Complexity Results for Aggregating Judgments using Scoring or Distance-Based Procedures

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ABSTRACT

Judgment aggregation is an abstract framework for studying collective decision making by aggregating individual opinions on logically related issues. Important types of judgment aggregation methods are those of scoring and distance-based methods, many of which can be seen as generalisations of voting rules. An important question to investigate for judgment aggregation methods is how hard it is to find a collective decision by applying these methods. In this article we study the complexity of this "winner determination" problem for some scoring and distance-based judgment aggregation procedures. Such procedures aggregate judgments by assigning values to judgment sets. Our work fills in some of the last gaps in the complexity landscape for winner determination in judgment aggregation. Our results reaffirm that aggregating judgments is computationally hard and strongly point towards the necessity of analyzing approximation methods or parameterized algorithms in judgment aggregation.

CCS Concepts

•**Theory of computation** → **Problems, reductions and completeness;** •**Computing methodologies** → *Artificial intelligence;*

Keywords

Judgment aggregation; computational complexity; winner determination; scoring procedures; distance-based procedures

1. INTRODUCTION

"Should we recommend buying the ticket now?", "Should we wait, would the tickets sell out?", "Does the customer want this ticket?". These questions, or *issues*, cannot be independently answered as they are logically related. For instance, a yes-answer on the first question implies a yes-answer on either the second or the third question. Aggregating binary answers from multiple sources to questions such as these is the focus of judgment aggregation studies [18]. Judgment aggregation is a social choice discipline that generalises voting and preference aggregation [7, 27], and has been investigated using computer science methods in the field of computational social choice [12].

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One of the topics studied in computational social choice is the computational complexity analysis of problems related to aggregation functions. One particular problem that is highly relevant is the problem of determining the outcomes of the aggregation procedure. In voting, the complexity of this problem indicates how difficult it is to find the winners of an election by using a particular voting method. Starting from the seminal work by Bartholdi, Tovey and Trick [2], numerous articles explore this topic and today the complexity landscape of the winner determination problem in voting is well covered. In judgment aggregation, the "winner" is a set of truth-value assignments, *i.e.*, *judgments*, to a set of logically related issues. Namely, judgment aggregation functions aggregate a *profile of judgments*, which is a collection of individual judgment sets, into a collective judgment set. Judgment aggregation is a much younger discipline with fewer aggregation methods investigated in the literature, but even so the complexity of the "winner determination" problem is not studied for every known method in the literature. The aim of this paper is to cover more of this gap.

The winner determination problem in judgment aggregation has been studied in various papers [14, 15, 28] covering the following judgment aggregation methods: Premise Based Procedure, Condorcet Admissible Set, Slater, Kemeny (also known as Median Rule), Young, Tideman, and *MNAC*[1]. Furthermore, Jamroga and Slavkovik [21] show some bounds for this problem for the class of distance-based judgment aggregation methods.

Contributions.

In this paper, we analyze the computational complexity of the winner determination problem for several judgment aggregation methods:

- the reversal scoring method introduced by Dietrich [6],
- the distance-based method for the Hamming distance and addition as aggregator function, introduced as the arbitration belief merging aggregator by Konieczny and Pino Pérez [23] and as judgment aggregator by Lang, Pigozzi, Slavkovik and Van der Torre [26], and
- the distance-based methods for the geodesic distance and addition as aggregator function, as introduced by Duddy and Piggins [9].

The first of these methods is a member of the class of *scoring judgment aggregators* and the second and third are from

[1]The reader can consult for example [26] for an integral overview of aggregation methods in judgment aggregation.

the class of *distance-based aggregators*. We show that deciding whether a judgment set can be selected as winner for a given profile by the first two of these methods is Θ_2^p-complete (Theorems 2 and 3). For the third method, this problem is Θ_3^p-complete (Theorem 4).

Both the scoring aggregators and the distance-based aggregators aggregate judgments by assigning values to show how similar a particular judgment set is to the aggregated profile. The distance-based aggregators are defined with respect to a distance that measures how different two judgment sets are. The scoring aggregators are defined with respect to a scoring function which attaches a value for each judgment in a judgment set. In this paper, we also show the general result that if the scoring function is computable in polynomial time, deciding whether a judgment set can be selected as winner for a given profile is in the class Θ_2^p (Theorem 1).

The completeness results that we establish for the winner determination problem, for various aggregators, are important steps for identifying what algorithmic approaches could work well to solve this problem in applied settings.

2. PRELIMINARIES

In this section, we begin by describing the formal framework that we use to model judgment aggregation scenarios (as used by, e.g., [8, 14, 28]), and briefly surveying some relevant notions from the theory of computational complexity.

Judgment Aggregation.

A propositional formula is *doubly-negated* if it is of the form $\neg\neg\psi$. For every propositional formula φ, we let $\sim\varphi$ denote the *complement* of φ, *i.e.*, $\sim\varphi = \neg\varphi$ if φ is not of the form $\neg\psi$, and $\sim\varphi = \psi$ if φ is of the form $\neg\psi$. For a propositional formula φ, the set $\text{Var}(\varphi)$ denotes the set of all variables occurring in φ.

An *agenda* is a finite, nonempty set Φ of formulas that does not contain any doubly-negated formulas and that is closed under complementation. Moreover, if $\Phi = \{\varphi_1, \dots, \varphi_n, \neg\varphi_1, \dots, \neg\varphi_n\}$ is an agenda, then we let $[\Phi] = \{\varphi_1, \dots, \varphi_n\}$ denote the *pre-agenda* associated to the agenda Φ. We denote the bitsize of the agenda Φ by $\text{size}(\Phi) = \sum_{\varphi \in \Phi} |\varphi|$. A *judgment set* J for an agenda Φ is a subset $J \subseteq \Phi$. A judgment set J is *complete* if $\varphi \in J$ or $\sim\varphi \in J$ for all $\varphi \in \Phi$; and it is *consistent* if there exists an assignment that makes all formulas in J true. Intuitively, the consistent and complete judgment sets are the opinions that individuals and the group can have.

We associate with each agenda Φ an integrity constraint Γ, that can be used to further restrict the set of feasible opinions. Such an *integrity constraint* consists of a single propositional formula. A judgment set J is Γ-*consistent* if there exists a truth assignment that simultaneously makes all formulas in J and Γ true. Let $\mathcal{J}(\Phi, \Gamma)$ denote the set of all complete and Γ-consistent subsets of Φ. We say that finite sequences $\mathbf{J} \in \mathcal{J}(\Phi, \Gamma)^+$ of complete and Γ-consistent judgment sets are *profiles*, and where convenient we equate a profile $\mathbf{J} = (J_1, \dots, J_p)$ with the (multi)set $\{J_1, \dots, J_p\}$.

A *judgment aggregation procedure* (or *rule*) for the agenda Φ and the integrity constraint Γ is a function F that takes as input a profile $\mathbf{J} \in \mathcal{J}(\Phi, \Gamma)^+$, and that produces a non-empty set of non-empty judgment sets. We call a judgment aggregation procedure F *resolute* if for any profile $\mathbf{J}$ it returns a singleton; otherwise, we call F *irresolute*.

The Polynomial Hierarchy.

We begin with reviewing some basic notions from computational complexity. We assume the reader to be familiar with the complexity classes P and NP, and with basic notions such as polynomial-time reductions. For more details, we refer to textbooks on computational complexity theory (see, e.g., [1]). We would like to remind you of the quintessential NP-complete problem SAT.

