Competitive Equilibrium For almost All Incomes

Erel Segal-Halevi
Ariel University
Ariel 40700, Israel

ABSTRACT

Competitive equilibrium from equal incomes (CEEI) is a well-known rule for fair allocation of resources among agents with different preferences. It has many advantages, among them is the fact that a CEEI allocation is both Pareto efficient and envy-free. However, when the resources are indivisible, a CEEI allocation might not exist even when there are two agents and a single item.

In contrast to this discouraging non-existence result, Babaioff, Nisan and Talgam-Cohen (2017) recently suggested a new and more encouraging approach to allocation of indivisible items: instead of insisting that the incomes be equal, they suggest to look at the entire space of possible incomes, and check whether there exists a competitive equilibrium for almost all income-vectors (CEFAI) — all income-space except a subset of measure zero. They show that a CEFAI exists when there are at most 3 items, or when there are 4 items and two agents. They also show that when there are 5 items and two agents there might not exist a CEFAI. They leave open the cases of 4 items with three or four agents.

This paper presents a new way to implement a CEFAI, as a subgame-perfect equilibrium of a sequential game. This new implementation allows us both to offer much simpler solutions to the known cases (at most 3 items, and 4 items with two agents), and to prove that a CEFAI exists even in the much more difficult case of 4 items and three agents. Moreover, we prove that a CEFAI might not exist with 4 items and four agents. Thus, this paper completes the characterization of CEFAI for monotone preferences.

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1 INTRODUCTION

Competitive equilibrium (hence CE) is a famous rule for allocating resources among agents with different preferences. In the simple *Fisher market* model [11], there are several goods to allocate and several agents, each of whom holds a certain amount of fiat money ("*income*", also known as "*budget*"). Based on the agents' preferences, a price-vector is determined, assigning a price to each good. Then, the items are partitioned among the agents such that each agent believes his/her bundle is better than all bundles that can be purchased with his/her income.

The CE rule has the two complementary virtues of *efficiency* and *fairness*. First, the CE allocation is always *weakly Pareto-efficient* — there is no other allocation which makes all agents happier. Second, if all incomes are equal, the CE allocation (which is then called *competitive equilibrium from equal incomes* or CEEI) is also *envy-free* — no agent believes that another agent has a better bundle. If the incomes are different, the CE allocation satisfies a generalized fairness property corresponding to agents with different entitlements, where the incomes are interpreted as the entitlements.

When the goods to allocate are divisible, CE exists under very general conditions. See [1, 6] for homogeneous goods and [24**?**] for a heterogeneous good ("cake"). When the goods to allocate are indivisible, CE still has strong efficiency and fairness properties (see Section 7), however, it might fail to exist even in very simple cases. For example, when there is one item and two agents with equal incomes, CE does not exist, since there is no price under which the demand exactly equals the supply: if the price is less than or equal to the agents' income, the demand is 2; if it is greater, the demand is 0 (Note that in this model money has no intrinsic value, so an agent always *strictly* prefers to buy an affordable item than to remain with no items).

The example above could make us think that we cannot enjoy the benefits of CE when there are indivisible goods. But a recent paper by Babaioff et al. [5] gives a new hope. They notice that, in the case of two agents, CE fails to exist *only when the incomes are exactly equal.* If one income is even slightly larger than the other, CE exists. So when there are two agents and one item, a CE exists in almost all the income-space (except a subset of measure zero). We say that in this case there exists a CEFAI — a Competitive-Equilibrium For almost All Incomes. This raises the following natural question:

In what cases does a CEFAI exist?

Babaioff et al. [5] proved that a CEFAI exists when there are at most 3 items, and when there are 4 items and 2 agents. Moreover, they proved that this is not true when there are 5 or more items: they presented a market with 2 agents and 5 items, in which the subset of the income-space where a CE does not exist has a strictly positive measure. Two cases are left open by [5]: the case of 4 items and 3 agents, and the case of 4 items and 4 agents.

1.1 Contributions

The first contribution of this paper is to resolve the two missing cases. It proves that a CEFAI exists when there are 4 items and 3 agents; in contrast, when there are 4 items and 4 agents, the subset in which a CE does not exist might have a positive measure. The following table summarizes the results; stars denote new results.

