Random Assignment with Optional Participation

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ABSTRACT

A central problem in multiagent systems concerns the fair assignment of objects to agents. We initiate the study of randomized assignment rules with optional participation and investigate whether agents always benefit from participating in the assignment mechanism. Our results are largely positive, irrespective of the strategyproofness of the considered rules. In particular, *random serial dictatorship*, the *probabilistic serial rule*, and the *Boston mechanism* strictly incentivize single agents to participate, no matter what their underlying utility functions are. Random serial dictatorship and the probabilistic serial rule also cannot be manipulated by groups of agents who abstain strategically. These results stand in contrast to results for the more general domain of voting where many rules suffer from the so-called "no-show paradox". We also show that rules that return *popular* random assignments may disincentivize participation for some (but never all) utility representations consistent with the agents' ordinal preferences.

General Terms

Economics, Theory

Keywords

random assignment; random serial dictatorship; probabilistic serial rule; Boston mechanism; popular random assignments; participation; no-show paradox; stochastic dominance

1. INTRODUCTION

A central problem in multiagent systems and microeconomic theory concerns the fair assignment of objects to agents based on the agents' ordinal preferences over the objects [e.g., 18, 27, 25, 11]. When objects are indivisible, it is impossible to deterministically assign objects such that agents with the same preferences receive the same objects. This problem is usually avoided by randomization, i.e., by assigning lotteries over objects to the agents. A number of random assignment rules such as *random serial dictatorship*, the *probabilistic serial rule*, and the *Boston mechanism* have been proposed and studied in the literature.

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An important property of such rules is strategyproofness, i.e., the immunity to strategic misrepresentation of preferences. In this paper, we consider a particularly severe form of manipulation where agents obtain a more preferred outcome by not participating in the assignment mechanism. This is modeled by assuming complete indifference for non-participating agents or, equivalently, by executing the assignment rule for participating agents and then assigning the remaining objects uniformly among non-participating agents. The effect of disincentivizing participation has been studied in the context of voting where it is widely known as the "no-show paradox" [see, e.g., 20, 26, 12, 13, 15]. A voting rule is said to satisfy *participation* if no voter can benefit from abstaining from an election. Assignment can be seen as a special case of voting where voters have preferences over assignments and are assumed to be indifferent between all assignments in which they receive the same object. This relationship allows the transfer of positive results from the voting to the assignment domain.

Randomized voting and assignment rules allow for a particularly strong form of participation: Rather than only disincentivizing abstention, they can even provide incentives for participation by offering participating agents a lottery that they strictly prefer (if only very slightly) to the lottery they would have received when abstaining. This is important because in most applications some cost is associated with participation (e.g., for figuring out one's own preferences).

As an example, consider a company that assigns office space to workers by using the probabilistic serial (PS) rule. The default preference pre-assigned to every worker is complete indifference and it is up to him to update his preferences before a given deadline or not. We prove that a worker is always *strictly* better off (whenever an improvement is possible at all) by updating his preferences and thus participating in the mechanism, no matter what his underlying von Neumann-Morgenstern utility function is. By contrast, it is well-known that PS fails to satisfy strategyproofness.[1]

All the rules we consider (including random serial dictatorship) violate group-strategyproofness, i.e., they can be manipulated by *groups* of agents who misrepresent their preferences. Yet we show that random serial dictatorship and the probabilistic serial rule *cannot* be manipulated by groups of abstaining agents. Our results (which also cover

[1]While PS satisfies strategyproofness for strict preferences, it violates strategyproofness for general preferences. Note that strategyproofness in our terminology is often referred to as *weak strategyproofness* in the literature. PS fails to satisfy the stronger notion of strategyproofness even for strict preferences.

	very strong participation		strong group-participation		group-participation	
Random Serial Dictatorship	✓	(Corollary 1)	✓	(Theorem 1)	✓	
Probabilistic Serial rule	✓	(Theorem 2)	✓	(Theorem 3)	✓	
Boston Mechanism	✓	(Theorem 4)	–	(Theorem 5)	✓	(Theorem 6)
Popular random assignment rules	–	(Theorem 7)	–	(Theorem 7)	✓	(Corollary 2)

Table 1: Overview of results. By definition, strong group-participation implies group participation, i.e., a checkmark in the second column implies a checkmark in the third column.

the Boston mechanism and so-called popular random assignments) are summarized in Table 1.

2. PRELIMINARIES

In this section, we define the notation and terminology required for our results.

2.1 Assignment Problems and Random Assignments

Let $N = \{1, \dots, n\}$ be a set of n agents and O be a set of n objects. Every agent $i \in N$ has a complete and transitive preference relation $\succsim_i$ over the elements of O. We represent a preference relation as an ordered list of indifference classes, e.g., the relation $\succsim_i$ with $a \sim_i b \succ_i c$ is represented as $i\colon \{a, b\}, c$. A preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ is an n-tuple of preference relations. The set of all preference profiles is denoted by $\mathcal{R}$. A triple $(N, O, \succsim)$ constitutes an *assignment problem*.

A *deterministic assignment (or pure matching)* is a one-to-one map from N to O. We identify a deterministic assignment m with a permutation matrix in $\mathbb{R}^{N\times O}$, where $m_{i,o} = 1$ if agent i is assigned object o and 0 otherwise. The set of all deterministic assignments is denoted by $\mathcal{M}$. A *random assignment* is a probability distribution over deterministic assignments. Thus, we represent a random assignment p as a bistochastic matrix in $\mathbb{R}^{N\times O}$,[2] where $p(i, o)$ is the probability that agent i is assigned object o. The set of all random assignments is thus denoted by $\Delta(\mathcal{M})$. Note that by the Birkhoff-von Neumann Theorem, every bistochastic matrix can be written as a probability distribution over deterministic assignments [see, e.g., 23]. To simplify notation, for $p \in \Delta(\mathcal{M})$ and $i \in N$, we write $p(i)$ for the ith row of p, i.e., the lottery over O assigned to agent i.

As an example, consider the following random assignment p where $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$.

$$p = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 1/12 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Here, $p(1, a) = 1/3$ and $p(3) = 1/4\, b + 3/4\, c$.

We extend the agents' preferences over objects to preferences over random assignments using two assumptions:

(i) agents only care about the lottery they are assigned, and

(ii) they compare lotteries via stochastic dominance.