SAT
Instance: A propositional formula φ.
Question: Is there an interpretation that satisfies φ?

We briefly review the classes of the Polynomial Hierarchy (PH) [29, 33, 36, 38]. In order to do so, we consider quantified Boolean formulas. A *quantified Boolean formula (in prenex form)* is a formula of the form $Q_1 x_1 Q_2 x_2 \dots Q_n x_n.\psi$, where all x_i are propositional variables, each Q_i is either an existential or a universal quantifier, and ψ is a (quantifier-free) propositional formula over the variables $x_1, \dots, x_n$. Truth for such formulas is defined in the usual way.

To consider the complexity classes of the PH, we restrict the number of quantifier alternations occurring in quantified Boolean formulas, *i.e.*, the number of times where $Q_i \neq Q_{i+1}$. We consider the complexity classes Σ_k^p, for each $k \geq 1$. Let $k \geq 1$ be an arbitrary, fixed constant. The complexity class Σ_k^p consists of all decision problems for which there exists a polynomial-time reduction to the problem QSAT$_k$, that is defined as follows. Instances of the problem QSAT$_k$ are quantified Boolean formulas of the form $\exists x_1 \dots \exists x_{\ell_1} \forall x_{\ell_1+1} \dots \forall x_{\ell_2} \dots Q_k x_{\ell_{k-1}+1} \dots Q_k x_{\ell_k}.\psi$, where $Q_k = \exists$ if k is odd and $Q_k = \forall$ if k is even, where $1 \leq \ell_1 \leq \dots \leq \ell_k$, and where ψ is quantifier-free. The problem is to decide if the quantified Boolean formula is true. The complementary class Π_k^p consists of all decision problems for which there exists a polynomial-time reduction to the problem co-QSAT$_k$, that is complementary to the problem QSAT$_k$. The Polynomial Hierarchy (PH) contains these classes Σ_k^p and Π_k^p.

Alternatively, one can characterize the class Σ_2^p using nondeterministic polynomial-time algorithms with access to an oracle for an NP-complete problem. Let O be a decision problem. A Turing machine $\mathbb{M}$ with access to an O *oracle* is a Turing machine with a dedicated *oracle tape* and dedicated states q_{query}, q_{yes} and q_{no}. Whenever $\mathbb{M}$ is in the state q_{query}, it does not proceed according to the transition relation, but instead it transitions into the state q_{yes} if the oracle tape contains a string x that is a yes-instance for the problem O, *i.e.*, if $x \in O$, and it transitions into the state q_{no} if $x \notin O$. Intuitively, the oracle solves arbitrary instances of O in a single time step. The class Σ_2^p consists of all decision problems that can be solved in polynomial time by a nondeterministic Turing machine that has access to an O-oracle, for some $O \in \text{NP}$.

We will also refer to the complexity class Θ_2^p, that consists of all decision problems that can be solved by a polynomial-time algorithm that queries an NP oracle $O(\log n)$ times. The following two problems are complete for the class Θ_2^p under polynomial-time reductions. The first is the MAX-MODEL [5, 24, 37].

MAX-MODEL
Instance: A satisfiable propositional formula φ, and a variable $w \in \text{Var}(\varphi)$.
Question: Is there a model of φ that sets a maximal number of variables in $\text{Var}(\varphi)$ to true (among all models of φ) and that sets w to true?

The second is the problem MaxCardScepticalInference from [28], which was proved to be Θ_2^{p}-complete [31] (under the name Cardinality-maximizing base revision). A supernormal default theory is a pair $D = \langle \Delta, \beta \rangle$ with $\Delta = \{\varphi_1, \ldots, \varphi_p\}$, where $\varphi_1, \ldots, \varphi_p, \beta$ are propositional logic formulas. A propositional formula α is a maxcard sceptical inference of D iff $S \cup \{\beta\} \models \alpha$ for every S that is a maximally consistent subset of Δ with respect to set cardinality. The MaxCard Sceptical Inference takes an instance consisting of D, β and α and returns yes if α is a maxcard sceptical inference of D.

MaxCardScepticalInference
Instance: A supernormal default theory $D = \langle \Delta, \beta \rangle$, propositional formula α.
Question: Is α a maxcard sceptical inference of D?

In addition, we will refer to the complexity class Θ_3^{p}, that consists of all decision problems that can be solved by a polynomial-time algorithm that queries an Σ_2^{p} oracle $O(\log n)$ times. The following problem is complete for the class Θ_3^{p} under polynomial-time reductions.

QSat$_2$-Max-Model
Instance: A satisfiable instance φ of QSat$_2$, where $\varphi = \exists X. \forall Y. \psi$, and a variable $w \in \mathrm{Var}(\varphi)$.
Question: Is there a truth assignment $\alpha : X \to \{0, 1\}$ s.t. $\psi[\alpha]$ is valid, that sets a maximal number of variables in X to true (among all such assignments making $\psi[\alpha]$ valid), and that sets w to true?

In the rest of the paper we are concerned with studying the computational complexity of the following problem defined for a judgment aggregator F.

Outcome(F)
Instance: An agenda Φ with an integrity constraint Γ, a profile $\mathbf{J} \in \mathcal{J}(\Phi, \Gamma)^+$ and subsets $L, L_1, \ldots, L_u \subseteq \Phi$ with $u \geq 0$.
Question: Is there a judgment set $J^* \in F(\mathbf{J})$ s.t. $L \subseteq J^*$ and $L_i \not\subseteq J^*$ for each $i \in \{1, \ldots, u\}$.

In the next two sections we consider members of two classes of judgment aggregators: *scoring methods* and *distance-based methods*.

3. SCORING METHODS

A score $s : \mathcal{J}(\Phi, \Gamma) \times \Phi \to \mathbb{R}^+$ is a function that assigns a nonnegative value for a judgment in Φ with respect to a rational judgment set from $\mathcal{J}(\Phi, \Gamma)$. A *scoring judgment aggregator* F_s is a function defined for a score s as:

$$F_s(\mathbf{J}) = \operatorname*{argmax}_{J \in \mathcal{J}(\Phi, \Gamma)} \sum_{\substack{J_i \in \mathbf{J} \\ \varphi \in J}} s(J_i, \varphi). \qquad (1)$$

Before we turn to the results that we develop in this section, we introduce some definitions and observations. An interpretation ω for the agenda Φ assigns a value true or false for each judgment in Φ. An interpretation satisfies a formula $\varphi \in \Phi$ if it evaluates it to true while also evaluating Γ to true, in the classic propositional logic sense; we write $\omega \models_\Gamma \varphi$. An interpretation satisfies a judgment set $J \in \mathcal{J}(\Phi, \Gamma)$ iff $\omega \models_\Gamma \bigwedge J$. The set Ω is the set of all interpretations that satisfy Φ with respect to Γ.

Algorithm 1: OutcomeScore($\mathbf{J^s}, k$) is in NP.

Set $k' := 0$
Set $J := \emptyset$
Guess an interpretation ω
For each $\langle \varphi_i, \sigma_i \rangle \in \mathbf{J^s}$ do
    if $\omega \models_\Gamma \varphi_i$ then do
        $J := J \cup \{\varphi_i\}$
Check $\omega \models_\Gamma J$
Check value($\mathbf{J^s}, J) \geq k$

Algorithm 2: Outcome(F_s) is in Θ_2^{p}.