Items:	1, 2, 3	4	5+
2 agents:	Yes	Yes	No
3 agents:		**Yes***	
4+ agents:		**No***	

The effort to solve the missing cases yielded a tool that may be interesting in its own right and can be considered a second contribution. In the cases in which a CEFAI exists (the cases marked by

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"Yes" or "Yes*" in the above table), we present a sequential game that implements the CE in subgame-perfect equilibrium. Thus, if the agents know each other's preferences, the divider can induce them to implement the CE even without knowing their preferences. The assumption that agents know each other's preferences is quite reasonable in some settings. For example, when dividing cabinet ministries among political parties, which is a common use-case of division with unequal entitlements [10], parties often have a good idea about the preferences of other parties.

The formal definitions are presented in Section 2. The sequential game that we use to implement CEFAI is presented in Section 3. The settings of three, four and five items are analyzed in Sections 4, 5 and 6 respectively. We view CE mainly as a rule for fair allocation. In Section 7 we prove that, indeed, a CE allocation has many fairness properties that are natural generalizations of envy-freeness for agents with different entitlements. Related work is surveyed in Section 8 and future work ideas are presented in Section 9.

2 PRELIMINARIES

There is a set $\mathcal{N}$ of agents, with $n = |\mathcal{N}|$. In this note $n \leq 4$. The agents are denoted $i \in \{1, \dots, n\}$ or Alice, Bob, Carl and Dana.

Each agent has a pre-determined positive *income*. The incomes are denoted t_i (for $i \in \{1, \dots, n\}$), or a, b, c, d.

There is a set $\mathcal{M}$ of items with $m = |\mathcal{M}|$. In this note $m \leq 5$, and the items are named z, y, x, w, v.

A *bundle* is a set of items. For brevity, we represent a bundle as a string of its items. I.g, xy represents the bundle $\{x, y\}$.

Each agent i has a total preference-relation $\succ_i$ on bundles of items. Only two assumptions are made on the preference relations:

- *Strict* — no agent is indifferent between any two bundles.
- *Monotone* — an agent prefers a bundle over all its subsets.

A *price-vector* p is a vector of positive numbers, one number per item. The price of a bundle is the sum of the prices of its items.

An *allocation* X is a partition of the m items among the n agents, such that $\mathcal{M} = X_1 \cup \dots \cup X_n$ and the X_i are pairwise-disjoint. Note that in this model all items should be allocated. This is in contrast to the approximate-CEEI mechanism of Budish [14], which may discard some items. Discarding items might be very inefficient in our setting since the initial number of items is small.

A *competitive equilibrium (CE)* is a pair (p, X), where p is a price-vector and X is an allocation and the following conditions hold.

Condition 1. For every agent with a non-empty bundle, the price of the agent's bundle exactly equals the agent's income:[1]

$$\forall i \in \mathcal{N}: \; X_i \neq \emptyset \implies p(X_i) = t_i$$

Condition 2. Each agent's bundle is better than all other bundles he can afford. Equivalently, for every agent i and bundle $Y \neq X_i$, at least one of the following holds:

The agent does not want Y:	$Y \prec_i X_i$,	or:
The agent cannot afford Y:	$p(Y) > p(X_i)$.	

The *income space* is the set of all possible income-vectors: $\mathbb{T} := (\mathbb{R}^+)^n$. Given a point $t \in \mathbb{T}$, we say that *CE exists in* t if, for *every* combination of the agents' preferences, there exist a price-vector and an allocation that satisfy the CE conditions given the income-vector t. We say that *there exists a CE for almost all incomes (CEFAI)* if the subset of $\mathbb{T}$ in which no CE exists has a measure of zero in $\mathbb{T}$.

In particular, if there is a finite set of equalities on the incomes such that a CE exists whenever none of these equalities is satisfied (i.e, the incomes are *generic*), then a CEFAI exists.

We emphasize that, when a CEFAI exists, the CE is allowed to depend on the income-vector. I.e, for every income-vector $t \in \mathbb{T}$ (except maybe a subset of measure zero), there may be a different allocation and a different price-vector that satisfy the CE conditions.

3 PICKING-SEQUENCES AND PIXEPS

Our algorithms for finding a CE are based on picking-sequences.

