\}$.

Definition 3. A *matching* is a subset of contracts $Y \subseteq X$ where each student s goes to at most one college[4]: $|Y_s| \leq 1$.

Given a matching $Y \subseteq X$, we abuse notation Y in a natural *functional* manner as follows. Let $Y(s) \in (C \times W) \cup \{(c_\emptyset, 0)\}$ denote the college (or home $c_\emptyset$) to which student s is assigned and the corresponding wage. Let $Y(c) \subseteq S$ denote the set of students assigned to college c.

Definition 4. A contract (s,c,w) is *feasible* if $(c,w) \succ_s (c_\emptyset, 0)$ and $s \succsim_c s_\emptyset$. A matching Y is *student-feasible* for student s if Y_s is feasible. A matching Y is *college-feasible* for college c if all students in Y_c are feasible, and if the sum of the wages is budget feasible: $\sum_{x \in Y_c} x_W \leq b_c$. A *feasible matching* $Y \subseteq X$ is a matching which is student-feasible for each student and college-feasible for each college.

[4]The students with no contract stay home.

Without loss of generality, we assume for each contract $(c,s,w) \in X$, $s \succsim_c s_\emptyset$ holds.

2.2 Stability

A pairwise stable matching is immune to pairwise deviations by blocking pairs.

Definition 5. For a matching Y, we say $(s,c) \in S \times C$ is a blocking pair if there exists $w \in W$ and $R \subseteq Y_c$ such that $(s,c,w) \in X \setminus Y$ and the following conditions hold:
1. $(c,w) \succ_s Y(s)$,
2. $(Y(c) \setminus R(c)) \cup \{s\} \tilde{\succ}_c Y(c)$, and
3. $\sum_{x \in Y_c \setminus R} x_W + w \leq b_c$.

In words, (s,c) is a blocking pair if s prefers (c,w) over her current contract, c is willing to reject the subset of its contracts R in order to accept s, and doing so satisfies its budget constraint.

Definition 6. We say a feasible matching Y is *pairwise stable* if it does not admit any blocking pair.

Similarly, a coalitionally stable matching is immune to coalitional deviations, as it does not admit any.

Definition 7. For a matching Y, we say $(S', c) \in 2^S \times C$ is a blocking coalition if there exists $w_s \in W$ for each $s \in S'$ and $R \subseteq Y_c$ such that $(s,c,w_s) \in X \setminus Y$ and the following conditions hold:
1. $\forall s \in S', (c, w_s) \succ_s Y(s)$,
2. $(Y(c) \setminus R(c)) \cup S' \tilde{\succ}_c Y(c)$, and
3. $\sum_{x \in Y_c \setminus R} x_W + \sum_{s \in S'} w_s \leq b_c$.

In words, (S', c) is a blocking coalition if each $s \in S'$ prefers (c, w_s) over her current contract, c is willing to reject the subset of its contracts R in order to accept S', and doing so satisfies its budget constraint.

Definition 8. We say a feasible matching Y is *coalitionally stable* if it does not admit any blocking coalition.

From the above definition, if Y is coalitionally stable, it is also pairwise stable, but not vice versa.

3. THE COMPLEXITY OF COALITIONAL STABILITY IN WEIGHTED MARKETS

In the field of computational complexity, a *decision problem* is modeled by an infinite set of instances and by a question that maps each instance to *yes* or *no*. The answer is the desired output. In this section, we assume additively separable utilities for colleges, so that condition 2 in Definitions 5 and 7 are rewritten with sums. First, we observe that a coalitionally stable matching is not guaranteed to exist in every weighted market. This fundamental observation lets us introduce *the coalitional stability in weighted market (CSWM) problem* to decide whether a given weighted market admits (yes or no) a coalitionally stable matching.

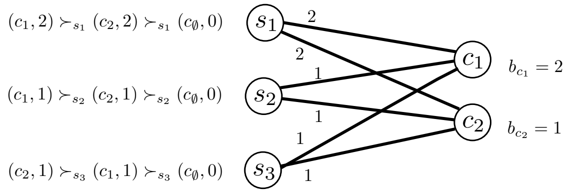
For verifications in the CSWM problem, we then address *the CSWM|Y problem* to decide whether in a given weighted market, a given matching is coalitionally stable. We show that the CSWM|Y problem is coNP-complete. Hence, verification for CSWM does not seem polynomial-time tractable and CSWM is likely to fall outside of NP and coNP.

Ultimately, we show that (indeed) the CSWM problem is $\mathrm{NP}^{\mathrm{NP}}$-complete. Therefore, coalitional stability is a computationally very hard requirement in weighted markets.

3.1 A coalitionally stable matching is not guaranteed to exist.

Theorem 1. *There exists a case in which no coalitionally stable matching exists.*

Example 1. Consider this simply weighted market in which each possible contract is represented by a weighted edge.



Possible contracts are $X = \{(s_1, c_1, 2), (s_1, c_2, 2), (s_2, c_1, 1), (s_2, c_2, 1), (s_3, c_1, 1), (s_3, c_2, 1)\}$ and the preference of each college c is $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_\emptyset$ and extends to:

$$\ldots \tilde{\succ}_c \{s_2, s_3\} \tilde{\succ}_c \{s_1\} \tilde{\succ}_c \{s_2\} \tilde{\succ}_c \{s_3\} \tilde{\succ}_c \emptyset.$$

Such a college preference could be obtained by additively separable utility $u_c(s_1) = 4, \ \ u_c(s_2) = 3, \ \ u_c(s_3) = 2$.

This example can also be modeled as a typed weighted market, where $\Theta = \{\theta_1, \theta_2\}$, $\tau(s_1) = \theta_1$, $\tau(s_2) = \tau(s_3) = \theta_2$, and for every college c, one has: $W_{c,\theta_1} = \{2\}$, $W_{c,\theta_2} = \{1\}$.