Set $m_n := 0$
Set $m_x := \sum_{\langle \varphi, \sigma \rangle \in \mathbf{J^s}} \sigma$
Set $k := \frac{m_x}{2}$
while $m_n \neq m_x$ do
    if OutcomeScore($\mathbf{J^s}, k$) then $m_n := k$
    else $m_x = k$
$k = \frac{m_x + m_n}{2}$

Observe first that given a score s, we can represent any profile $\mathbf{J}$ as a set of pairs $\mathbf{J^s}$ in the following way:

$$\mathbf{J^s} = \{\langle \varphi, \sum_{J_i \in \mathbf{J}} s(J_i, \varphi) \rangle \mid \varphi \in \Phi\} \qquad (2)$$

Let us define the *value* of a judgment set $J \in F_s(\mathbf{J})$ with respect to a profile $\mathbf{J}$ and a score s as:

$$\mathrm{value}(\mathbf{J^s}, J) = \sum_{\substack{\langle \varphi, \sigma \rangle \in \mathbf{J^s} \\ \varphi \in J}} \sigma. \qquad (3)$$

In this section, we will also consider the following auxiliary problem, defined for a score s.

OutcomeScore($\mathbf{J^s}, k$)
Instance: An agenda Φ, constraints Γ, a profile $\mathbf{J^s} \in \Phi \times \mathbb{R}^+$ and a real nonnegative number $k \in \mathbb{R}^+$.
Question: Is there a judgment set $J^* \in \mathcal{J}(\Phi, \Gamma)$ s.t. value($\mathbf{J^s}, J^*) \geq k$?

3.1 Scoring methods with polynomial scores

In this section, we show the general result that for any score s that is computable in polynomial time—that is, $\mathbf{J^s}$ can be computed from $\mathbf{J}$ in polynomial time—the winner determination problem Outcome(F_s) is in Θ_2^{p}.

Theorem 1. *If $\mathbf{J^s}$ can be computed from $\mathbf{J}$ in polynomial time, then* Outcome(F_s) *is in* Θ_2^{p}.

Proof. The proof is similar to the proof of Property 4.20 in [11]. We first consider the problem OutcomeScore($\mathbf{J^s}, k$). Algorithm 1 shows that OutcomeScore($\mathbf{J^s}, k$) is in NP.

We then show that Outcome(F_s) is in Θ_2^{p} via Algorithm 2. Algorithm 2 finds the maximal value k for which there is some judgment set J with value($\mathbf{J^s}, J) = k$. Then, checking whether there is such a judgment set s.t. $L \subseteq J$ and $L_i \not\subseteq J$ for all i can be done with one additional NP oracle query. $\square$

Matching Θ_2^{p}-hardness results.

We showed that for any score s for which $\mathbf{J}^{\mathbf{s}}$ can be computed from $\mathbf{J}$ in polynomial time, the winner determination problem $\textsc{Outcome}(F_s)$ for the corresponding judgment aggregation procedure is contained in the class Θ_2^{p}. This is an upper bound—the complexity of winner determination for such scoring procedures is not higher than Θ_2^{p}.

Not every polynomial-time computable score s leads to a judgment aggregation procedure that is also complete for the class Θ_2^{p} (taking some common complexity-theoretic assumptions into account). There are trivial counterexamples. For instance, the trivial score s_0 that always returns 0—that is, for which $s_0(J, \varphi) = 0$ for all J and all φ—results in a judgment aggregation procedure F_{s_0} for which the winner determination problem is in NP. Namely, all complete and Γ-consistent judgment sets J get the same total score $\sum_{J_i \in \mathbf{J}, \varphi \in J} s_0(J_i, \varphi) = 0$. Therefore, deciding whether there is a complete and Γ-consistent judgment set J^* with maximum score and with $L \subseteq J^*$ and $L_i \not\subseteq J^*$ for all i can be done in nondeterministic polynomial time. In particular, this tells us that $\textsc{Outcome}(F_{s_0})$ is not Θ_2^{p}-complete, unless the Polynomial Hierarchy collapses.

There are, however, natural scoring judgment aggregation procedures F_s for which the score s is computable in polynomial-time and for which the problem $\textsc{Outcome}(F_{s_0})$ is Θ_2^{p}-complete. One example of such a procedure is the *simple scoring procedure* [6], based on the score spl defined by letting $\mathrm{spl}(J, \varphi) = 1$ if $\varphi \in J$ and $\mathrm{spl}(J, \varphi) = 0$ if $\varphi \notin J$. This procedure is also known as the Kemeny judgment aggregation procedure and its winner determination problem is known to be Θ_2^{p}-complete [14, 15, 28].

3.2 Reversal scoring aggregator

In this section, we consider the *reversal scoring procedure* [6]. This procedure can be seen as a generalisation of the Borda rule in voting (for more details, we refer to Section 5). The reversal scoring procedure is defined as follows.

Let $J \in \mathcal{J}(\Phi, \Gamma)$ be a complete and Γ-consistent judgment set for an agenda Φ and an integrity constraint Γ. Moreover, let $\varphi \in \Phi$. Then the *reversal score* $\mathrm{rev}(J, \varphi)$ of φ for J is defined as the minimal number of formulas in J that need to be negated to get a complete and Γ-consistent judgment set J' that contains $\sim\varphi$. Formally:

$$\mathrm{rev}(J, \varphi) = \min_{\substack{J' \in \mathcal{J}(\Phi,\Gamma) \\ \sim\varphi \in J'}} d_{\mathrm{H}}(J', J).$$

(Here d_{H} denotes the Hamming distance between two complete judgment sets, *i.e.*, $d_{\mathrm{H}}(J, J') = |(J \setminus J')| = |(J' \setminus J)|$.) Based on the reversal score, we get the *reversal scoring judgment aggregation procedure* F_r, that is defined as follows:

$$F_r(\mathbf{J}) = \operatorname*{argmax}_{J^* \in \mathcal{J}(\Phi,\Gamma)} \sum_{\substack{J_i \in \mathbf{J} \\ \varphi \in J^*}} \mathrm{rev}(J_i, \varphi).$$

The reversal scoring judgment aggregation procedure is irresolute, complete and Γ-consistent.

We show that the winner determination problem for the reversal scoring judgment aggregation procedure is Θ_2^{p}-complete.

Since the reversal score is not computable in polynomial time (unless P = NP), we cannot directly apply Theorem 1. Nevertheless, $\textsc{Outcome}(F_r)$ is in Θ_2^{p}.

Lemma 1. *$\textsc{Outcome}(F_r)$ is in Θ_2^{p}.*

Proof. We show membership in Θ_2^{p} by describing a polynomial-time algorithm that solves $\textsc{Outcome}(F_r)$ by making a logarithmic number of queries to (different) NP oracles. Let $(\Phi, \Gamma, \mathbf{J}, L, L_1, \dots, L_u)$ specify an instance of $\textsc{Outcome}(F_r)$, where $\mathbf{J} = (J_1, \dots, J_p)$.