Definition 3.1. A *picking-sequence* is a sequence of m agent-names. It is interpreted as a sequential game in which, at each step, the current agent in the sequence may pick a single item.

For example, with $m = 3$ items, a possible picking-sequence is *ABA*, which denotes a game in which Alice picks an item, then Bob picks an item, then Alice receives the last remaining item.

We analyze these games assuming *complete information*, i.e, each agent knows the preferences of all other agents.

We use the following *backward induction* analysis. The m-th picker just picks the single remaining item. The $m-1$-th picker picks one of the two remaining items that results in a better bundle for him. For every possible pair of remaining items we know what the $m-1$-th picker is going to pick; based on this knowledge, the $m-2$-th picker picks one of the three remaining items that results in a better final bundle for her. We proceed in the same way down to step 1. Every sequence of picks that results from this process is called a *subgame-perfect equilibrium* (SPE). [2]

For example, consider again the game *ABA*. In step 3 Alice takes the last remaining item. In step 2 Bob chooses the single item he prefers. Suppose w.l.o.g. that for Bob: $x \succ y \succ z$, then Bob will never take z. This means that Alice's bundle will be either xz or yz. So in step 1, Alice decides which of these two bundles she prefers and chooses accordingly. For example, if for Alice: $yz \succ xz$, then in the 1st step Alice picks y. Then, Bob picks x and Alice gets z, and the final allocation is: yz, x. Note that there can be more than one SPE. In this case, Alice can also pick z in the 1st step; she will get y in the 3rd step anyway.

Definition 3.2. (a) A *picking-sequence-with-prices* (*pixep* for short) is a picking-sequence in which a price is attached to each position. The interpretation is that, whenever an agent picks an item, the corresponding price is attached to that item.

(b) Let I be a pixep and Q a subgame-perfect equilibrium in the sequential game defined by I. The pair (I, Q) is called an *execution* of the pixep I. We denote the allocation induced by this execution by $X(I, Q)$, and the induced price-vector by $p(I, Q)$.

For example, with three items, a possible pixep is:

A	B	A	
4	2	1	(*)

[1] Apparently, it is more general to assume that the price of an agent's bundle is *at most* the agent's income; however, Babaioff et al. [5] observe that for every CE allocation there exists a price-vector by which all agents with non-empty bundles exhaust their incomes. Therefore, the assumption that the price equals the income loses no generality.

[2] It is known that every SPE is also a *Nash equilibrium*, and moreover, a Nash equilibrium is played in each sub-game (including unreachable ones). See Aumann [2] for the connection between backward induction and common knowledge of rationality.

which means that the first item picked by Alice is priced 4, the item picked by Bob is priced 2, and the last item received by Alice is priced 1. A pixep can be seen as a shorthand for an allocation rule; (*) is a shorthand for the rule: "Give Alice her most preferred pair from the two pairs that contain Bob's worst item; price Alice's two items as 1 and 4; give Bob the remaining item and price it as 2".

3.1 Pixeps implementing a CE

The most important feature we require from a pixep is that it should implement a competitive equilibrium.

Definition 3.3. Let I be a pixep and t an income-vector. I *implements CE given income-vector* t if, whenever the income-vector of the agents is t, for *every* combination of their preferences, there *exists* a SPE Q of the sequential game defined by I, such that the allocation $X(I,Q)$ with the price-vector $p(I,Q)$ are a CE.

When does a pixep implement CE? Consider the two conditions in the definition of CE.

Condition 1 requires that the price of each agent's bundle equals the agent's income. For example, pixep (*) implements CE only when the income of Alice is 5 and the income of Bob is 2. Therefore we impose the following requirement:

(R1) The sum of all prices appearing below agent i is t_i.

Condition 2 requires that, for each agent and each bundle not picked by that agent, the agent either doesn't want or cannot afford the bundle. To make checking this condition easier, we impose the following *decreasing prices* requirements:

(R2) The sequence of prices should be decreasing, and strictly-decreasing whenever the picking-sequence switches between agents. For example, in the pixep:

A	B	B	A	
p_1	p_2	p_3	p_4	(**)

we require that $p_1 > p_2 \geq p_3 > p_4$. This ensures that no agent can afford to switch the item he picked in his turn with a better item picked by another agent in a previous turn.