Proof. We discuss all possible matchings Y of Example 1. Due to budget constraint, s_1 cannot be assigned to c_2.
case 1 $Y(c_1) = \emptyset$: (s_1, c_1) or (s_2, c_1) blocks Y.
case 2 $Y(c_1) = \{s_2\}$ or $Y(c_1) = \{s_3\}$: (s_1, c_1) blocks Y.
case 3 $Y(c_1) = \{s_2, s_3\}$: (s_3, c_2) blocks Y.
case 4 $Y(c_1) = \{s_1\}$, $Y(c_2) \neq \{s_2\}$: (s_2, c_2) blocks Y.
case 5 $Y(c_1) = \{s_1\}$,$Y(c_2) = \{s_2\}$: $(\{s_2, s_3\}, c_1)$ blocks Y.
Since every possible matching admits a blocking coalition, there is no coalitionally stable matching in Example 1. □

3.2 Reminders on computational complexity

Class P (polynomial-time) corresponds to the decision problems that can be answered in polynomial-time. Traditionally, we regard these problems as *easy* or *tractable*.

Class NP (non-deterministic polynomial-time) corresponds to the set of decision problems which 'yes'-instances have a certificate verifiable in polynomial-time. For instance, consider the SubsetSum problem: given a target $\alpha \in \mathbb{N}$ and a set $\mathcal{S} = \{w_1, \ldots, w_n\}$ of weights, the question asks whether there exists a subset of items $\mathcal{T} \subseteq \mathcal{S}$ that satisfies the constraint $\sum_{w \in \mathcal{T}} w = \alpha$ (hits the target). For 'yes'-instances, providing such a solution is an easy-to-check yes certificate[5], hence the SubsetSum problem is in NP.

Complementation consists in transposing the yes and no answers, e.g., the coSubsetSum problem asks whether $\forall \mathcal{T} \subseteq \mathcal{S}, \ \sum_{w \in \mathcal{T}} w \neq \alpha$. The 'no'-instances are polynomial-time verifiable. This defines the problems of *class coNP*.

Furthermore, the SubsetSum problem is known to be part of the most difficult problems of class NP, for which a polynomial-time algorithm is suspected not to exist. Indeed, SubsetSum is *NP-complete* [19]:

1. it is in NP,
2. it is *NP-hard* in the sense that every problem in NP can be reduced in polynomial time to SubsetSum.

[5]Guessing subset $\mathcal{T}$ is the non-deterministic part.

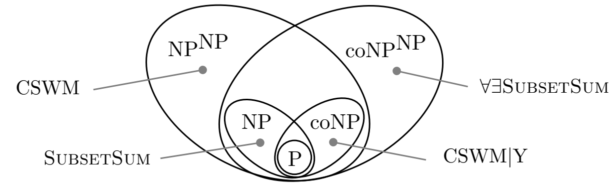


Figure 2: Inclusions of decision problem classes.

Hence, the existence of a polynomial-time algorithm for SubsetSum would imply P=NP, which is assumed wrong and argues for the intractability of SubsetSum. Similarly, one can show that a problem is *coNP-complete* by proving that it is in coNP and that it is the complement of an NP-hard problem, since NP and coNP are symmetric classes.

For some decision problems, neither yes nor no certification is polynomial-time tractable. In that case, the problem falls outside of NP and coNP. *Class NP^{NP}* corresponds[6] to the decision problems in which 'yes'-instances have proofs verifiable in polynomial time by using a constant-time NP-oracle. *Class $coNP^{NP}$* is its complement. For instance, let us introduce the following new decision problems:

Definition 9. Given a target $\alpha \in \mathbb{N}$ and two multi-sets of integers $\mathcal{S}^\forall$ and $\mathcal{S}^\exists$, the $\forall\exists$SubsetSum problem asks whether

$$\forall \mathcal{T}^\forall \subseteq \mathcal{S}^\forall, \quad \exists \mathcal{T}^\exists \subseteq \mathcal{S}^\exists, \quad s.t. \quad \sum_{w \in \mathcal{T}^\forall} w + \sum_{w \in \mathcal{T}^\exists} w = \alpha.$$

Conversely, the $\exists\forall$SubsetSum problem asks whether formula $\exists \mathcal{T}^\forall \subseteq \mathcal{S}^\forall, \forall \mathcal{T}^\exists \subseteq \mathcal{S}^\exists, \sum_{w \in \mathcal{T}^\forall} w + \sum_{w \in \mathcal{T}^\exists} w \neq \alpha$ is true. The latter is simply the complement of the former.

The $\exists\forall$SubsetSum problem lies in class $\mathrm{NP}^{\mathrm{NP}}$. Indeed, by guessing the right set $\mathcal{T}^\forall$, one can use the NP-oracle to solve the remaining coSubsetSum problem and verify the 'yes' answer. Similarly, $\forall\exists$SubsetSum is in class $\mathrm{coNP}^{\mathrm{NP}}$. *Completeness* is defined in a standard manner with polynomial-time reductions. Showing that problem $\forall\exists$SubsetSum is $\mathrm{coNP}^{\mathrm{NP}}$-complete will be a middle step in the proof below.

3.3 Complexity of verification

We now address the complexity of a classical yes verification. The CSWM|Y problem, given a weighted market $\pi = (S, C, W, X, b_C, \succ_S, \tilde{\succsim}_C)$ and a feasible matching Y, asks whether Y is (yes or no) coalitionally stable.