Firstly, we determine the following value:

$$b_1 = \sum_{\substack{J_i \in \mathbf{J} \\ \varphi \in \Phi}} \min_{\substack{J' \in \mathcal{J}(\Phi,\Gamma) \\ \sim\varphi \in J'}} d_{\mathrm{H}}(J', J_i).$$

We will determine b_1 by means of binary search using one NP oracle. After having determined the value b_1, we can use this value to determine the maximum reversal score b_2 for any complete and Γ-consistent judgment set $J^* \in \mathcal{J}(\Phi, \Gamma)$ by means of binary search using another NP oracle. This second oracle uses the value b_1. Finally, after having determined the value b_2, we can decide whether the instance is a yes-instance by using one last query to the second oracle.

The NP oracle Q_1 that we will use to determine b_1 decides—for a given natural number b—whether $\sum_{J_i \in \mathbf{J}} \sum_{\varphi \in \Phi} \min_{J' \in \mathcal{J}(\Phi,\Gamma), \sim\varphi \in J'} d_{\mathrm{H}}(J', J_i) \leq b$. It is straightforward to verify that the problem Q_1 that this oracle solves is in NP.

We have a straightforward upper bound of $2pn^2$ for the value of b_1. Therefore, we can find the exact value of b_1 using $O(\log p + 2 \log n)$ queries to the oracle Q_1.

Next, we will determine the following value:

$$b_2 = \max_{J^* \in \mathcal{J}(\Phi,\Gamma)} \sum_{\substack{\varphi \in J^* \\ J_i \in \mathbf{J}}} \min_{\substack{J' \in \mathcal{J}(\Phi,\Gamma) \\ \sim\varphi \in J'}} d_{\mathrm{H}}(J', J_i).$$

The NP oracle Q_2 that we will use for this decides—for given sets $L', L'_1, \dots, L'_u \subseteq \Phi$ and a given natural number b—whether there exists a complete and Γ-consistent judgment set $J^* \in \mathcal{J}(\Phi, \Gamma)$ s.t. $L' \subseteq J^*$, $L'_i \not\subseteq J^*$ for each L'_i, and $\sum_{J_i \in \mathbf{J}} \sum_{\varphi \in J^*} \min_{J' \in \mathcal{J}(\Phi,\Gamma), \sim\varphi \in J'} d_{\mathrm{H}}(J', J_i) \geq b$. The problem Q_2 that this oracle solves is in NP. Because we know the value of b_1, we can guess for each $\varphi \in \Phi$ and each $J_i \in \mathbf{J}$ a judgment set J with $\neg\varphi \in J$, and verify that the total Hamming distance from these judgment sets to the sets $J_i \in \mathbf{J}$ is exactly b_1. This allows us to show that the problem Q_2 is in NP.

We can now determine the value of b_2 by using binary search. We have a straightforward upper bound of pn^2 for b_2. Therefore, we can find the exact value of b_2 using $O(\log p + 2 \log n)$ queries to the oracle Q_2 (where we take $L' = \emptyset$ and where we use no sets L'_i, *i.e.*, $u = 0$).

Finally, using one more query to the oracle Q_2, we can decide whether there exists a complete and Γ-consistent judgment set $J^* \in \mathcal{J}(\Phi, \Gamma)$ s.t.: (1) $L \subseteq J^*$, (2) $L_i \not\subseteq J^*$ for each L_i, and (3) $\sum_{J_i \in \mathbf{J}} \sum_{\varphi \in J^*} \min_{J' \in \mathcal{J}(\Phi,\Gamma), \sim\varphi \in J'} d_{\mathrm{H}}(J', J_i) \geq b_2$. For this query, we use $b = b_2$, $L = L'$, and $L_i = L'_i$ for each i. The instance of $\textsc{Outcome}(F_r)$ is a yes-instance iff this this last oracle query returns "yes." This completes our description of the algorithm that witnesses membership in Θ_2^{p}. $\square$

Theorem 2. *$\textsc{Outcome}(F_r)$ is Θ_2^{p}-hard.*

Proof. Membership in Θ_2^{p} follows from Lemma 1. We show Θ_2^{p}-hardness by giving a reduction from the Θ_2^{p}-complete problem $\textsc{Max-Model}$ [5, 24, 37]. Let (φ, w) be an instance of MAX-MODEL, where φ is a satisfiable propositional formula with $\text{Var}(\varphi) = \{x_1, \dots, x_n\}$ and $w \in \text{Var}(\varphi)$. Without loss of generality, we may assume that the truth assignment that sets every variable x_i to false satisfies φ. We construct an agenda Φ, an integrity constraint Γ, a profile $\mathbf{J}$, and a set $L \subseteq \Phi$ s.t. there is some $J^* \in F_r(\mathbf{J})$ with $L \subseteq J^*$ iff there is a model of φ that sets a maximum number of variables in $\text{Var}(\varphi)$ to true (among all models of φ) and that sets w to true.

Let $m = 3n + 2$. We introduce new propositional variables $z_{\ell,j}$ for each $\ell \in [3]$ and each $j \in [m]$. We define Φ by letting $[\Phi] = \{x_1, \dots, x_n\} \cup \{\, z_{\ell,j} : \ell \in [3], j \in [m] \,\}$. Moreover, we define Γ as follows:

$$\Gamma = \bigwedge_{\substack{j_1, j_2 \in [m] \\ j_1 \neq j_2}} \Big(\big(\bigwedge_{\substack{\ell \in [3] \\ j \in \{j_1, j_2\}}} \neg z_{\ell,j} \big) \rightarrow \varphi \Big).$$

We let $L = \{w\}$. Finally, we define the profile $\mathbf{J} = (J_1, J_2, J_3)$ as defined in Figure 1.

$\mathbf{J}$	J_1	J_2	J_3
x_i	1	1	1
$z_{1,j}$	1	0	0
$z_{2,j}$	0	1	0
$z_{3,j}$	0	0	1

Figure 1: The profile $\mathbf{J} = (J_1, J_2, J_3)$ in the proof of Theorem 2. Here i ranges over $[n]$ and j ranges over $[m]$.

We observe the following equalities, for each $i \in [n]$, each $\ell \in [3]$, and each $j \in [m]$: $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, x_i) = 3$, $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, \neg x_i) = 0$, $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, z_{\ell,j}) = 1$, and $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, \neg z_{\ell,j}) = 2$. We also observe that each of the complete and Γ-consistent judgment sets $J \in \mathcal{J}(\Phi, \Gamma)$ satisfies at least one of the following conditions: (1) for $m-1$ of the indices $j \in [m]$, it holds that $z_{\ell,j} \in J$ for some $\ell \in [3]$, or (2) $J \models \varphi$. This follows directly from the construction of Γ.

We show that for each $J^* \in F_r(\mathbf{J})$ it holds that $J^* \models \varphi$. Take an arbitrary $J^* \in F_r(\mathbf{J})$. To derive a contradiction, suppose that $J^* \not\models \varphi$. Then, by our previous observation, we know that for $m-1$ of the indices $j \in [m]$, it holds that $z_{\ell,j} \in J$ for some $\ell \in [3]$. Therefore, the total score $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, J^*)$ can be at most $5m + 3n + 1$. However, consider the complete judgment set J' that includes all formulas $z_{\ell,j}$, for $\ell \in [3]$ and $j \in [m]$, and that includes $\neg x_i$ for all $i \in [n]$. Clearly, J' is Γ-consistent, since $J' \models \varphi$. Moreover, the total score $\sum_{J_i \in \mathbf{J}} \text{rev}(J_i, J')$ of J' is $6m > 5m + 3n + 1$. This contradicts our assumption that J^* has maximum score. Therefore, we can conclude that $J^* \models \varphi$.