(R3) The last price must be strictly larger than the income of any agent who does not appear in the sequence. For example, in the pixep (**) we require that p_4 be larger than the income of Carl. This ensures that Carl, who is allocated an empty bundle, cannot afford any non-empty bundle.

Henceforth we consider only pixeps satisfying (R1) (R2) and (R3).

3.2 Domination of bundles

Given an execution (I,Q), we define a *domination* relation on bundles based on the positions of items in the sequence. Given two different bundles $X \neq Y$, we say that X is dominated by Y if there exists an injection $f : X \to Y$ such that, for each item $x \in X$, $f(x)$ appears (weakly) earlier than x in the sequence I. For example, in a sequence of four items, the pair of items in positions #1 and #4:

- Is dominated by the pair of items #1 and #2, as well as by the triplet of items #1 #3 #4;
- Dominates the pair of items #3 and #4, as well as the singleton containing item #1;
- Is unrelated to the triplet of items #2 #3 and #4 (none of them dominates the other).

Given an execution (I,Q) and an agent i, the *dominating bundles / dominated bundles / unrelated bundles* of i are the bundles that dominate / are dominated by / are unrelated to X_i, respectively. We will verify Condition 2 for these three types of bundles separately.

Lemma 3.4. *Suppose a pixep I satisfies (R1,R2,R3). Then in any execution (I,Q), no agent can afford a* dominating bundle.

Proof. If X_i is empty, then (R3) implies that any non-empty bundle costs more than the income of agent i.

Otherwise, in any bundle dominating X_i, each item appears either at the same location or earlier than a corresponding item in X_i. Moreover, by definition a dominating bundle is different than X_i so it has at least one item selected by a different agent than i. Therefore, (R2) implies that it is more expensive than X_i. (R1) implies that the agent's income exactly equals $p(X_i)$, so he cannot afford a more expensive bundle. □

Lemma 3.5. *Suppose in a pixep I all the turns of agent i are in a single contiguous sequence. Then in any execution (I,Q), agent i does not want any* dominated bundle.

Proof. Suppose the turns of i are a contiguous sequence of length k. Then, the best strategy of i is to pick the best k-tuple from among the items remaining on the table, and it is better than any dominated bundle. □

Example 3.6. In both pixeps (*) and (**), Lemma 3.5 holds for Bob. In (*), he picks the best remaining item and obviously does not want the other item; in (**), he picks the best remaining pair and does not want any other remaining pair or singleton. □

Lemmas 3.4 and 3.5 imply that, to verify that a pixep implements CE, we only have to check the unrelated bundles of each agent, and the dominated bundles of agents with non-contiguous turns. Moreover, (R3) implies that we do not have to check any bundle for an agent who does not appear in the pixep.

4 WARM-UP: THREE ITEMS

As a warm-up, we show in this section how to design pixeps implementing CE for the case of three items and any number of agents. Babaioff et al. [5] already proved that in this case there exists a CEFAI, but the algorithm presented here (Algorithm 1) is shorter.[3]

We can assume that all incomes are different, since this assumption removes from the income-space a set of measure zero. We also assume w.l.o.g. that $a > b > c >$ all other incomes.

We now examine some picking-sequences to see if they can be made into a pixep that implements CE. Consider first the sequence *AAA*, giving Alice all three items. (R3) implies that the last price must be $b+\epsilon$ for some $\epsilon > 0$. (R2) implies that the second price must be at least $b+\epsilon$, so we set it to $b+\epsilon$. (R1) implies that the sum of all prices must equal a, so we set the first price to $a-2b-2\epsilon$. (R2) implies that $a-2b-2\epsilon \geq b+\epsilon$, which implies that $a \geq 3b+3\epsilon$.

For brevity, from now on we will omit the ϵ from the notation. I.e, instead of $b+\epsilon$ we will write b^+, instead of $a-2b-2\epsilon$ we will write $a-2b^{--}$, etc. So the above discussion can be summarized as:

		A	A	A
If $a > 3b$	then	$a-2b^{--}$	b^+	b^+

[3] Note that they also prove existence of CE in some subsets of measure zero, like $a = b + c$. We ignored such subsets to keep the focus on CE for almost-all incomes

The interpretation of this notation is: "If $a > 3b$, then there exists some $\epsilon > 0$ such that the sequence AAA with prices $a - 2b - 2\epsilon, b + \epsilon, b + \epsilon$ implements CE". Clearly there are no unrelated bundles, so by Lemmas 3.4,3.5 this pixep indeed implements CE when $a > 3b$.