Theorem 2. *The CSWM|Y problem is coNP-complete, even for a simply weighted market with only one college that has an additively separable utility.*

Proof. First, the CSWM|Y problem is in coNP, since providing a blocking coalition (T, c) is a no-certificate that can be verified in polynomial-time. Secondly, the complement of CSWM|Y (which answers 'yes' if there is a blocking coalition) is NP-hard, as we reduce SubsetSum to co-CSWM|Y.

Let set $\mathcal{S} = \{w_1, \ldots, w_n\}$ and target $\alpha \in \mathbb{N}$ be an instance of SubsetSum. We construct in polynomial-time the following CSWM|Y instance addressing it. In this simply weighted market, there are students $S = \{s_1, \ldots, s_n, s_\alpha\}$ and one college c. College c has budget α. The wages and utilities are the same $w(c, s_i) = u_c(s_i) = w_i$ for $1 \leq i \leq n$ and $w(c, s_\alpha) = u_c(s_\alpha) = \alpha - 1/2$ for the last student[7]. The pref-

[6]Class Σ_2^P in the second level of the *Polynomial Hierarchy.*
[7]To have only integers, as in the model, one might multiply all numbers by 2 and obtain a strategically equivalent market, or allow for half integers in the model.

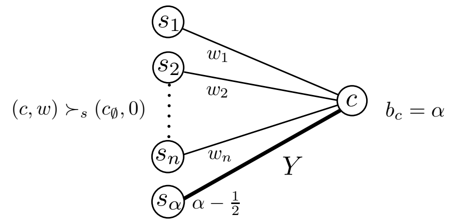


Figure 3: Reducing SubsetSum to co-CSWM|Y.

erences of students are to go to college c rather than going home. The preference of the college is to maximize its utility, which here precisely corresponds to maximize its budget consumption. In the given matching $Y = \{(s_\alpha, c, \alpha - 1/2)\}$, student s_α goes to college c, and all other students go home. This reduction is depicted in Figure 3. The college's budget consumption corresponds to its utility. A blocking coalition exists if and only if a subset of items hits target α.

If there is a subset $\mathcal{T} \subseteq \mathcal{S}$ which hits the target α, then there is the corresponding blocking coalition (T, c) which would improve the college's interest from $\alpha - 1/2$ to α. If no subset of items hits target α, then no feasible coalition of students is better for college c than $u_c(s_\alpha) = \alpha - 1/2$. □

3.4 The complexity of coalitional stability

The previous subsection suggests that certification is hard, hence that problem CSWM falls outside of NP and coNP. Indeed, it is even harder than NP-complete.

Theorem 3. *The CSWM problem is NP^{NP}-complete. It is even the case for a number of colleges in $O(1)$.*

Proof sketch. The CSWM problem is in $\mathrm{NP}^{\mathrm{NP}}$ since 'yes'-instances can be certified by this two steps meta-algorithm.

1. Guess a coalitionally stable matching Y.
2. By using the NP-oracle on the corresponding CSWM|Y instance, prove that Y is coalitionally stable.

For completeness in $\mathrm{NP}^{\mathrm{NP}}$, we equivalently show that problem coCSWM is $\mathrm{coNP}^{\mathrm{NP}}$-complete. The proof proceeds in two steps. First, we reduce the $\mathrm{coNP}^{\mathrm{NP}}$-complete problem ∀∃3CNF to problem ∀∃SubsetSum (Lemma 2). Second, we reduce problem ∀∃SubsetSum to problem coCSWM (Lemma 3), achieving the proof. □

Let $(\mathbb{B} = \{0, 1\}, \vee, \wedge, \neg)$ denote the usual *Boolean algebra*. Given a set of variables V, an *instantiation* $I : V \to \mathbb{B}$ maps each variable $v \in V$ to a Boolean value $I(v) \in \mathbb{B}$. Given a Boolean variable v, the *literals* it induces are $\{v, \neg v\}$. A *3-clause* is the disjunction of 3 literals. A Boolean formula is in *3 conjunctive normal form* (3CNF) if it is the conjunction of a set of 3-clauses.

Definition 10. An *instance of the* ∀∃3CNF *problem* is defined by two sets of Boolean variables $V^\forall, V^\exists$ $(V^\forall \cap V^\exists = \emptyset)$ and by a 3CNF formula ϕ defined as the conjunction of a set of 3-clauses C on the literals induced by $V^\forall \cup V^\exists$. It asks if

$$\forall I^\forall : V^\forall \to \mathbb{B}, \quad \exists I^\exists : V^\exists \to \mathbb{B}, \quad \bigwedge_{c \in C} c(I^\forall, I^\exists).$$

Example 2. Let $V^\forall = \{v_1, v_2\}$, $V^\exists = \{v_3, v_4\}$ and $\phi =$

$$\underbrace{(v_1 \vee \neg v_2 \vee \neg v_3)}_{c_1} \wedge \underbrace{(\neg v_1 \vee v_3 \vee \neg v_4)}_{c_2} \wedge \underbrace{(v_2 \vee v_3 \vee v_4)}_{c_3}.$$

Does for every instantiation of $\{v_1, v_2\}$, there exists an instantiation of $\{v_3, v_4\}$, such that formula ϕ is true?