From this, it straightforwardly follows that there is some $J^* \in F_r(\mathbf{J})$ with $\{w\} = L \subseteq J^*$ iff there is a model of φ that sets a maximum number of variables in $\text{Var}(\varphi)$ to true (among all models of φ) and that sets w to true. This concludes our proof of Θ_2^{p}-hardness. $\square$

To illustrate the reversal scoring rule, but also the particular *distance-based methods* we consider in the next section, we use the so-called "doctrinal paradox" as an example. Consider the agenda $\Phi = \{p, \neg p, q, \neg q, r, \neg r\}$ and the integrity constraint $\Gamma = r \leftrightarrow (p \wedge q)$. We consider the profile $\mathbf{J}$ given in Table 1. The set $m(\mathbf{J})$, constructing by adding those judgments from Φ that are supported by a majority of the agents in $\mathbf{J}$, is inconsistent with Γ. The last two columns of the table list all the judgment sets that are selected by $F_r(\mathbf{J})$. For details of the intermediary calculations we refer to the literature [6, Table 3].

$\mathbf{J}$	J_1	J_2	J_3	$m(\mathbf{J})$	$F_r(\mathbf{J})$	
p	1	1	0	1	1	0
q	1	0	1	1	0	1
r	1	0	0	0	0	0

Table 1: Profile for the doctrinal paradox example.

4. DISTANCE-BASED METHODS

A distance $d : \mathcal{J}(\Phi, \Gamma) \times \mathcal{J}(\Phi, \Gamma) \rightarrow \mathbb{R}^+$ is a function that satisfies identity of the indiscernible, symmetry and triangle inequality[2]. A *distance-based judgment aggregator* $F_d^\star$ is a function defined for a distance d and an aggregation function $\star$ as:

$$F_d^\star(\mathbf{J}) = \underset{J \in \mathcal{J}(\Phi, \Gamma)}{\text{argmin}} \star(d(J, J_1), \dots, d(J, J_n)). \tag{4}$$

The aggregation function $\star$ usually used is Σ or max. To ease the notation, we will use the subscript H instead of d_H to indicate that the aggregator is defined using the Hamming distance.

The aggregator F_H^Σ is well studied in judgment aggregation; it appears in many places under different names: PROTOTYPE [30], *median rule* [32], *maximum weighted agenda rule* [25], *simple scoring rule* [6] and *distance-based procedure* [14], see also [26]. Variants of this rule have been defined by Konieczny and Pino-Pérez [22] and Pigozzi [34]. The F_H^Σ aggregator generalises the Kemeny voting rule in judgment aggregation [27]. The Kemeny voting method is one of the first problems to be shown to be Θ_2^{p}-complete [20]. The winner determination problem for F_H^Σ has also been shown to be Θ_2^{p}-complete [15, 14, 28].

In this section, we will also consider the following auxiliary problem, defined for a distance d and an aggregation function $\star$.

OUTCOMEDISTANCE$(d, \star)$
Instance: An agenda Φ, an integrity constraint Γ, a profile $\mathbf{J} = (J_1, \dots, J_p)$, a real nonnegative number $k \in \mathbb{R}^+$, and subsets $L, L_1, \dots, L_u \subseteq \Phi$ with $u \geq 0$.
Question: Is there a judgment set $J^* \in \mathcal{J}(\Phi, \Gamma)$ s.t. $L \subseteq J^*$, $L_i \not\subseteq J^*$ for all i, and $\star(d(J^*, J_1), \dots, d(J^*, J_p)) \leq k$?

4.1 The max+Hamming distance aggregator

We show Θ_2^{p}-completeness for OUTCOME$(F_H^{\max})$—the problem of computing outcomes for the judgment aggregation procedure based on the Hamming distance with maximization as aggregator function.

For the profile in Table 1 we obtain $F_H^{\max}(\mathbf{J}) = \{\{p, q, r\}, \{p, \neg q, \neg r\}, \{\neg p, q, \neg r\}\}$. Namely, all of the profile judgment sets are selected, because in this case, as in other

[2]For any three $J^a, J^b, J^c \in \mathcal{J}(\Phi, \Gamma)$, d satisfies identity of the indiscernible when $d(J^a, J^b) = 0$ iff $J^a = J^b$; symmetry when $d(J^a, J^b) = d(J^b, J^a)$; triangle inequality when $d(J^a, J^c) \leq d(J^a, J^b) + d(J^b, J^c)$.

	x_1	y_1	$(x_1 \wedge y_1) \to \varphi_1$	x_2	y_2	$(x_2 \wedge y_2) \to \varphi_2$	$\cdots$	x_p	y_p	$(x_p \wedge y_p) \to \varphi_p$
J_1	1	1	1	1	0	1	$\cdots$	1	0	1
J_2	1	1	1	0	1	1	$\cdots$	0	1	1
J_3	1	0	1	1	1	1	$\cdots$	1	0	1
J_4	0	1	1	1	1	1	$\cdots$	0	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
J_{2p-1}	1	0	1	1	0	1	$\cdots$	1	1	1
J_{2p}	0	1	1	0	1	1	$\cdots$	1	1	1

Table 2: The profile J used in the proof of Theorem 3.

cases with small agendas and profiles with small number of agents, the $F_H^{\max}$ is not very selective.

THEOREM 3. OUTCOME($F_H^{\max}$) *is* Θ_2^{p}*-complete.*

PROOF. Let $\mathbf{J} = (J_1, \dots, J_p)$. To show that the problem OUTCOME($F_H^{\max}$) is in Θ_2^{p} we can use Algorithm 2 with one minor adjustment—we set $m_x = |\Phi|/2$ in line 2, since the maximal Hamming distance that one judgment set can have from another is $|\Phi|/2$ and thus, for any rational J we have that $\max(d(J, J_1), \dots, d(J, J_p)) \leq |\Phi|/2$.

To show that OUTCOME($F_H^{\max}$) is Θ_2^{p}-hard, we make a reduction from MAXCARD SCEPTICAL INFERENCE to a judgment aggregation problem, as done in [28].

Consider $\Delta = \{\varphi_1, \dots, \varphi_p\}$, β, and propositional formula α. We construct an agenda $[\Phi_\Delta] = \{\, x_i, y_i, (x_i \wedge y_i) \to \varphi_i : i \in [1, p], \varphi_i \in \Delta \,\}$, where x_i, y_i are new propositional variables. We construct the profile $\mathbf{J}_\Delta$ given in Table 2. Moreover, set $\Gamma = \beta$, set $L = \{\neg\alpha\}$, and let $u = 0$.

Let $|\Delta| = n$. Consider a rational judgment set $J \in \mathcal{J}(\Phi_\Delta, \beta)$ s.t $((x_i \wedge y_i) \to \varphi_i) \notin J$ for any $i \in [n]$. Let $D_m(J, \mathbf{J}) = \max(d(J, J_1), \dots, d(J, J_p))$. Observe that if J' is s.t. there exists a $i \in [n]$ for which $(x_i \wedge y_i) \to \varphi_i \in J'$, we have that $D_m(J', \mathbf{J}) = 2p + D_m(J, \mathbf{J})$, thus such J' will for sure not be in $F_H^{\max}(\mathbf{J}_\Delta)$, therefore we do not consider them.