As a second example, consider the sequence *AAB*. (R1) implies that the last price is b, which is by assumption larger than c, so (R3) is satisfied too. (R2) implies that the second price should be more than b so we set it to b^+; (R1) implies that the first price should be $a - b^-$, and (R2) then implies that $a - b \geq b^{++}$. Summarizing:

		A	A	B
If $a > 2b$	then	$a - b^-$	b^+	b

Again there are no unrelated bundles, and by Lemma 3.5 no agent wants a dominated bundle, so this pixep implements CE if $a > 2b$.

Remark 4.1. When $a > 3b$, the two above pixeps, based on *AAA* and *AAB*, both implement CE. This raises the question which CE is "better". Intuitively, if Alice's income is much larger than Bob's, it seems "fairer" to give all items to Alice, while if the difference is not so high, it seems plausible to leave the last item for Bob. However, these intuitions are not supported by the definition of CE. From the point-of-view of CE, which is the one taken in the present paper, both pixeps are equally good. Moreover, the allocations yielded by both pixeps satisfy the fairness properties described in Section 7. We leave the question of selecting a single CE to future work.

As a third example, consider the sequence *ABC*. (R1) implies that the prices must be a, b, c, and by assumption $a > b > c >$ all other incomes, so (R2,R3) are satisfied. Lemma 3.5 holds for all three agents, so to verify CE we only need to consider unrelated bundles. Only Alice has an unrelated bundle, and it is the bundle of items #2 and #3. To ensure that Alice cannot afford that bundle, it is sufficient to require that $a < b + c$:

		A	B	C
If $a < b + c$	then	a	b	c

Finally, consider the sequence *ABA*. To satisfy the conditions we set the prices to $a - c^-, b, c^+$. (R2) then implies that $a - c^- > b$, so:

		A	B	A
If $a > b + c$	then	$a - c^-$	b	$c+$

Here no agent has unrelated bundles. Lemma 3.5 holds for Bob, so to verify that the above pixep implements CE, we only need to verify that Alice does not want a dominated bundle. The only dominated bundle that Alice might want is the pair of the two items picked last. Suppose w.l.o.g. that Bob's ranking of singletons is: $x > y > z$. Then Bob never picks z so Alice gets either xz or yz. If for Alice $xz \succ yz$ then she certainly picks x first, so she prefers her bundle over the dominated bundle yz. If for Alice $yz \succ xz$ then she has two options: pick y first and get z last, or pick z first and get y last. Both options lead to the same final allocation. In the first option she prefers her bundle to the dominated bundle xz, while in the second option she might prefer the dominated bundle xy. However, to prove that the pixep implements CE, it is sufficient to prove that *there exists* a SPE in which the allocation satisfies the CE conditions, so we can assume that Alice picks the first option.

Looking at the last two pixeps, *ABC* and *ABA*, reveals that the conditions under which they implement CE cover all the income space except the hyperplane $a = b + c$, which has a measure of zero in $\mathbb{T}$. This proves that a CEFAI exists when there are 3 items. The proof and its SPE implementation are summarized in Algorithm 1.

Algorithm 1 Implementing Competitive Equilibrium with $m = 3$ items. The algorithm works in almost all the income space, i.e, for all income-vectors $(a, b, c, \ldots)$ in which $a > b > c > \ldots$ and $a + b \neq c$.

If $a > b + c$	then	A: $a - c^-$	B: b	A: c^+
If $a < b + c$	then	A: a	B: b	C: c

Algorithm 2 Implementing Competitive Equilibrium with $m = 4$ items and $n = 2$ agents. Works for all income-vectors (a, b) with $a > b$, $a \neq 2b$.