Weights:		v_1	v_2	v_3	v_4	c_1	c_2	c_3	goes in:
$V^\forall$	w_{v_1}	1	0	0	0	1	0	0	$\mathcal{S}^\forall$
	$w_{\neg v_1}$	1	0	0	0	0	1	0	$\mathcal{S}^\exists$
	w_{v_2}	0	1	0	0	0	0	1	$\mathcal{S}^\forall$
	$w_{\neg v_2}$	0	1	0	0	1	0	0	$\mathcal{S}^\exists$
$V^\exists$	w_{v_3}	0	0	1	0	0	1	1	$\mathcal{S}^\exists$
	$w_{\neg v_3}$	0	0	1	0	1	0	0	$\mathcal{S}^\exists$
	w_{v_4}	0	0	0	1	0	0	1	$\mathcal{S}^\exists$
	$w_{\neg v_4}$	0	0	0	1	0	1	0	$\mathcal{S}^\exists$
slack	w_{c_1}	0	0	0	0	1	0	0	$\mathcal{S}^\exists$
	$w_{c'_1}$	0	0	0	0	1	0	0	$\mathcal{S}^\exists$
	w_{c_2}	0	0	0	0	0	1	0	$\mathcal{S}^\exists$
	$w_{c'_2}$	0	0	0	0	0	1	0	$\mathcal{S}^\exists$
	w_{c_3}	0	0	0	0	0	0	1	$\mathcal{S}^\exists$
	$w_{c'_3}$	0	0	0	0	0	0	1	$\mathcal{S}^\exists$
	α	1	1	1	1	3	3	3	Target

Table 1: Reducing Example 2 to ∀∃SubsetSum: each line represents a weight; last line is the target.

Lemma 1. *Problem* ∀∃3CNF *is $coNP^{NP}$-complete [23].*

Problem ∀∃3CNF is prototypical for the second level of the polynomial hierarchy, as it uses two groups of quantifiers.

Lemma 2. *Problem* ∀∃SubsetSum *is $coNP^{NP}$-complete.*

Proof of Lemma 2. We will encode a given instance of problem ∀∃3CNF into the numerical weights and target of a ∀∃SubsetSum instance. It helps to represent the reduction as in Table 1 where each line represents a weight and each column is a component of the weight in some base B. Each variable and each 3-clause indices a column; so there are $|V^\forall \cup V^\exists| + |C|$ columns. To never have overflows in any addition of weights, the numbers are represented in a base B which is large enough, so that each column has to precisely sum to the same column of the target to satisfy Equation $\sum_{w_i \in \mathcal{T}^\forall} w_i + \sum_{w_j \in \mathcal{T}^\exists} w_j = \alpha$ from Definition 9. It is sufficient to take $B = 2(|V^\forall \cup V^\exists| + |C|) + 1$.

Intuitively, the quantified Boolean variables and their instantiations are precisely modeled by the following $2|V^\forall| + 2|V^\exists|$ weights and their quantifications. Two weights are associated to each variable v, one per induced literal: w_v and $w_{\neg v}$. Both have their variable-columns v equal to 1 and the other variable-columns equal to 0. Also, for column v, the target is set to 1; so that exactly one literal-weight per-variable will be in $\mathcal{T}^\forall \cup \mathcal{T}^\exists$. For universally quantified variables $v \in V^\forall$, exactly one weight (for instance w_v) goes in the universally quantified set of items $\mathcal{S}^\forall$ and the other (for instance $w_{\neg v}$) goes in the existentially quantified set of items $\mathcal{S}^\exists$, so that selecting a subset $\mathcal{T}^\forall \subseteq \mathcal{S}^\forall$ is equivalent to choosing an instantiation of $V^\forall$ and the same universal quantification is modeled. For existentially quantified variables $v \in V^\exists$, both weights go to the set of items $\mathcal{S}^\exists$.

For the clause columns, each clause that literal v (or $\neg v$) makes true is set to 1 and the others to 0. Then, in the column of clause c, the target would be that the clause is made true at least once. Note also that a clause cannot be made true more than 3 times. Consequently, we introduce slack-weights to reach target 3: for each clause c, we add 2 weights w_c and $w_{c'}$ with a 1 on clause-column c.

By construction, the ∀∃3CNF instance is a 'yes' one if and only if this ∀∃SubsetSum instance is also a 'yes' one. Moreover, the reduction is polynomial. □

Lemma 3. *Problem* $\forall\exists$SubsetSum *reduces to coCSWM.*

Proof of Lemma 3. Let integer multisets $\mathcal{S}^\forall = \{\ldots, w_i, \ldots\}$ and $\mathcal{S}^\exists = \{\ldots, w_j, \ldots\}$, and integer target $\alpha \in \mathbb{N}$ define an instance of $\forall\exists$SubsetSum that we reduce to the following instance of coCSWM. Recall that problem coCSWM asks whether for all matchings there exists a blocking coalition.

Without loss of generality, we rule out the case in which $\sum_{w_i \in \mathcal{S}^\forall} w_i > \alpha$. We introduce a number M that is large enough. From multisets $\mathcal{S}^\forall$, $\mathcal{S}^\exists$ of the $\forall\exists$SubsetSum instance, we make two sets of students $S^\forall$ and $S^\exists$ in the coCSWM instance: for each item $w_i \in \mathcal{S}^\forall$, we introduce a student $s_i \in S^\forall$; and for each item $w_j \in \mathcal{S}^\exists$, we introduce a student $s_j \in S^\exists$. Then, let us define 3 colleges: $c_{\forall\emptyset}$, $c_{\forall\exists}$ and $c_{\exists\emptyset}$. The budgets of the colleges are: $b_{c_{\forall\emptyset}} = M$, $b_{c_{\forall\exists}} = \alpha$ and $b_{c_{\exists\emptyset}} = M$. Finally, we also insert Example 1, by allowing student s_1 to go to college $c_{\forall\exists}$ with wage $w(s_1, c_{\forall\exists}) = 1/2$ and giving to college $c_{\forall\exists}$ additional utility $u_{c_{\forall\exists}}(s_1) = 1/2$. Crucially, if student s_1 is matched to college $c_{\forall\exists}$, then there is a coalitionally stable matching in Example 1; and otherwise, if s_1 is not matched to $c_{\forall\exists}$, then there is no coalitionally stable matching in Example 1, nor in the whole coCSWM instance, which is then a 'yes' instance.