For an agent $i \in N$ and two lotteries $p(i), q(i)$ over O, $p(i)$ *stochastically dominates* $q(i)$ if, for every object o, the probability that $p(i)$ yields an object at least as good as o is at least as large as the probability that $q(i)$ yields an object at least as good as o. Formally,

$$p(i) \succsim_i^{SD} q(i) \text{ iff } \sum_{o' \succsim_i o} p(i, o') \geq \sum_{o' \succsim_i o} q(i, o') \text{ for all } o \in O.$$

Then, for $p, q \in \Delta(\mathcal{M})$,

$$p \succsim_i q \quad \text{iff} \quad p(i) \succsim_i^{SD} q(i),$$

i.e., agent i prefers the random assignment p to the random assignment q if $p(i)$ stochastically dominates $q(i)$. Note that preferences over random assignments may be incomplete, as some lotteries are not comparable via stochastic dominance. The importance of stochastic dominance stems from the fact that $p(i) \succsim_i^{SD} q(i)$ iff the expected utility for $p(i)$ is at least as large as that for $q(i)$ for every von-Neumann-Morgenstern utility function compatible with $\succsim_i$.

To illustrate this definition, consider the example above and assume agent 1 has preferences $1\colon a, b, c$. Then, $p(1) \succ_1^{SD} p(3)$ and $p(2) \succ_1^{SD} p(3)$. $p(1)$ and $p(2)$ are not comparable for agent 1 using stochastic dominance.

A *random assignment rule* is a map from the set of preference profiles $\mathcal{R}$ to the set of random assignments $\Delta(\mathcal{M})$. In the remainder of the paper we consider four common random assignment rules and study to which extent they provide incentives for agents to participate in the assignment process.

We introduce some more notation that is needed for our proofs. Readers not interested in the proofs may skip this paragraph without loss of continuity. For all $i \in N$, we denote by k_i the number of indifference classes of agent i and by O_i^k the union of the upper k indifference classes for $k \in [k_i] = \{1, \dots, k_i\}$. For a set of agents $C \subseteq N$ and a set of objects $O' \subseteq O$, we denote by $p(C, O')$ the sum of probabilities of agents in C for objects in O' in the random assignment p. Formally, $p(C, O') = \sum_{(i,o)\in C\times O'} p(i, o)$. In case either C or O' is a singleton, we write $p(i, O')$ and $p(C, o)$ for convenience, respectively.

2.2 Participation

Based on earlier observations by Fishburn and Brams [20], Moulin [26] introduced the axiom of *participation* in voting. A voting rule is said to satisfy participation if no voter can benefit by abstaining from an election. Brandl et al. [13] extended participation to randomized voting rules, i.e., rules that return lotteries over alternatives, by defining three different degrees of participation: participation, strong participation, and very strong participation. We transfer these concepts to the assignment domain and stick to the notation and terminology of Brandl et al. [13].

First note that by definition, we require the number of agents and objects to be equal in any assignment problem.

[2] A matrix is bistochastic if all entries are non-negative and every row and every column sums up to 1.

We therefore define abstention by letting an agent declare complete indifference. This leads to a natural notion of participation in settings where agents always receive an object (no matter whether they participate in the mechanism or not). Another justification for this interpretation of abstention is that for all assignment rules we consider in this paper, if a group $C \subseteq N$ abstains, objects are first assigned to agents in $N \setminus C$ and whichever objects remain are distributed uniformly among the agents in C.[3]

In this sense, given an assignment problem $(N, O, \succsim)$ and a group of agents $C \subseteq N$, we say that $(N, O, \succsim_{-C})$ is the assignment problem in which C abstains. Here, $\succsim_{-C}$ is defined by $(\succsim_{-C})_i = \succsim_i$ if $i \in N \setminus C$ and $(\succsim_{-C})_i = \sim$ if $i \in C$ with $\sim$ denoting the completely indifferent preference relation. If an assignment rule yields the random assignment p for $(N, O, \succsim)$, we write p_{-C} for the random assignment resulting for $(N, O, \succsim_{-C})$. Slightly abusing notation we write $\succsim_{-i}$ and p_{-i} if only a single agent i abstains.

We say that an assignment rule satisfies

- *participation* iff there is no assignment problem $(N, O, \succsim)$ and no agent $i \in N$ with $p_{-i} \succ_i p$,
- *strong participation* iff for all $(N, O, \succsim)$, $i \in N$, $p \succsim_i p_{-i}$, and
- *very strong participation* iff for all $(N, O, \succsim)$, $i \in N$, $p \succ_i p_{-i}$ whenever there exists $q \in \Delta(\mathcal{M})$ with $q \succ_i p_{-i}$ and $p \succsim_i p_{-i}$ otherwise.

In game-theoretic terms, for rules that satisfy participation, participating is a strictly undominated strategy. Similarly, for rules that satisfy strong participation, participating is a very weakly dominant strategy. For rules that satisfy very strong participation, participating guarantees an agent a strictly preferred result whenever this is possible.

Clearly, participation is related to strategyproofness. An assignment rule satisfies

- *strategyproofness* iff there is no assignment problem $(N, O, \succsim)$ and $\succsim'_i$, $i \in N$, with $p' \succ_i p$, and
- *strong strategyproofness* iff for all $(N, O, \succsim)$ and $\succsim'_i$, $i \in N$, $p \succsim_i p'$,

where p' is the random assignment resulting of $(N, O, (\succsim_1, \ldots, \succsim_{i-1}, \succsim'_i, \succsim_{i+1}, \ldots, \succsim_n))$.[4]

In analogy to the definitions for single agents we also define participation for groups. An assignment rule is said to satisfy

- *group-participation* iff there is no assignment problem $(N, O, \succsim)$ and agents $C \subseteq N$ such that $p_{-C} \succ_i p$ for all $i \in C$, and
- *strong group-participation* iff for all $(N, O, \succsim)$ and $C \subseteq N$ we have that $p \succsim_i p_{-C}$ for all $i \in C$.

[3]We will use both interpretations of abstention—declaring indifference and being awarded what remains after a first round of non-abstainers—in our proofs, depending on which idea fits our needs best. As mentioned before, both concepts coincide for all rules considered in the present paper.

[4]Note that what we call *strategyproofness* and *strong strategyproofness* is often called *weak strategyproofness* and *strategyproofness* in the literature. We chose the present terminology to be in line with the different notions of participation defined by Brandl et al. [13].