Let us define the sets K_1, K_2, K_3 and K_4, of cardinalities k_1, k_2, k_3 and k_4 respectively, as follows:

$$\begin{array}{l} K_1 = \{\, \{x_i, y_i, (x_i \wedge y_i) \to \varphi_i\} \subset J : i \in [n] \,\} \\ K_2 = \{\, \{x_i, \neg y_i, (x_i \wedge y_i) \to \varphi_i\} \subset J : i \in [n] \,\} \\ K_3 = \{\, \{\neg x_i, y_i, (x_i \wedge y_i) \to \varphi_i\} \subset J : i \in [n] \,\} \\ K_4 = \{\, \{\neg x_i, \neg y_i, (x_i \wedge y_i) \to \varphi_i\} \subset J : i \in [n] \,\} \end{array} \tag{5}$$

Observe that for any $J_j \in \mathbf{J}$, $J_j \cap K_1$ is a singleton, $J_j \cap K_4$ is the empty set, for j odd $J_j \cap K_3$ is the empty set and for j even $J_j \cap K_2$ is the empty set. Given a judgment set $J_j \in \mathbf{J}$, let k be the index of the unique positive triple in J_j, namely $\{x_k, y_i, (x_k \wedge y_k) \to \varphi_k\} \subset J_j$. Let us define the number $k_d(J_j, J)$ as:

- $k_d(J_j, J) = 0$ iff $\{x_k, y_k, (x_k \wedge y_k) \to \varphi_k\} \subset J$,
- $k_d(J_j, J) = 1$ iff $\{\neg x_k, y_i, (x_k \wedge y_k) \to \varphi_k\} \subset J$ or $\{x_k, \neg y_i, (x_k \wedge y_k) \to \varphi_k\} \subset J$, and
- $k_d(J_j, J) = 2$ iff $\{\neg x_k, \neg y_i, (x_k \wedge y_k) \to \varphi_k\} \subset J$

The cells in Table 3 gives the possible distances between J and a judgment set $J_j \in \mathbf{J}$ for all cases.

Observe now that judgment sets $J \in \mathcal{J}(\Phi_\Delta, \beta)$ for which there exists a $J_j \in \mathbf{J}$ s.t. $k_d(J_j, J) = 2$ cannot be included in $F_H^{\max}(\mathbf{J}_\Delta)$. But since each $J_j \in \mathbf{J}$ has exactly one positive triple, then if $k_4 \neq 0$, there necessarily will exist at least one $J_j \in \mathbf{J}$ s.t. $k_d(J_j, J) = 2$. Thus we conclude that $J \in F_H^{\max}(\mathbf{J}_\Delta)$ iff $k_4 = 0$.

	j odd	j even
$k_d(J_j, J) = 0$	$k_1 + 2k_3 + k_4 - 1$	$k_1 + 2k_2 + k_4 - 1$
$k_d(J_j, J) = 1$	$k_1 + 2k_3 + k_4 + 1$	$k_1 + 2k_2 + k_4 + 1$
$k_d(J_j, J) = 2$	$k_1 + 2k_3 + k_4 + 2$	$k_1 + 2k_2 + k_4 + 2$

Table 3: Hamming distances from rational judgment sets to the profile from Table 2.

Assume, without the loss of generality that $k_2 \geq k_3$, thus $D_m(J, \mathbf{J}) = k_1 + 2k_2 + 1$. Observe that $k_1 + k_2 + k_3 = n$. Consequently, the larger the k_1 is, the smaller the $D_m(J, \mathbf{J})$ becomes. The judgment set with maximal k_1 is the one containing all the triples $\{x_i, y_i, (x_i \wedge y_i) \to \varphi_i\}$ for which φ_i is in the maximally consistent subset in Δ. $\square$

4.2 Geodesic distance aggregator

The *geodesic distance aggregator* [9] is based on the geodesic distance d_g, that is defined as follows. Take an agenda Φ and an integrity constraint Γ. We then define the graph $G(\Phi, \Gamma) = (V, E)$ as follows. The set V of vertices of $G(\Phi, \Gamma)$ consists of all complete and Γ-consistent judgment sets J, *i.e.*, $V = \mathcal{J}(\Phi, \Gamma)$. Moreover, two vertices $J, J' \in V$ are connected in the graph $G(\Phi, \Gamma)$ if there exists no third vertex J'' s.t. (1) J'' is distinct from J and J', and (2) on each proposition $\varphi \in \Phi$, J'' agrees with J or J' (or both). The *geodesic distance* d_g is then defined by letting $d_g(J, J')$ be the length of the shortest path from J to J' in the graph $G(\Phi, \Gamma)$. The geodesic distance aggregator F_g^Σ (or F_g for short) uses addition as its aggregator function $\star$. That is:

$$F_g(\mathbf{J}) = \underset{J \in \mathcal{J}(\Phi,\Gamma)}{\operatorname{argmin}} \sum_{J_i \in \mathbf{J}} d_g(J, J_i). \tag{6}$$

For example, for the profile $\mathbf{J}$ in Table 1 we obtain $F_g(\mathbf{J}) = \{\{p, q, r\}\}$. For intermediary steps of the calculation, the reader can consult [9], in particular, the last paragraph of the Section 3.1 there.

We show that the problem of computing outcomes for the judgment aggregation procedure based on the geodesic distance with addition as aggregator function is Θ_3^{p}-complete.

LEMMA 2. OUTCOME(F_g) *is in* Θ_3^{p}.

PROOF. In order to show membership in Θ_3^{p}, we firstly show that the problem OUTCOMEDISTANCE(d_g, Σ) is in Σ_2^{p}, by describing a nondeterministic polynomial-time algorithm that has access to an NP oracle, and that decides OUTCOMEDISTANCE(d_g, Σ). The algorithm gets as input an agenda Φ, an integrity constraint Γ, a profile $\mathbf{J}$, two complete and consistent judgment sets $J, J' \in \mathcal{J}(\Phi, \Gamma)$, and a number k. The algorithm guesses some natural number $0 \leq \ell \leq k$, ℓ many judgment sets $J_1, \dots, J_\ell \subseteq \Phi$, and ℓ many truth assignments $\omega_1, \dots, \omega_\ell : \mathrm{Var}(\Phi) \to \{0,1\}$. The algorithm then verifies (1) whether each judgment set J_i is complete, and (2) whether the assignment ω_i satisfies the judgment set J_i as well as Γ, for each $i \in [\ell]$. Moreover, it verifies (3) whether for no $0 \leq i < \ell$, there is some complete and consistent judgment set that lies between J_i and J_{i+1}. It does so by querying the NP oracle whether there exists some set $J' \subseteq \Phi$ and some $i \in [\ell - 1]$ s.t. (a) J' is a complete and Γ-consistent judgment set, and (b) for each $\varphi \in \Phi$, J' agrees with J_i or J_{i+1} (or both). We know that condition (3) is satisfied iff the oracle answers "no." The algorithm accepts iff all conditions (1)–(3) are satisfied. It is straightforward to verify that $\sum_{J_i \in \mathbf{J}} d(J^*, J_i) \leq k$ iff there exists some guess that satisfies conditions (1)–(3). Thus, $\textsc{OutcomeDistance}(d_g, \Sigma)$ is in Σ_2^{p}.