For college $c_{\forall\emptyset}$, hiring a student from $S^\forall$ costs 0 and adds utility 0. For college $c_{\exists\emptyset}$, hiring a student from $S^\exists$ costs 0 and adds utility 0. Hence colleges $c_{\forall\emptyset}$ and $c_{\exists\emptyset}$ can hire every student that comes, but are indifferent to the sets of students that they receive. For college $c_{\forall\exists}$, hiring student s_i from $S^\forall$ costs w_i (the corresponding weight in the $\forall\exists$SubsetSum instance) and adds utility M. Also, hiring student s_j from $S^\exists$ costs w_j and adds utility w_j. As a consequence, the preference of college $c_{\forall\exists}$ is lexicographically to:

1. take all the students from $S^\forall$ who come,
2. maximize budget consumption, trying to hit budget α.
3. If budget consumption does not hit α, hire student s_1.

For every student s_i in $S^\forall$, her preference $(c_{\forall\emptyset}, 0) \succ_{s_i} (c_{\forall\exists}, w_i) \succ_{s_i} (c_\emptyset, 0)$ means that her first choice is to go to college $c_{\forall\emptyset}$, while $c_{\forall\emptyset}$ is indifferent between hiring her or not. In a matching, let $T^\forall$ denote the subset of students from $S^\forall$ matched to $c_{\forall\exists}$, and let $S^\forall \setminus T^\forall$ denote those matched to college $c_{\forall\emptyset}$. Note that no student from $S^\forall$ may form a blocking coalition: First, students in $T^\forall$ will not provide a strict interest to college $c_{\forall\emptyset}$ by deviating to it. Second, students in $S^\forall \setminus T^\forall$ are not interested into deviating to $c_{\forall\exists}$.

For every student s_j in $S^\exists$, her preference $(c_{\forall\exists}, w_j) \succ_{s_j} (c_{\exists\emptyset}, 0) \succ_{s_j} (c_\emptyset, 0)$ means that her first choice is to go to college $c_{\forall\exists}$, which enthusiastically welcomes her. Similarly, let $T^\exists$ denote the subset of students from $S^\exists$ matched to college $c_{\forall\exists}$, and let $S^\exists \setminus T^\exists$ denote those matched to $c_{\exists\emptyset}$.

(no $\Rightarrow$ no.) Assume that the $\forall\exists$SubsetSum instance is a 'no'-instance, which means that

$$\exists \mathcal{T}^\forall \subseteq \mathcal{S}^\forall, \quad \forall \mathcal{T}^\exists \subseteq \mathcal{S}^\exists, \quad \sum_{w_i \in \mathcal{T}^\forall} w_i + \sum_{w_j \in \mathcal{T}^\exists} w_j \neq \alpha$$

and let us show that there exists a coalitionally stable matching. We construct this matching as follows. The set of students $T^\forall$ given by $\mathcal{T}^\forall$ in the formula above goes to college $c_{\forall\exists}$. Then, college $c_{\forall\exists}$ hires the subset of students $T^\exists$ that maximizes its budget consumption, but does not hit target α, because of the conditions on isomorphic sets $\mathcal{T}^\exists$ in the formula above. Finally, college $c_{\forall\exists}$ hires student s_1, and there is a coalitionally stable matching in Example 1. This matching is coalitionally stable.

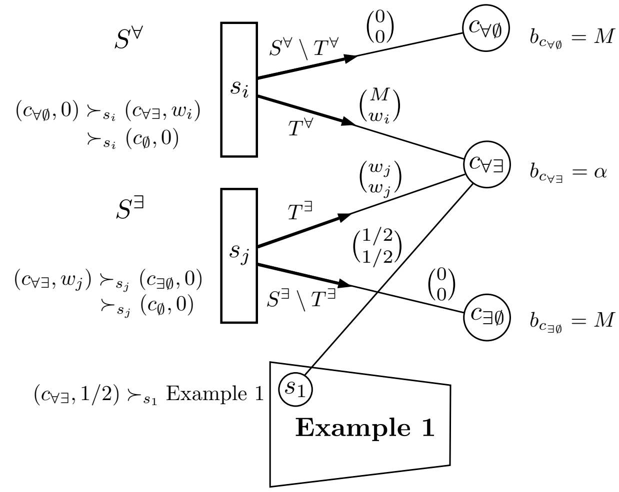


Figure 4: Reducing $\forall\exists$SubsetSum to coCSWM.

(yes $\Rightarrow$ yes.) Assume that the $\forall\exists$SubsetSum instance is a 'yes'-instance, which means that

$$\forall \mathcal{T}^\forall \subseteq \mathcal{S}^\forall, \quad \exists \mathcal{T}^\exists \subseteq \mathcal{S}^\exists, \quad \sum_{w_i \in \mathcal{T}^\forall} w_i + \sum_{w_j \in \mathcal{T}^\exists} w_j = \alpha$$

and let us show that every matching admits a blocking coalition. Assume for the sake of contradiction that there exists a coalitionally stable matching. Then college $c_{\forall\exists}$ hired student s_1 and achieves at best budget consumption $\alpha - 1/2$. However, there exists a blocking coalition $(T, c_{\forall\exists})$ in which T is a set of students corresponding to $\mathcal{T}^\exists$; the budget consumption of $c_{\forall\exists}$ is α. □

4. MECHANISM DESIGN IN TYPED WEIGHTED MARKETS

The previous section suggests that coalitional stability is a vain requirement. In this section, we discuss a strategy-proof and pairwise stable mechanism for typed weighted markets[8] called the sequential deferred acceptance (SDA) mechanism.