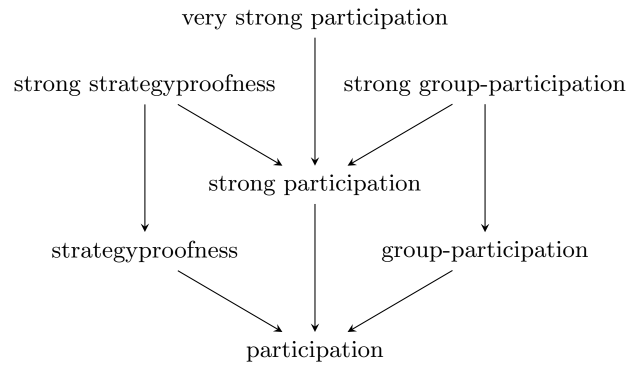

Figure 1: Implications between the different notions of participation and strategyproofness. An arrow from one notion to another signifies the former implies the latter.

While *very* strong group-participation could be defined analogously, we believe this notion to be too demanding. To see this, consider the following assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$ and $\succsim$ as depicted below. Any reasonable assignment rule satisfying mild fairness and efficiency assumptions will return the random assignment p which is identical to p_{-C} for $C = \{1, 2\}$. This would violate very strong group-participation.

$$\succsim = \begin{array}{ll} 1\colon & a, b, c \\ 2\colon & a, b, c \\ 3\colon & c, b, a \end{array} \qquad p = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the sake of completeness, we also define strategyproofness for groups. An assignment rule satisfies

- *group-strategyproofness* iff there is no assignment problem $(N, O, \succsim)$ and $\succsim'_i$, $i \in C \subseteq N$, with $p' \succ_i p$ for all $i \in C$, and
- *strong group-strategyproofness* iff for all $(N, O, \succsim)$ and $\succsim'_i$, $i \in C \subseteq N$, $p \succsim_i p'$ for all $i \in C$,

where p' is the random assignment resulting of $(N, O, (\succsim_{N \setminus C}, \succsim'_C))$.

The logical relationships between the different notions of participation are as follows: Very strong participation implies strong participation which in turn implies participation. Similarly, strong group-participation implies group-participation and the two variants of group-participation obviously imply the corresponding notion of participation. Since strategic abstention is defined by declaring complete indifference, we have that strong strategyproofness and strategyproofness imply strong participation and participation, respectively, and analogously strong group-strategyproofness and group-strategyproofness imply strong group-participation and group-participation. All implications important for this paper are illustrated in Figure 1.

3. RESULTS AND DISCUSSION

We consider four prominent assignment rules. The two most studied rules for ordinal preferences are *random serial dictatorship* and the *probabilistic serial* rule. The *Boston mech-*

anism is frequently used for real-world applications. However, it has lost support in the last few years for its high vulnerability to strategic manipulation. *Popular random assignments* have recently gained increasing attention among researchers.

We now briefly explain these rules and investigate which degrees of participation they satisfy.

3.1 Random Serial Dictatorship

The characteristic feature of *random serial dictatorship (RSD)*, also known as *random priority*, is its resistance to strategic manipulation by a single agent, i.e., *RSD* satisfies strong strategyproofness [see, e.g., 8, 9]. This directly implies that *RSD* also satisfies strong participation. However, *RSD* violates group-strategyproofness. By contrast and perhaps surprisingly, we will show that *RSD* satisfies strong group-participation.

Typically, *RSD* is defined for the special case where all agents have strict preferences over objects. Our definition extends *RSD* to the full preference domain [cf. 10, 5], For better exposition, we start by defining *RSD* for agents with strict preferences: first a permutation of agents is drawn uniformly at random, then the agents successively choose their most preferred object among the remaining objects according to the order given by the permutation. For general preferences, this process is not well-defined as agents may have multiple most preferred objects. In this case an agent narrows down the set of assignments to assignments in which he is assigned one of his most preferred objects.

Formally, let Π_N be the set of all permutations of N. For a preference relation $\succsim_i$ and a set of deterministic assignments $\mathcal{M}' \subseteq \mathcal{M}$, let

$$\max_{\succsim_i} \mathcal{M}' = \{M \in \mathcal{M}' \colon M \succsim_i M' \text{ for all } M' \in \mathcal{M}'\}$$

be the set of most preferred assignments according to $\succsim_i$ in $\mathcal{M}'$. For $\succsim \in \mathcal{R}$, $\pi \in \Pi_N$, and $k \in \{1, \dots, n\}$, we define inductively

$$\sigma^k(\succsim, \pi) = \begin{cases} \max_{\succsim_{\pi(1)}} \mathcal{M} & \text{if } k = 1, \text{ and} \\ \max_{\succsim_{\pi(k)}} \sigma^{k-1}(\succsim, \pi) & \text{if } k \in \{2, \dots, n\}. \end{cases}$$

Then, $\sigma^n(\succsim, \pi)$ is the outcome of *serial dictatorship* according to the permutation π. Note that this set may contain more than one deterministic assignment. We resolve this ambiguity by randomizing uniformly over these assignments and define $\sigma(\succsim, \pi)$ as the uniform distribution over $\sigma^n(\succsim, \pi)$. Then *RSD* is defined by randomizing uniformly over all permutations of agents, i.e.,

$$RSD(\succsim) = 1/n! \sum_{\pi \in \Pi_N} \sigma(\succsim, \pi).$$

Consider the following example with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.

$$\succsim = \begin{array}{ll} 1\colon & \{a, b\}, c \\ 2\colon & a, b, c \\ 3\colon & b, a, c \end{array} \qquad RSD(\succsim) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

To illustrate the definition, we explain the computation for the permutation $\pi = (1, 2, 3)$. Agent 1 narrows down the set of assignments to all assignments where he is assigned either object a or object b. Out of these, agent 2 prefers the assignments where he is assigned object a (and hence agent 1 is assigned object b). In the only remaining assignment agent 3 is assigned object c.

Our first result states that *RSD* satisfies very strong participation.

Corollary 1. *RSD satisfies very strong participation.*

Proof. The statement follows directly from Brandl et al. [13, Thm. 4] and the observation that assignment is a special case of voting [cf., e.g., 6]. □

We proceed by showing that, in contrast to Theorem 1, *RSD* violates group-strategyproofness.[5] To this end, consider the assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3, 4\}$, $O = \{a, b, c, d\}$ and $\succsim$ as follows.

$$\succsim = \begin{array}{ll} 1\colon & a, b, c, d \\ 2\colon & a, b, c, d \\ 3\colon & b, a, d, c \\ 4\colon & b, a, d, c \end{array} \qquad RSD(\succsim) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}$$

Then, the group consisting of all four agents can manipulate by reporting the preferences $\succsim'$.

$$\succsim' = \begin{array}{ll} 1\colon & a, c, b, d \\ 2\colon & a, c, b, d \\ 3\colon & b, d, a, c \\ 4\colon & b, d, a, c \end{array} \qquad RSD(\succsim') = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

Next, we consider group-participation.