Then, we describe how to solve $\textsc{Outcome}(F_g)$ in polynomial time by querying an oracle in Σ_2^{p} at most a logarithmic number of times. We know that the maximum value of $\sum_{J_i \in \mathbf{J}} d_g(J, J_i)$ for any judgment set J is upper bounded by $|\mathbf{J}| \cdot |\Phi|$. Therefore, by using a logarithmic number of queries to an oracle for the problem $\textsc{OutcomeDistance}(d_g, \Sigma)$, we can determine the minimum value k^* for $\sum_{J_i \in \mathbf{J}} d_g(J, J_i)$ using binary search. Then, with one additional query to this oracle we can determine whether there exists a judgment set $J \in \mathcal{J}(\Phi, \Gamma)$ s.t. (1) $\sum_{J_i \in \mathbf{J}} d_g(J, J_i) \leq k^*$, (2) $L \subseteq J$, and (3) $L_i \not\subseteq J$ for all i. Thus, the problem $\textsc{Outcome}(F_g)$ is in Θ_3^{p}. $\square$

Theorem 4. $\textsc{Outcome}(F_g)$ *is* Θ_3^{p}*-complete.*

Proof. Membership in Θ_3^{p} is shown in Lemma 2. To show Θ_3^{p}-hardness, we give a polynomial-time reduction from $\textsc{QSat}_2$-$\textsc{Max-Model}$. Let $\varphi = \exists X.\forall Y.\psi$ be a true quantified Boolean formula, where ψ is quantifier-free, and where $X = \{x_1, \dots, x_n\}$. We may assume without loss of generality that for each assignment $\omega : X \to \{0,1\}$ we have that $\exists Y.\psi[\omega]$ is true. We construct an agenda Φ and a profile $\mathbf{J}$ s.t. the complete and consistent judgment sets J selected by F_g correspond exactly to the maximal models of φ.

We define $m_1 = 3$, and $m_2 = (m_1+1) \cdot n$, and we introduce auxiliary variables z_j^i for each $i \in [m_1]$ and each $j \in [m_2]$. Intuitively, for each i, the variables z_j^i will act as a cluster of switches, that can be triggered individually (and in several cases, collectively). We then construct the propositional formula $\chi = \psi \to \bigvee_{i \in [m_1]} \chi_i$, where for each $i \in [m_1]$ we let $\chi_i = \bigwedge_{j \in [m_2]} z_j^i \wedge \bigwedge_{i' \in [m_1], i' \neq i, j \in [m_2]} \neg z_j^{i'}$. Intuitively, this formula χ ensures that whenever ψ is true, it holds that the switches z_j^i are "on" for exactly one cluster.

We now define the agenda Φ by letting $[\Phi] = \{x_1, \dots, x_n\} \cup \{\, z_j^i : i \in [m_1], j \in [m_2] \,\} \cup \{\chi\}$. Moreover, we let $\Gamma = \top$. For the sake of convenience, we will introduce the following property Q. We say that a complete and consistent judgment set $J \in \mathcal{J}(\Phi)$ has property Q if there exists some $i \in [m_1]$ s.t. $\{z_1^i, \dots, z_{m_2}^i\} \subseteq J$ and for each $i' \in [m_1]$ s.t. $i' \neq i$ it holds that $\{\neg z_1^{i'}, \dots, \neg z_{m_2}^{i'}\} \subseteq J$. Because $\exists Y.\psi[\omega]$ is true for each assignment $\omega : X \to \{0,1\}$, it is straightforward to verify that each $J \in \mathcal{J}(\Phi)$ belongs to exactly one of three classes (A)–(C), defined by the following conditions: (A): $\chi \in J$, and J has property Q; (B): $\chi \in J$, J does not have property Q, and $J \cup \{\neg\psi\}$ is satisfiable; and (C): $\neg\chi \in J$, and J does not have property Q.

Next, for each $i \in [m_1]$, we define the complete and consistent judgment set $J_i = \{\chi, \neg x_1, \dots, \neg x_n\} \cup \{\, z_j^i : j \in [m_2] \,\} \cup \{\, \neg z_j^{i'} : i' \in [m_1], i' \neq i, j \in [m_2] \,\}$. Moreover, we let $\mathbf{J} = (J_1, \dots, J_{m_1})$. In the remainder of the proof, we will show that F_g selects (all and only) judgment sets J that correspond to the assignments $\omega : X \to \{0,1\}$ of minimal weight s.t. $\forall Y.\psi[\omega]$ is true. For the sake of convenience, we will assume without loss of generality that $\exists X.\forall Y.\psi$ is true.

Let $\omega : X \to \{0,1\}$ be an assignment of weight w s.t. $\forall Y.\psi[\omega]$ is true (we will call such assignments *satisfying assignments*). For any integer $i \in [m_1]$, let the complete and consistent judgment set J_i^ω be defined by $J_i^\omega = \{\chi\} \cup \{\, x_i : \omega(x_i) = 1 \,\} \cup \{\, \neg x_i : \omega(x_i) = 0 \,\} \cup \{\, z_j^i : j \in [m_2] \,\} \cup \{\, \neg z_j^{i'} : i' \in [m_1], i' \neq i, j \in [m_2] \,\}$. We now claim the following.

Claim 1: For any $i_1, i_2 \in [m_1]$ with $i_1 \neq i_2$, and any satisfying assignment ω, it holds that $d_g(J_{i_1}^\omega, J_{i_2}^\omega) = 1$.

Claim 2: For any $i \in [m_1]$, and any $i' \in [m_1]$ s.t. $i' \neq i$, it holds that $d_g(J_i, J_i^\omega) = w$ and $d_g(J_{i'}, J_i^\omega) = w + 1$.

Claim 3: Let w_0 be the minimum weight of any satisfying assignment $\omega : X \to \{0,1\}$. Then for any such ω of weight w_0, and any $i \in [m_1]$, it holds that $\sum_{J_j \in \mathbf{J}} d_g(J_i^\omega, J_j) = m_1 \cdot w_0 + 1 = d_0$.

Claim 4: For any complete and consistent judgment set that does not coincide with J_i^ω for any satisfying assignment ω of weight w_0, it holds that $\sum_{J_j \in \mathbf{J}} d_g(J_i^\omega, J_j) > d_0$.

Suppose for a moment that we have proven these claims. We can then complete our proof as follows, by specifying how to produce a satisfying assignment for φ that has minimum weight from any J selected by F_g. Let J be an arbitrary judgment set selected by F_g. By Claim 3 and Claim 4, we know that J must be of the form J_i^ω, for some satisfying assignment ω of minimum weight w_0 and for some $i \in [m_1]$. It is straightforward to extract the assignment ω from J_i^ω.

All that remains now is to prove our claims.

Proof of Claim 1: This follows from the facts that $\psi[\omega]$ is a tautology and that $\chi \in J_{i_1}^\omega \cap J_{i_2}^\omega$.

Proof of Claim 2: The conclusion that $d_g(J_i, J_i^\omega) = w$ follows directly from the fact that all complete judgment sets $J' \supseteq J_i \cap J_i^\omega$ are consistent. The conclusion that $d_g(J_{i'}, J_i^\omega) = w + 1$ then follows directly by using Claim 1.

Proof of Claim 3: This follows from Claims 1 and 2.