4.1 Mechanism

A mechanism φ is a function that takes a profile of preferences of students $\succ_S$ as an input and returns a matching Y. Let $\succ_{S\setminus\{s\}}$ denote a profile of preferences of students except s, and $(\succ_s, \succ_{S\setminus\{s\}})$ denote a profile of preferences of all students, where s's preference is $\succ_s$ and the profile of preferences of other students is $\succ_{S\setminus\{s\}}$.

Definition 11. Mechanism φ is *strategy-proof* for students, if it holds that $Y(s) \succ_s Y'(s)$ or $Y(s) = Y'(s)$, for every s, $\succ_s$, $\succ'_s$ and $\succ_{S\setminus\{s\}}$, where $Y = \varphi((\succ_s, \succ_{S\setminus\{s\}}))$ and $Y' = \varphi((\succ'_s, \succ_{S\setminus\{s\}}))$.

The SDA mechanism sequentially applies the (student-proposing) deferred acceptance mechanism (DA) [12], from the highest type θ_1 to the lowest type θ_k. The DA mechanism makes use of the following crafted choice functions.

Definition 12 (*Choice function of students*)**.** For each student s, her *choice function* Ch_s maps any subset of contracts

[8] Theorems 1, 2 and 3 also hold for typed weighted markets.

Mechanism 1 (Sequential Deferred Acceptance)

Let $Y \leftarrow \emptyset$ and $i \leftarrow 1$.
Round i :

1. Let $\hat{S} \leftarrow \{s \in S \mid \tau(s) = \theta_i\}$, i.e., the set of all type θ_i students, and run the DA.
2. Let Y^i be the obtained matching. $Y \leftarrow Y \cup Y^i$.
3. **If** $i = k$ then **return** Y, otherwise:
   $\forall c \in C, \quad b_c \leftarrow b_c - \sum_{x \in Y_c^i} x_W$;
   $i \leftarrow i + 1$; Go to **Round** i.

$X' \subseteq X$ to contract $\{x\}$, which is the most preferred contract in X'_s based on $\succ_s$ if one exists, otherwise $\emptyset$ if no feasible contract exists. The choice function of a set of students $\hat{S}$, denoted as $Ch_{\hat{S}}$, is defined as $Ch_{\hat{S}}(X') = \bigcup_{s \in \hat{S}} Ch_s(X')$, i.e., the union of choice functions of $\hat{S}$.

Definition 13 (*Choice function of colleges*)**.** For each college c, choice function $Ch_c(X')$ is defined as:

1. $Z \leftarrow \emptyset$, $Y \leftarrow X'_c$.
2. Repeat the following procedure: If $Y = \emptyset$, return Z. Otherwise, remove $(s, c, w) \in Y$ with the highest priority ranking from Y, s.t. $s \succ_c s_\emptyset$, based on preference $\succsim_c$ on students (ties are broken in some deterministic way, e.g., based on the alphabetical order of students' identifiers). If $\sum_{x \in Z} x_W + w \leq b_c$, add (s, c, w) to Z.

The choice function of all colleges is defined as $Ch_C(X') = \bigcup_{c \in C} Ch_c(X')$.

Note that the choice function for each college c is crafted such that it does not reflect $\tilde{\succsim}_c$ exactly. Actually, it is defined based on preference $\succsim_c$ over individual students. This fact is considered as an advantage, since as discussed in Section 2, obtaining $\tilde{\succsim}_c$ is difficult in general. As we show, we can guarantee strategy-proofness for students and pairwise stability by using the choice functions defined this way.

We use the following (student-proposing) DA as a component of our SDA mechanism. For a given set of students $\hat{S}$, it is defined as follows, where $X_{\hat{S}} = \bigcup_{s \in \hat{S}} X_s$.

Definition 14 (*Deferred Acceptance mechanism* (DA))**.**

1. $Re \leftarrow \emptyset$.
2. $Y \leftarrow Ch_{\hat{S}}(X_{\hat{S}} \setminus Re)$, $Z \leftarrow Ch_C(Y)$.
3. If $Y = Z$, then return Y, otherwise:
   $Re \leftarrow Re \cup (Y \setminus Z)$, go to Step 2.

Here, Re represents the set of rejected contracts, Y represents the contracts proposed by students $\hat{S}$ within $X_{\hat{S}} \setminus Re$, and Z represents the contracts in Y accepted by colleges. Thus, $Y \setminus Z$ represents a set of newly rejected contracts.

The SDA is defined in Mechanism 1. It repeatedly applies the DA for students of each type, from θ_1 to θ_k.

Example 3. Let us describe the execution of the SDA on the market illustrated in Figure 1.

Round 1. We run the DA for $\hat{S} = \{s_1, s_2, s_3\}$, i.e., all type θ_1 students, under original budgets b_C. The iterations in the DA are as follows:

1. $Y = \{(s_1, c_2, 3), (s_2, c_1, 3), (s_3, c_1, 3)\}$ and $Z = \{(s_1, c_2, 3), (s_2, c_1, 3)\}$; c_1 rejects $(s_3, c_1, 3)$ because $s_2 \succ_{c_1} s_3$ and its budget is 5.
2. $Y = \{(s_1, c_2, 3), (s_2, c_1, 3), (s_3, c_2, 3)\}$ and $Z = \{(s_2, c_1, 3), (s_3, c_2, 3)\}$; c_2 rejects $(s_1, c_2, 3)$ because $s_3 \succ_{c_2} s_1$ and its budget is 5.
3. $Y = \{(s_1, c_1, 3), (s_2, c_1, 3), (s_3, c_2, 3)\}$ and $Z = \{(s_1, c_1, 3), (s_3, c_2, 3)\}$; c_1 rejects $(s_2, c_1, 3)$ because $s_1 \succ_{c_1} s_2$ and its budget is 5.
4. $Y = Z = \{(s_1, c_1, 3), (s_2, c_1, 2), (s_3, c_2, 3)\}$. All colleges satisfy their budget constraints. Therefore, we obtain $Y^1 = \{(s_1, c_1, 3), (s_2, c_1, 2), (s_3, c_2, 3)\}$.