Theorem 1. *RSD satisfies strong group-participation.*

Proof. The proof relies on considering the outcome of serial dictatorship for all possible permutations of agents in N. We make use of the fact that if C abstains, it is irrelevant whether we consider permutations of $N \setminus C$ and distribute the remaining probabilities uniformly among agents in C or instead consider all permutations of N with agents in C being completely indifferent. Hence, for every permutation of agents in $N \setminus C$ where C receives some probability $\alpha > 0$ of $o \in O$, we have $\binom{|N|}{|C|}C!$ permutations of agents in N where C receives the very same probability α.[6] Note that agent $i \in C$ precedes all other agents in C in exactly $1/|C|$ of those permutations. Assuming object o is i's first choice, he would thus have received o with probability at least α in all permutations where he is first out of C had he participated. Going back to all $\binom{|N|}{|C|}C!$ permutations of agents in N, we hence have that when C participates, i receives o with probability at least $1/|C|\,\alpha$ while i's probability for o is exactly $1/|C|\,\alpha$ when C abstains.

Formally, we show that for all assignment problems $(N, O, \succsim)$ and agents $i \in N$ we have that $p(i, O_i^k) \geq p_{-C}(i, O_i^k)$ for all $C \subseteq N$, $i \in C$, and $k \in [k_i]$. Here we use $p = RSD(\succsim)$. To this end, let $(N, O, \succsim)$ be an assignment problem and choose $C \subseteq N$, $i \in C$, and $k \in [k_i]$ arbitrary. We begin with the case where agents in C abstain, i.e., they are completely indifferent. Recall that under this circumstance, for *RSD* it is irrelevant whether we include

[5]This statement is included to illustrate the contrast with group-participation. It follows from *RSD*'s well-documented lack of *SD*-efficiency Bade [see, also, 7].

[6]We consider a single object $o \in O$ and not a set of objects $O' \subseteq O$ as would be required in order to show strong group-participation for reasons of exposition. Please see the formal proof for arguments employing sets.

agents in C in the sequence of agents or only focus on $N \setminus C$ and distribute the remaining probabilities uniformly.

Consequently, we first consider permutations of $N \setminus C$ only. By $\Pi_{N\setminus C}$ denote the set of all permutations of $N \setminus C$. Let $\pi \in \Pi_{N\setminus C}$ and let α_π be the corresponding probability share of O_i^k given to C.

Note that for each π there exist $\binom{|N|}{|C|}C!$ permutations of N using which serial dictatorship yields the same probability share α_π of O_i^k for C. Exactly $1/|C|$ of these sequences list i as jth agent out of C. Thus, i precedes all agents in C in $1/|C|$ of the sequences and had C participated, he would have received a probability share of at least $\min\{1, \alpha_\pi\}$ of O_i^k in these cases. In another $1/|C|$ of the sequences i comes second and had C participated, he would have received a probability share of at least $\max\{0, \min\{1, \alpha_\pi - 1\}\}$ of O_i^k. For the general case of i being at the lth position of agents in C, he would have received a probability share of at least $\max\{0, \min\{1, \alpha_\pi - (l-1)\}\}$ of O_i^k had C participated.

Summing all possible positions with respect to agents in C, we obtain that had C participated, i would have received a probability share of at least $\lfloor\alpha_\pi\rfloor/|C| + 1/|C|\,(\alpha_\pi - \lfloor\alpha_\pi\rfloor)$ of O_i^k. Here, the first summand corresponds to positions where a would receive a full probability share of 1 while the second summand models the situations in which i would receive a probability share of only $\alpha_\pi - \lfloor\alpha_\pi\rfloor < 1$. Note that since

$$\lfloor\alpha_\pi\rfloor/|C| + 1/|C|\,(\alpha_\pi - \lfloor\alpha_\pi\rfloor) = 1/|C|\,\alpha_\pi,$$

we have that had C participated, i would have received at least the same probability share of objects in O_i^k for all orderings $\pi \in \Pi_{N\setminus C}$. We consequently have that $p_{-C}(i, O_i^k) \leq p(i, O_i^k)$.

For the sake of clarity, we put all (in)equalities together and obtain

$$\begin{aligned} p_{-C}(i, O_i^k) &= 1/|C|\, p_{-C}(C, O_i^k) \\ &= 1/(|N\setminus C|)! \sum_{\pi\in\Pi_{N\setminus C}} 1/|C|\,\alpha_\pi \\ &= 1/(|N\setminus C|)! \sum_{\pi\in\Pi_{N\setminus C}} \lfloor\alpha_\pi\rfloor/|C| + 1/|C|\,(\alpha_\pi - \lfloor\alpha_\pi\rfloor) \\ &\leq p(i, O_i^k), \end{aligned}$$

which completes the proof. $\square$

3.2 Probabilistic Serial

In contrast to *RSD*, the *probabilistic serial (PS)* rule is a relatively new assignment rule that was proposed in 2001 by Bogomolnaia and Moulin [9]. They consider assignment problems with strict preferences and show that in this domain, *PS* is efficient, envy-free, and (weakly) strategyproof, but violates strong strategyproofness. Intuitively, *PS* works as follows:

Assume all objects are edible and of equal size 1. At time $t = 0$, agents begin at their favorite object and start eating at uniform speed. As soon as an object is completely consumed, all agents involved at this point continue on to their most preferred remaining object and resume eating. At time $t = 1$, when there are no more objects available, all agents have consumed a total amount of 1. Finally, the fraction an agent has eaten of some object is equal to the probability he receives for it.

Consider the following example with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as given below.

$$\succsim\, = \begin{array}{rl} 1: & a, b, c \\ 2: & a, c, b \\ 3: & b, a, c \end{array} \qquad PS(\succsim) = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

In the beginning, agents 1 and 2 are eating a while agent 3 is eating b. At time $t = 1/2$, a is completely consumed while half of b has been eaten by agent 3. 1 and 2 continue on to b and c, their respective second most preferred objects. At time $t = 3/4$, b is completely consumed as well and all agents simultaneously finish c. Put together, the random assignment returned by *PS* is as given above.