Proof of Claim 4: For any satisfying assignment ω, and any $i_1, i_2 \in [m_1]$ with $i_1 < i_2$, we call the edge in $G(\Phi, \Gamma)$ between $J_{i_1}^\omega$ and $J_{i_2}^\omega$ a *shortcut*. We claim the following. Let $i_1, i_2 \in [m_1]$ with $i_1 < i_2$, and let J_1, J_2 be complete and consistent judgment sets s.t. $\{\, z_j^{i_1} : j \in [m_2] \,\} \subseteq J_1$, $\{\, \neg z_j^{i'} : i' \in [m_1], i' \neq i_1, j \in [m_2] \,\} \subseteq J_1$, $\{\, z_j^{i_2} : j \in [m_2] \,\} \subseteq J_2$, and $\{\, \neg z_j^{i'} : i' \in [m_1], i' \neq i_2, j \in [m_2] \,\} \subseteq J_2$. Then any path from J_1 to J_2 in $G(\Phi, \Gamma)$ that does not involve any shortcuts is of length at least $m_2 > d_0$. This claim holds, because any intermediate complete judgment set, *i.e.*, any set J' with $\{\, \neg z_j^{i'} : i' \in [m_1], i' \notin \{i_1, i_2\}, j \in [m_2] \,\} \subseteq J'$, is consistent. This is straightforward to verify. $\square$

5. RELATED WORK

The problem of finding the collective judgment sets has been approached by formulating various decision problems. One decision problem that has been studied [21, 28] is the following: given a judgment φ, a profile of judgments $\mathbf{J}$ and an aggregation function f is it true that the judgment φ is in all $J \in f(\mathbf{J})$. In contrast, the decision problem studied by Endriss et al. [15, 14] is the same as we study it here. The class of distance based aggregators has been considered by Jamroga and Slavkovik [21], and they have shown that it is possible for the decision problem to be undecidable. In the literature [15, 14, 28], all of the better known judgment aggregation functions have been analyzed with the exception of the aggregators that we study in this paper. With this work, we thus—for now—complete the complexity analysis of the "winner determination" problem for the judgment aggregators considered in the literature.

It is well known that judgment aggregation generalises preference aggregation [7] and in particular that judgment aggregators generalise voting aggregators [27]. A preference aggregation problem is defined with a finite set of candidates $\mathbf{C}$ and a finite set of voters represented as total, strict and transitive preference orders $\succ_i$ over the set of candidates. Such preference aggregation problems can be represented as judgment aggregation problems: the agenda $\Phi_{\mathbf{C}}$ is a set of issues xPy, each interpreted as "alternative x is preferred to alternative y", while the set of constraints is the transitivity constraint Tr expressing that you cannot have a judgment xPy and a judgment yPz without having a judgment xPz, for any triple of alternatives x, y, z. The voters can be represented as profile of rational judgment sets $\mathbf{J_V} \in \mathcal{J}(\Phi_{\mathbf{C}}, \mathrm{Tr})^+$.

It is interesting to compare the complexity results for the "winner determination" problem that we obtain with the complexity of the winner determination in the case of the respective preference aggregation methods. Dietrich [6] shows that the reversal scoring method generalises the Borda preference aggregation method [3], and also, as we discussed in Section 4, that F_H^{Σ} is a generalisation of the Kemeny method. The problem of checking if a candidate is a Kemeny winner for a profile of preferences is one of the first Θ_2^{p}-complete problems, and here we observe no "complexity jump" when generalising the method to judgment aggregation. In contrast to its judgment aggregation generalisation, the Borda method is computationally easy, finding winners can be done in linear time of the number of agents and alternatives.

Interestingly, the F_g^{Σ} aggregator is also a generalisation of the Kemeny method. Let us construct the Hamming graph $G(\Phi) = (V, E)$ where V is the set of all complete, but possibly inconsistent judgment sets from Φ and there exists an edge in E between two sets J and J' iff $d_H(J, J') = 1$. Observe that if $G(\Phi)$ is s.t. every rational judgment set (node) in $G(\Phi)$ is connected to at least one other rational judgment set in $G(\Phi)$, then for every $J, J' \in \Gamma(\Phi, \Gamma)$, we have that $d_H(J, J') = d_g(J, J')$. Now observe that the graph $G(\Phi_{\mathbf{C}}, \mathrm{Tr})$ for any set of alternatives $\mathbf{C}$ is a permutohedron with every node being connected to $|\mathbf{C}| - 1$ nodes that are at Hamming distance one from it. Therefore, for any two judgment sets $J, J' \in \mathcal{J}(\Phi_{\mathbf{C}}, \mathrm{Tr})$, we have that $d_H(J, J') = d_g(J, J')$. Consequently for profiles $\mathbf{J_V}$ for the preference agenda it holds that $F_g^{\Sigma}(\mathbf{J_V}) = F_H^{\Sigma}(\mathbf{J_V})$. Thus F_g^{Σ} would aggregate to judgment sets corresponding to preference orders that the Kemeny method would produce.

Binary Aggregation.

In addition to the judgment aggregation framework that we consider in this paper (as defined in Section 2), another framework has been considered in the literature to model judgment aggregation settings (see, e.g., [13, 17])—this other framework is known under the name *binary aggregation (with integrity constraints)*. The main difference with the judgment aggregation framework that we consider in this paper is that no additional propositional variables (beyond those representing the issues) can be used to specify the logical relations between issues. In general, the complexity of the winner determination problem can differ between these two different judgment aggregation frameworks [13]. In particular, when considering the complexity from the more detailed perspective of parameterized complexity, complexity results tend to differ (see, e.g., [19]). All the membership results (for the classes Θ_2^{p} and Θ_3^{p}) that we presented in this paper straightforwardly carry over to the setting of binary aggregation. The hardness proofs in this paper cannot all directly be used for the setting of binary aggregation. The reason for this is that transforming an agenda from the judgment aggregation framework that we use in this paper to the framework of binary aggregation, in general, leads to an exponential blow-up [13] (under some common complexity-theoretic assumptions). However, whenever the agenda Φ used in a hardness proof contains all the variables occuring in Φ as formulas in the agenda, the hardness proof carries over to the setting of binary aggregation. This is the case for our proof of Theorem 2, for instance. Extending the other hardness results that we obtained in this paper to the setting of binary aggregation remains a topic for future research.

6. CONCLUSION

We studied the complexity bounds of the winner determination problem in judgment aggregation for three aggregation methods so far unconsidered for such analysis in the literature: the reversal scoring rule and the two distance-based methods $F_H^{\max}$ and F_g^{Σ}. Judgment aggregation is a relatively new area of social choice and new methods for aggregating profiles of judgments are being actively developed, such as [4, 10, 16, 35] as well as other scoring methods given in [6], therefore we cannot claim to have closed the chapter on winner determination complexity analysis in judgment aggregation with our work. However, the methods we study here were the last of the methods frequently referenced in the literature for which a complexity analysis was not done.

We showed that the winner determination problem for the reversal scoring rule and $F_H^{\max}$ is Θ_2^{p}-complete, while that problem for F_g^{Σ} is Θ_3^{p}-complete. For judgment aggregation methods that satisfy non-dictatorship and the universal domain property, *i.e.*, nontrivial methods that can aggregate any profile of rational judgment sets, the complexity of the winner determination problem has so far not been below the Θ_2^{p} level. We thus have a very strong indicator that the problem of aggregating judgments is computationally very hard. We do not yet have a clear picture of what the parameterized complexity landscape of this problem looks like, with the exception of the $F_H^{\max}$ method studied in [19], nor how approximable this problem is for the various aggregation methods. These are immediate directions for future work.

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