Round 2. We run the DA for $\hat{S} = \{s_4, s_5\}$ with the remaining budget, i.e., $b_{c_1} = 5 - 5 = 0$ and $b_{c_2} = 5 - 3 = 2$. The iterations in the DA are as follows:

1. $Y = \{(s_4, c_1, 1), (s_5, c_1, 1)\}$ and $Z = \emptyset$, because c_1 has no budget to accept any student.
2. $Y = Z = \{(s_4, c_2, 1), (s_5, c_2, 1)\}$. All colleges satisfy their budget constraints. Therefore, we obtain $Y^2 = \{(s_4, c_2, 1), (s_5, c_2, 1)\}$.

To conclude, the SDA returns the following matching:
$Y^1 \cup Y^2 = \{(s_1, c_1, 3), (s_2, c_1, 2), (s_3, c_2, 3), (s_4, c_2, 1), (s_5, c_2, 1)\}$.

4.2 Pairwise stability

Theorem 4. *The SDA always returns a pairwise stable matching.*

To prove this theorem, we use the following lemmas.

Lemma 4. *Let $s \in S$, $S' \subseteq S \setminus \{s\}$ s.t. $s' \succsim_c s_\emptyset$ holds for all $s' \in S' \cup \{s\}$. Assume there exists $S'' \subseteq S'$ s.t. $S' \setminus S'' \cup \{s\} \tilde{\succ}_c S'$ holds. Then, $s \succ_c s'$ holds for all $s' \in S''$.*

In words, if college c, which currently has S', prefers adding s by removing S'', then c prefers s over any student $s' \in S''$. This is intuitively natural; if s can win against coalition S'', she can also win against each individual in S''. We formally prove this from the fact that $\tilde{\succsim}_c$ is responsive.

Proof of Lemma 4. Assume by way of contradiction, there exists $\hat{s} \in S''$ such that $\hat{s} \succsim_c s$ holds. Since we assume $\hat{s} \succsim_c s$ holds, from responsiveness, when we add either $\hat{s}$ or s to $S' \setminus \{\hat{s}\}$, we have $S' \tilde{\succsim}_c S' \setminus \{\hat{s}\} \cup \{s\}$. From the assumption, $s' \succsim_c s_\emptyset$ holds for all $s' \in S' \cup \{s\}$. Thus, from responsiveness, by adding students in $S'' \setminus \{\hat{s}\}$ one by one to $S' \setminus S'' \cup \{s\}$, we have $S' \setminus \{\hat{s}\} \cup \{s\} \tilde{\succsim}_c S' \setminus S'' \cup \{s\}$. From these facts, we obtain $S' \tilde{\succsim}_c S' \setminus S'' \cup \{s\}$. However, this contradicts the assumption $S' \setminus S'' \cup \{s\} \tilde{\succ}_c S'$. □

Lemma 5. *Assume Y is the obtained matching of SDA, while for student s, where $s \succsim_c s_\emptyset$, contract (s, c, w) is rejected. Let $Z = \{(s', c, w') \in Y_c \mid w' \geq w\}$. Then, $b_c - \sum_{(s', c, w') \in Z} w' < w$ holds.*

In words, if (s, c, w) is rejected, college c does not have enough budget to accept it even when all contracts whose weights are less than w are rejected.

Proof of Lemma 5. Each student s', whose type is θ, proposes (s', c, w') only after she has proposed (s', c, w'') (and it is rejected) for all $w'' \in W_{c,\theta}$ such that $w'' > w'$ holds. Thus, the fact that (s, c, w) is rejected implies that there exists contract (s', c, w) (s' can be either identical to s or different from s) that was rejected while no contract whose weight is less than w is proposed yet. Thus, all contracts accepted so far have weights larger than or equal to w. Then, $b_c - \sum_{(s', c, w') \in Z} w' < w$ must hold. □

Proof of Theorem 4. Assume by way of contradiction, there exists blocking pair (s, c) for the obtained matching Y. More precisely, we assume there exist $R \subseteq Y_c$ and $w \in W_{c,\tau(s)}$ such that (i) $(c, w) \succ_s Y(s)$, (ii) $(Y(c) \setminus R(c)) \cup \{s\} \tilde{\succ}_c Y(c)$, and (iii) $\sum_{x \in Y_c \setminus R} x_W + w \leq b_c$ hold. From (i), s must have proposed (s, c, w) and it was rejected. Then, by Lemma 5, we have $b_c - \sum_{(s',c,w') \in Z} w' < w$, where $Z = \{(s', c, w') \in Y_c \mid w' \geq w\}$. Then, we obtain the following inequality:

$$b_c < \sum_{(s',c,w') \in Z} w' + w. \quad (1)$$

From (ii) and Lemma 4, we have $\forall s' \in R(c), s \succ_c s'$. Then, for all $s' \in R(c)$, where $(s', c, w') \in Y_c$, $w' < w$ holds (otherwise, (s, c, w) must be accepted instead of (s', c, w')). Thus, $\sum_{x \in Y_c \setminus R} x_W \geq \sum_{(s',c,w') \in Z} w'$ holds, since $Y_c \setminus R \supseteq Z$. Combining this and (iii), we obtain $b_c \geq \sum_{x \in Y_c \setminus R} x_W + w \geq \sum_{(s',c,w') \in Z} w' + w$, which contradicts with (1). □