PS has been extended to the full preference domain by Katta and Sethuraman [22]. They call their algorithm (which is based on maximal flows in networks) the *extended probabilistic serial* rule and prove that it still satisfies efficiency and envy-freeness, but fail to satisfy (weak) strategyproofness.[7] Whenever we refer to *PS* in the sequel, we mean the generalization by Katta and Sethuraman [22].

To get a first taste consider the preference profile $\succsim$ depicted below. In contrast to a naive generalization, agent 1 is not 'eating' a and b simultaneously but instead he is reserving some probability of $\{a, b\}$ with identical uniform speed. At $t = 2/3$ we arrive at a point where three agents have each reserved a share of $2/3$ of $\{a, b\}$—or at least one object out of it—meaning that the set $\{a, b\}$ is completely consumed. We therefore say that $\{a, b\}$ is a 'bottleneck' and *PS* proceeds by gradually identifying subsequent bottlenecks and dividing the included objects to the competing agents in a fair way. For the preference profile $\succsim$ below, *PS* thus finds the bottlenecks $\{a, b\}$, $\{c\}$, $\{d\}$ in chronological order.

$$\succsim\, = \begin{array}{rl} 1: & \{a, b\}, c, d \\ 2: & a, c, b, d \\ 3: & b, a, d, c \\ 4: & c, d, a, b \end{array} \qquad PS(\succsim) = \begin{pmatrix} 1/3 & 1/3 & 1/9 & 2/9 \\ 2/3 & 0 & 1/9 & 2/9 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 7/9 & 2/9 \end{pmatrix}$$

As stated before, the implementation makes use of flows on networks that are redesigned after each bottleneck. We omit a more detailed and formal explanation of *PS* for general preferences in the interest of space and refer to the paper by Katta and Sethuraman [22].

Recently, *PS* was generalized to the more general domain of voting by Aziz and Stursberg [3], who called the resulting rule the *egalitarian simultaneous reservation (ESR)* rule. Interestingly, this rule violates strong participation [2].

We show that within the domain of assignment, *PS* fares better with respect to manipulation by strategic abstention, i.e., *PS* satisfies very strong participation and strong group-participation. This stands in contrast to the voting domain as well as to the related concept of strategyproofness.

THEOREM 2. *PS satisfies very strong participation.*

PROOF. It follows from Theorem 3 below that *PS* satisfies strong participation. We now prove that even the stronger notion of *very* strong participation holds. Therefore focus on the first level of preferences and let p be the random assignment returned by *PS*. First note that $p_{-i}(i, O_i^1) \leq p(i, O_i^1) \leq 1$ is implied by strong participation.

[7] More so, Katta and Sethuraman show that efficiency and envy-freeness are incompatible with (weak) strategyproofness on the full preference domain.

Furthermore, if $p(i, O_i^1) = 1$, very strong participation is trivially satisfied. Thus, the only case remaining for examination is $p_{-i}(i, O_i^1) \leq p(i, O_i^1) < 1$. We will show that it indeed always holds that $p_{-i}(i, O_i^1) < p(i, O_i^1)$, which implies very strong participation.

Note that by the algorithm used for *PS* [cf. 22], we have that $0 < p(i, O_i^1)$. In addition, $p(i, O_i^1) < 1$ by the above assumption, thus, O_i^1 has to be part of some bottleneck $B \subseteq O$ that occurs at time t_B. Let the set of agents who cause said bottleneck to occur be C_B. We additionally use the notation Γ taken from Katta and Sethuraman [22] and slightly modify it to better fit our needs: $\Gamma_t(C)$ is the union of objects agents C are eating (or reserving) at time t.

We distinguish whether B is the first bottleneck or not:

(i) O_i^1 is part of the first bottleneck. We have that $\Gamma_{t_B}(C_B) = \Gamma_{t_B}(C_B \setminus \{i\})$ because otherwise a different bottleneck would have occurred earlier for agents $C_B \setminus \{i\}$. To see this assume for contradiction $\Gamma_{t_B}(C_B) \neq \Gamma_{t_B}(C_B \setminus \{i\})$. Trivially,

$$|\Gamma_{t_B}(C_B \setminus \{i\})| \leq |\Gamma_{t_B}(C_B)| - 1$$

and thus

$$\frac{|\Gamma_{t_B}(C_B \setminus \{i\})|}{|C_B \setminus \{i\}|} < \frac{|\Gamma_{t_B}(C_B)|}{|C_B|}.$$

This contradicts the first bottleneck appearing for C_B—a different one would have appeared earlier for $C_B \setminus \{i\}$. Hence, $\Gamma_{t_B}(C_B) = \Gamma_{t_B}(C_B \setminus \{i\})$.

As $|C_B| - |C_B \setminus \{i\}| = 1$ we conclude that ${}^1\!/_{|C_B \setminus \{i\}|}\, |\Gamma_{t_B}(C_B \setminus \{i\})| \leq 1$. Hence, given that i abstains, we still have a bottleneck that includes agents $C_B \setminus \{i\}$ (not necessarily the first) and it holds that $0 = p_{-i}(i, O_i^1) < p(i, O_i^1)$.[8]

(ii) O_i^1 is not part of the first bottleneck. For the bottleneck including O_i^1 we have that $p(C_B, B) = |\Gamma_{t_B}(C_B)| = |B|$ and trivially $p(C_B, B) < |C_B|$. Consequently $p(i', B) < 1$ for all $i' \in C_B$ and for similar arguments as above we have that $\Gamma_{t_B}(C_B) = \Gamma_{t_B}(C_B \setminus \{i\})$. Hence, $p(C_B \setminus \{i\}, B) < |\Gamma_{t_B}(C_B \setminus \{i\})|$ which means that the bottleneck B' including O_i^1 will occur strictly later at time $t = t_{B'} \leq 1$ for a possibly different group of agents $C_{B'} \supseteq C_B \setminus \{i\}$.[9] At this point we have $p_{-i}(C_{B'}, B') = |\Gamma_{t_{B'}}(C_{B'})|$. Since $t_{B'} > t_B$ and thus $p_{-i}(i', B') > p(i', B')$ for all $i' \in C_{B'}$ as well as $p_{-i}(i', B') = p(i', B)$ for all $i' \in C_B$, it holds that $p(C_B, B) < p_{-i}(C_B, B)$. Putting everything together we obtain

$$\begin{aligned} p_{-i}(i, O_i^1) &\leq |\Gamma_{t_{B'}}(C_{B'})| - p_{-i}(C_{B'} \setminus \{i\}, B') \\ &< |\Gamma_{t_B}(C_B)| - p(C_B \setminus \{i\}, B) \\ &= p(i, B) \\ &= p(i, O_i^1). \end{aligned}$$

Thus, $p_{-i}(i, O_i^1) < p(i, O_i^1)$ for both cases and very strong participation is satisfied. □

Theorem 3. *PS satisfies strong group-participation.*

Proof. Let $(N, O, \succsim)$ be an assignment problem with $O = \{o_1, \ldots, o_n\}$, $C \subseteq N$ the group of agents that abstains, and $p = PS(\succsim)$. In the case where C participates, we call the bottlenecks that appear when executing the algorithm in order to determine *PS* $B_1, B_2, B_3, \ldots \subseteq O$ where the naming is done in chronological order with arbitrary tie-breaking. Denote by $\beta(O_i^k)$ the minimal $l \in \mathbb{N}$ such that $O_i^k \subseteq \bigcup_{j \in [l]} B_j$. We want to show that $p \succsim_i p_{-C}$ for all $i \in C$ which is equivalent to $p(i, O_i^k) \geq p_{-C}(i, O_i^k)$ for all $i \in C, k \in [k_i]$.

First, we claim that $p_{-C} \succsim_i p$ for all $i \in N \setminus C$. This holds true as all original bottlenecks either remain unchanged when C abstains or occur later (in a possibly changed version). In particular, they cannot appear earlier, as less agents compete for the objects. Hence, $p_{-C}(i, O_i^k) \geq p(i, O_i^k)$ for all $i \in N \setminus C, k \in [k_i]$ which proves the claim.

Now consider any agent $i \in C$, $k \in [k_i]$ and define $B = \bigcup_{j \in [\beta(O_i^k)]} B_j$, the set of all objects that are part of some bottleneck up to $B_{\beta(O_i^k)}$. We have that $p(i, O_i^k) = p(i, B)$ since for all B_j, $j \in [\beta(O_i^k)]$, such that $B_j \cap O_i^k = \emptyset$, i is not awarded any probability. However, they are completely consumed by other agents until all objects in B are consumed, hence, $p(i, B_j) = 0$ holds for them.

Note that i is awarded some probability in bottleneck $B_{\beta(O_i^k)}$, which means that at this point no other agent can have received more total probability of B than i. In particular, this holds for all agents in C. We thus conclude that $p(i, B) \geq {}^1\!/_{|C|}\, p(C, B)$.

Concerning the total probability awarded up to the moment of bottleneck $B_{\beta(O_i^k)}$, we trivially have that

$$p(C, B) + p(N \setminus C, B) = |B|$$

and consequently

$$p(C, B) = |B| - p(N \setminus C, B).$$

We now make use of our initial claim about agents not in C preferring p_{-C} to p and conclude

$$|B| - p(N \setminus C, B) \geq |B| - p_{-C}(N \setminus C, B).$$

A variant of the sum formula of $|B|$ which we used before yields

$$|B| - p_{-C}(N \setminus C, B) = p_{-C}(C, B).$$

Recall that by our definition of abstention, a group C that does not participate is given the 'remaining' probability of all objects which is then distributed evenly among agents in C. Thus,

$${}^1\!/_{|C|}\, p_{-C}(C, B) = p_{-C}(i, B)$$

and since $O_i^k \subseteq B$, we have that $p_{-C}(i, B) \geq p_{-C}(i, O_i^k)$.

[8]Theoretically, agents $C_B \setminus \{i\}$ do not necessarily belong to the same bottleneck. However, they will each be part of some bottleneck before the algorithm terminates. We omit details for the sake of readability.

[9]As before, $C_B \setminus \{i\}$ do not necessarily contribute to the same bottleneck, they may be part of different ones. However, all of them occur at some point between t_B and the hypothetical $t_{B'}$. We once more omit details for the sake of readability.

Putting everything together, we obtain the following chain of (in)equalities:

$$\begin{aligned} p(i, O_i^k) &= p(i, B) \\ &\geq {}^1/_{|C|}\, p(C, B) \\ &= {}^1/_{|C|}\, [|B| - p(N \setminus C, B)] \\ &\geq {}^1/_{|C|}\, [|B| - p_{-C}(N \setminus C, B)] \\ &= {}^1/_{|C|}\, p_{-C}(C, B) \\ &= p_{-C}(i, B) \\ &\geq p_{-C}(i, O_i^k) \end{aligned}$$

This proves strong group-participation of PS. $\square$

3.3 Boston Mechanism

The *Boston mechanism* BM originates from the domain of school choice. In this context, BM is arguably one of the simplest rules: Consider only top-ranked schools in the first round and assign the top-ranked school to every agent as long as there are enough available seats; if not break ties randomly. Now, remove all students who have been assigned a seat and their respective seats, and consider the students' second most-preferred schools in the next round. Again, seats are assigned to students and ties are broken randomly. This procedure continues until no students are left.[10]

In our framework, we assume there is an equal number of schools and students with only one seat per school. In addition, we require that individual preferences are strict—it is unclear how to define BM for general preferences.

Unfortunately, the relative straightforwardness of BM comes at a price: Among other shortcomings, BM may yield unstable assignments and is easily manipulable by a large number of agents [1, 17, 19]. These findings reduced BM's popularity among researchers and practitioners. Nevertheless, BM is still considered an important assignment rule, which is also reflected by a recent axiomatic characterization due to Kojima and Ünver [24].

With respect to participation, it turns out that, when only single agents abstain, it fares equally well as RSD and PS, i.e., it satisfies very strong participation. When considering abstention by groups of agents, results are mixed. While BM satisfies group-participation, it violates strong group-participation (which is satisfied by both RSD and PS).

THEOREM 4. *BM satisfies very strong participation.*

PROOF. Let $(N, O, \succsim)$ be an assignment problem with $N = \{1, \dots, n\}$, $O = \{o_1, \dots, o_n\}$, $i \in N$, and $p = BM(\succsim)$. Without loss of generality, we may assume that i has preferences $o_1 \succ_i \cdots \succ_i o_n$. First assume that i is the only agent who ranks o_1 at the top. In this case we have that $p(i, o_1) = 1$ and very strong participation is trivially satisfied as i gets the best possible result when participating.

Now, suppose there exists another agent who also lists o_1 as top choice. We have that $p_{-i}(i, o_1) = 0$ and $p(i, o_1) > 0$. In addition, for all agents $i' \in N \setminus \{i\}$ and $k \in [k_{i'}]$, we have that $p_{-i}(i', O_{i'}^k) \geq p(i', O_{i'}^k)$. This holds true as reduced competition cannot 'harm' the remaining agents.