4.3 Strategy-proofness

When the choice functions of all colleges satisfy the following three properties, the DA is guaranteed to be strategy-proof for students [16]. Informally, the *irrelevance of rejected contracts* means if contract x is rejected when it is added to X', it does not affect the outcomes of other contracts in X'. Also, the *substitutability* means if some contract x is rejected when $x \in X'$, it is also rejected when another contract is added to X'. Furthermore, the *law of aggregate demand* means if the set of contracts expands, the number of accepted contracts weakly increases. Although our choice functions satisfy the irrelevance of rejected contracts, they fail to satisfy the rest. For example, assume there are four students s.t. $s_1 \succ_c s_2 \succ_c s_3 \succ_c s_4$, and $b_c = 5$. From $\{(s_2, c, 2), (s_3, c, 3), (s_4, c, 1)\}$, contract $(s_4, c, 1)$ is rejected. However, from $\{(s_1, c, 2), (s_2, c, 2), (s_3, c, 3), (s_4, c, 1)\}$, $(s_4, c, 1)$ is accepted. Thus, the substitutability is violated. Also, from contracts $\{(s_2, c, 2), (s_3, c, 2), (s_4, c, 1)\}$, all three contracts are accepted. However, from $\{(s_1, c, 3), (s_2, c, 2), (s_3, c, 2), (s_4, c, 1)\}$, only first two contracts are accepted. Thus, the law of aggregated demand is violated.

Theorem 5. *The SDA is strategy-proof for students.*

Proof. Assume student s is a type θ_i student, i.e., she is assigned in **Round** i. It is clear that s has no influence on the outcomes of **Round** j, where $j < i$. Also, the outcome of later rounds is irrelevant to i. Thus, to show the strategy-proofness of the SDA, it is sufficient to show the strategy-proofness of the DA used for each round. To show this fact, we introduce an alternative market in which each (sub-)college has its maximum quota/capacity limit (but no budget constraint). In this market, the standard DA is guaranteed to be strategy-proof. We show the equivalence of the outcomes in these markets.

The alternative market is defined as follows. Let us assume W_{c,θ_i}, i.e., the possible set of c's weights for type θ_i students, is given as $\{w_c^1, w_c^2, \ldots, w_c^{\ell_c}\}$, where $w_c^1 > \ldots > w_c^{\ell_c}$ for all $c \in C$. We divide college c into ℓ_c sub-colleges, i.e., $c^1, c^2, \ldots, c^{\ell_c}$. The maximum quota q_{c^i} for each sub-college c^i is recursively defined as follows, where $r_1 = b_c$ (more precisely, b_c is the budget obtained in each round of the SDA).

$$q_{c^i} = \lfloor r_i / w_c^i \rfloor, r_{i+1} = r_i - q_{c^i} \times w_c^i.$$

Contract (s, c, w_c^i) in the original market is translated into contract (s, c^i) in the alternative market. The preference of each student in the alternative market is identical to the original market according to the above translation. The preference of each sub-college c^i is defined based on $\succsim_c$, i.e., c^i will accept students according to $\succsim_c$ until its maximum quota q_{c^i}, using the same tie-breaking method as Ch_c.

In the original market, c can accept at most q_{c^1} contracts with weight w_c^1 due to its budget constraint. Also, each student s proposes contract (s, c, w_c^2) only after (s, c, w_c^1) is rejected. This implies that c already accepts q_{c^1} contracts with weight w_c^1. Then, c can accept at most q_{c^2} contracts with weight w_c^2 due to its budget constraint. Also, each student s proposes contract (s, c, w_c^3) only after (s, c, w_c^2) is rejected. This implies that c accepts q_{c^2} contracts with weight w_c^2, and so on. From these facts, the outcome in the alternative market and that in the original market must be identical. Then, from the fact that the standard DA in the alternative market is strategy-proof, the DA (Definition 14) in the original market must be strategy-proof. □

Let us show an example of the alternative market using the original market illustrated in Figure 1. In **Round** 1, we create two sub-colleges for c_1, i.e., c_1^1 and c_1^2. The maximum quotas of these sub-colleges are 1. There exists only one sub-college for c_2, which we denote c_2^1, whose maximum quota is also 1. Then, s_1 is accepted for c_1^1, s_2 is accepted for c_1^2, and s_3 is accepted for c_2^1. In **Round** 2, since the sub-college for c_1 has no capacity, its maximum quota is 0. There exists one sub-college for c_2, which we denote c_2^1, whose maximum quota is 2. Then, s_4 and s_5 are accepted for c_2^1.

From Theorem 4, we immediately obtain the following.

Theorem 6. *In a typed weighted market, a pairwise stable matching is guaranteed to exist, and it can be calculated in the time linear to $|X|$, assuming the calculation of Ch_C and $Ch_{\hat{S}}$ can be done in a constant time.*

Proof. We can always find a pairwise stable matching using the SDA. Also, during the iteration of the DA in Definition 14, at least one contract must be rejected; otherwise, the procedure terminates. Thus, assuming the calculation of Ch_C and $Ch_{\hat{S}}$ can be done in a constant time, the run-time of the SDA is linear in $|X|$. □

5. CONCLUSION

This paper examined two-sided matchings with budget constraints and showed computational hardness results for problems related to coalitional stability. Then, we designed a strategy-proof mechanism that achieves pairwise stability.

Our future works include examining $(1 + \sigma)$-coalitional stability, which is an intermediate concept between pairwise and coalitional stability, i.e., the number of students in a coalition is at most σ (where $1 \leq \sigma \leq n$). Also, we suspect problem CSWM to be easier for a constant number of types.

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