Going back to the abstaining agent i we compare $p(i, o_j)$ to $p_{-i}(i, o_j)$ for $2 \leq j \leq k_i$. We have that if $p_{-i}(i, o_j) > p(i, o_j)$ for some j, then $p_{-i}(N \setminus \{i\}, o_j) < p(N \setminus \{i\}, o_j)$. By the observation above it however holds that $p_{-i}(i', O_{i'}^k) \geq p(i', O_{i'}^k)$ for all $i' \neq i$ and $k \in [k_{i'}]$ which means that $p(N \setminus \{i\}, O_i^{j-1}) < p_{-i}(N \setminus \{i\}, O_i^{j-1})$ where

$$p_{-i}(i, o_j) - p(i, o_j) \leq p_{-i}(N \setminus \{i\}, O_i^{j-1}) - p(N \setminus \{i\}, O_i^{j-1}).$$

Hence, $p_{-i}(i, O_i^k) \leq p(i, O_i^k)$ for all $k \in [k_i]$ and $p_{-i}(i, o_1) = 0 < p(i, o_1)$ by the initial assumption.

Put less formally, even though i's probability for some object o_j can rise, his maximum gain in probability is capped by the sum of probabilities he has lost for objects $\{o_1, \dots, o_{j-1}\}$. Together with the fact that by abstaining, i loses all probability for o_1, this shows very strong participation. $\square$

THEOREM 5. *BM does not satisfy strong group-participation.*

PROOF. Consider the following assignment problem $(N, O, \succsim)$ with $N = \{1, 2, 3, 4\}$, $O = \{a, b, c, d\}$, and $\succsim$ as given below and the corresponding random assignment $p = BM(\succsim)$:

$$\succsim = \begin{array}{ll} 1: & a, b, c, d \\ 2: & a, b, c, d \\ 3: & b, a, c, d \\ 4: & d, a, b, c \end{array} \qquad p = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If agents 1, 2, and 3 abstain, i.e., $C = \{1, 2, 3\}$, each of them is assigned the uniform lottery over objects a, b, and c, i.e.,

$$p_{-C} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But $p \not\succsim_i p_{-C}$ for $i \in \{1, 2\}$. Hence, BM violates strong group-participation. $\square$

THEOREM 6. *BM satisfies group-participation.*

PROOF. In order to prove group-participation of BM, we have to show that for no assignment problem $(N, O, \succsim)$ and $C \subseteq N$ it holds that $p_{-C} \succ_i p$ for all $i \in C$ where $p = BM(\succsim)$.

We first consider the case where at least two agents i, i' out of C have disjoint most preferred objects. For reasons of readability assume i's favorite object is o_1. In this instance we have that C's total probability for objects top-ranked among C cannot increase when C abstains, i.e., $p_{-C}(C, o) \leq p(C, o)$ for all $o \in O_j^1, j \in C$. Since it trivially also holds that $p(i, o_1) \geq p(j, o_1)$ for all $j \in C$ and $p(i, o_1) > p(i', o_1) = 0$ we obtain in total

$$\begin{aligned} p_{-C}(i, o_1) &= {}^1/_{|C|}\, p_{-C}(C, o_1) \\ &\leq {}^1/_{|C|}\, p(C, o_1) \\ &< p(i, o_1). \end{aligned}$$

Consequently, $p_{-C} \not\succ_i p$.

Now assume all agents in C have identical first k levels of preferences. If $p_{-C}(C, O_i^k) = |C|$, $i \in C$, then also $p(C, O_i^k) = |C|$ and $p \succsim_i p_{-C}$ for all $i \in C$. If on the other hand $p_{-C}(C, O_i^k) < |C|$, $i \in C$, then either $p(C, O_i^k) = |C|$ and consequently $p \succsim_i p_{-C}$ for all $i \in C$ or $p(i, O_i^{k+1}) > p_{-C}(i, O_i^{k+1})$ for similar reasons as above and thus $p_{-C} \not\succ_i p$. This completes the proof. $\square$

[10]Note that what we describe here is sometimes also called *naive* Boston mechanism in contrast to the *adaptive* Boston mechanism where students apply to their most-preferred school that still has free seats. We here consider the naive BM for simplicity. Our results, however, also hold for the adaptive BM.

3.4 Popular Random Assignments

We finally consider a class of random assignment rules that is based on the notion of popularity. Popularity was first considered in the context of deterministic assignments by Gärdenfors [21]. An assignment is *popular* if there exists no other assignment that is preferred by a majority of those agents that do not receive identical objects in both assignments. Popular assignments correspond to weak Condorcet winners in social choice theory and unfortunately do not have to exist.

The problem of potential non-existence was addressed in 2011 by Kavitha et al. [23] who introduced popular *random* assignments. A random assignment is *popular* if there does not exist another random assignment that is preferred by an *expected* majority of agents. In contrast to RSD, whose outcome was shown to be $\#P$-complete to compute [4], popular random assignments can be found efficiently via linear programming [23]. However, popular random assignments need not be unique. A popular random assignment rule is a rule that always returns popular random assignments.

The axiomatic study of popular random assignment rules was initiated by Aziz et al. [6], who showed that all popular random assignment rules satisfy efficiency and there always exists at least one popular random assignment satisfying equal treatment of equals. On the other hand, popularity is incompatible with envy-freeness and strong strategyproofness if $n \geq 3$—impossibilities that were recently strengthened to weak envy-freeness and strategyproofness by Brandt et al. [16].

Aziz et al. [6] also pointed out that popular random assignment rules are a special case of randomized voting rules returning so-called *maximal lotteries*. Brandl et al. [14] proved that all rules that return maximal lotteries satisfy group-participation. We therefore directly obtain the following statement.

Corollary 2. *All popular random assignment rules satisfy group-participation.*

Proof. The statement follows directly from Brandl et al. [14, Cor. 1] and the observation that assignment is a special case of voting [cf., e.g., 6]. □

Again, this result stands in contrast to results about strategyproofness because popular random assignment rules are manipulable.

For the remainder of this section, we make the reasonable assumption that popular random assignment rules assign the same lottery to all abstaining agents, i.e., to all agents that are indifferent between all objects. It turns out that the strongest notions of participation and group-participation we consider are not satisfied by popular random assignments.

Theorem 7. *All popular random assignment rules violate very strong participation and strong group-participation.*

Proof. We start with very strong participation. To this end, let $(N, O, \succsim)$ be an assignment problem with $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, and $\succsim$ as depicted below.

$$\succsim = \begin{array}{rl} 1: & a, c, b \\ 2: & a, b, c \\ 3: & a, b, c \end{array} \qquad p = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 1-\lambda & 0 \\ 1-\lambda & \lambda & 0 \end{pmatrix}$$

For this assignment problem, all popular random assignments are of the form p with $0 \leq \lambda \leq 1$. Hence, $p(1, c) = p_{-1}(1, c) = 1$ even though $c \notin O_1^1$ which violates very strong participation.

For strong group-participation consider the assignment problem $(N', O', \succsim')$ with $N' = \{1, 2, 3, 4\}$, $O' = \{a, b, c, d\}$ and $\succsim'$ as depicted below.

$$\succsim' = \begin{array}{rl} 1: & a, b, c, d \\ 2: & a, b, c, d \\ 3: & b, a, c, d \\ 4: & d, a, b, c \end{array}$$

All popular random assignments for this assignment problem are of the form p' with $0 \leq \lambda \leq 1$. Now, if $C = \{1, 2, 3\}$ abstains, only agent 4 remains and p'_{-C}, as given below, is the unique popular random assignment.

$$p' = \begin{pmatrix} \lambda & 0 & 1-\lambda & 0 \\ 1-\lambda & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad p'_{-C} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For at least one agent $i \in \{1, 2\}$ we have that $p'(i, O_i^2) \leq 1/2 < 2/3 = p'_{-C}(i, O_i^2)$ contradicting strong group-participation. □

4. CONCLUSIONS

We studied well-known random assignment rules under the assumption that participation is optional. Our main concern are not agents who deliberately abstain to improve their assignment (because this requires the agents to be very well-informed about the others' preferences). Rather, we think of settings where participation is associated with a small effort or cost (e.g., for figuring out one's own preferences). Our positive results show that participation is encouraged because it can only lead to more utility (sometimes even strictly). Participation is also desirable from the planner's perspective because it is required to identify efficient assignments of the objects.

Our results show that all considered rules satisfy a weak notion of participation (even for groups of agents). Perhaps surprisingly, RSD and PS even satisfy a strong notion of group-participation that is prohibitive in the more general voting domain [13]. Moreover, all considered rules except popular random assignment rules even provide strict incentives to participate. Whether popular random assignments satisfy strong participation remains an interesting, but presumably challenging, open problem.

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REFERENCES

[1] A. Abdulkadiroğlu and T. Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

[2] H. Aziz. Computational social choice: Some current and new directions. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 4054–4057, 2016.

[3] H. Aziz and P. Stursberg. A generalization of probabilistic serial to randomized social choice. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI)*, pages 559–565, 2014.

[4] H. Aziz, F. Brandt, and M. Brill. The computational complexity of random serial dictatorship. *Economics Letters*, 121(3):341–345, 2013.

[5] H. Aziz, F. Brandt, and M. Brill. On the tradeoff between economic efficiency and strategyproofness in randomized social choice. In *Proceedings of the 12th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 455–462. IFAAMAS, 2013.

[6] H. Aziz, F. Brandt, and P. Stursberg. On popular random assignments. In *Proceedings of the 6th International Symposium on Algorithmic Game Theory (SAGT)*, volume 8146 of *Lecture Notes in Computer Science (LNCS)*, pages 183–194. Springer-Verlag, 2013.

[7] S. Bade. Fairness and group-strategyproofness clash in assignment problems. *Journal of Economic Theory*, 165:257–262, 2016.

[8] S. Barberà, A. Bogomolnaia, and H. van der Stel. Strategy-proof probabilistic rules for expected utility maximizers. *Mathematical Social Sciences*, 35(2): 89–103, 1998.

[9] A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

[10] A. Bogomolnaia, H. Moulin, and R. Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

[11] S. Bouveret, Y. Chevaleyre, and N. Maudet. Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 12. Cambridge University Press, 2016.

[12] F. Brandl, F. Brandt, C. Geist, and J. Hofbauer. Strategic abstention based on preference extensions: Positive results and computer-generated impossibilities. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 18–24. AAAI Press, 2015.

[13] F. Brandl, F. Brandt, and J. Hofbauer. Incentives for participation and abstention in probabilistic social choice. In *Proceedings of the 14th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1411–1419. IFAAMAS, 2015.

[14] F. Brandl, F. Brandt, and J. Hofbauer. Welfare maximization entices participation. Technical report, http://arxiv.org/abs/1508.03538, 2015.

[15] F. Brandt, C. Geist, and D. Peters. Optimal bounds for the no-show paradox via SAT solving. In *Proceedings of the 15th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 314–322. IFAAMAS, 2016.

[16] F. Brandt, J. Hofbauer, and M. Suderland. Majority graphs of assignment problems and properties of popular random assignments. In *Proceedings of the 16th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 2017. Forthcoming.

[17] Y. Chen and T. Sönmez. School choice: An experimental study. *Journal of Economic Theory*, 127: 202–231, 2006.

[18] Y. Chevaleyre, P. E. Dunne, U. Endriss, J. Lang, M. Lemaître, N. Maudet, J. Padget, S. Phelps, J. A. Rodríguez-Aguilar, and P. Sousa. Issues in multiagent resource allocation. *Informatica*, 30:3–31, 2006.

[19] H. Ergin and T. Sönmez. Games of school choice under the Boston mechanism. *Journal of Public Economics*, 90:215–237, 2006.

[20] P. C. Fishburn and S. J. Brams. Paradoxes of preferential voting. *Mathematics Magazine*, 56(4): 207–214, 1983.

[21] P. Gärdenfors. Match making: Assignments based on bilateral preferences. *Behavioral Science*, 20(3): 166–173, 1975.

[22] A.-K. Katta and J. Sethuraman. A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, 131(1):231–250, 2006.

[23] T. Kavitha, J. Mestre, and M. Nasre. Popular mixed matchings. *Theoretical Computer Science*, 412(24): 2679–2690, 2011.

[24] F. Kojima and M. U. Ünver. The "Boston" school-choice mechanism: An axiomatic approach. *Economic Theory*, 55:515–544, 2014.

[25] D. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific Publishing Company, 2013.

[26] H. Moulin. Condorcet's principle implies the no show paradox. *Journal of Economic Theory*, 45:53–64, 1988.

[27] T. Sönmez and M. U. Ünver. Matching, allocation, and exchange of discrete resources. In J. Benhabib, M. O. Jackson, and A. Bisin, editors, *Handbook of Social Economics*, volume 1, chapter 17, pages 781–852. Elsevier, 2011.