(1) If $a > 2b$ then

A	A	B	A
$a - b^{--}$	b^+	b	0^+

(2) If $a < 2b$ then play the sequential game below:

Alice may choose:	A: a^{--}	B: b^-	A: 0^{++}	B: 0^+
Else, Bob may choose:	B: b	A: b^{--}	A: $(a-b)/2^+$	A: $(a-b)/2^+$
Else:	A: $a/2$	A: $a/2$	B: $b/2$	B: $b/2$

5 FOUR ITEMS

5.1 Two agents

In this section there are $m = 4$ items. Initially we assume there are only two agents — Alice and Bob — with incomes $a > b$. Babaioff et al. [5] already proved that in this case CE exists in almost all income space, but the algorithm presented here (Algorithm 2) is shorter. The case $a > 2b$ is handled by *AABA*:

		A	A	B	A
If $a > 2b$	then	$a - b^{--}$	b^+	b	0^+

All three requirements on the price-sequence are clearly satisfied. No agent has any unrelated bundles. It only remains to check that Alice does not want any dominated bundle. This can be verified similarly to the case *ABA* in the previous section: there exists a SPE in which Bob's worst item is picked (by Alice) at the last step. Alice receives the best of the three triplets that contain this item, so it is better for her than any dominated triplet.

The case $a < 2b$ is more complicated. It requires letting agents choose between different pixeps. This leads to the following three-step sequential game.

Step #1: Alice may choose the following pixep based on *ABAB*:

A	B	A	B
a^{--}	b^-	0^{++}	0^+

Step #2: If Alice does not choose *ABAB*, then Bob may choose the following pixep based on *BAAA*:

B	A	A	A
b	b^{--}	$(a-b)/2^+$	$(a-b)/2^+$

Step #3: If Bob does not choose *BAAA*, then we play the following pixep based on *AABB*:

A	A	B	B
$a/2$	$a/2$	$b/2$	$b/2$

Intuitively, in Step 1 Alice chooses between her 2nd-best pair and her worst triplet, and in Step 2 Bob chooses between his best singleton and worst pair. The agents' choices guarantee that, in the chosen pixep, they don't want an unrelated bundle they can afford.

Formally, we analyze the game using backward induction. We rename the items such that Bob's best item is w, and for Alice: $wx \succ wy \succ wz$.

In Step 3, Alice gets her best pair and Bob gets its complement. By Lemma 3.5 no agent wants a dominated bundle. Moreover, Alice cannot afford any of her unrelated bundles (triplets), since $a/2+b > a$. Bob *can* afford his two unrelated bundles (singletons). But, if he wants a singleton, he could choose *BAAA* in the previous step and get his best singleton. So in step 3 it is safe to assume that Bob does not want an unrelated bundle.

In Step 2, Bob has to choose between w (his best singleton), and the complement to Alice's best pair. There are three cases: (a) If Alice's best pair is xy or xz or yz, then the complement contains w so Bob certainly prefers *AABB*. (b) Otherwise, Alice's best pair is wx and its complement is yz; if for Bob $yz \succ w$, then again he prefers *AABB*. (c) Only if Alice's best pair is wx and for Bob $w \succ yz$, does Bob choose *BAAA*. In the latter case, Bob's bundle is w. There exists a SPE in which Alice chooses her three items in the order: x, y, z. Then, Bob cannot afford xy or xz since they cost more than b. The only unrelated bundle he can afford is yz. However, in case (c) Bob prefers w to yz, so Bob does not want any unrelated bundle. Alice can afford only two unrelated bundles — wy and wz. However, if she wants any of these pairs, she could choose in the previous step *ABAB* and pick w first; this would guarantee her at least wy. So there exists a SPE in which, in step 2, Alice does not want any unrelated bundle that she can afford.

In Step 1, in cases (a-b) above, Alice never chooses *ABAB*, since she can get her best pair by waiting for Step 3. In case (c), Alice chooses *ABAB* iff she prefers the pair she is going to get over the triplet xyz. This pair must contain w, so we assume that if *ABAB* is played, Alice picks w first. In her second turn, Alice picks x (if it is available) or y (if x is not available). Now, Bob has only one unrelated bundle w, which he cannot afford since $a > b$. Alice has one unrelated bundle xyz, which by assumption she does not want.

It remains to check the dominated bundles in the case *ABAB*, since they are not covered by Lemma 3.5. Alice receives a pair that she prefers over xyz, so it is certainly better than any dominated pair. From Bob's point of view, the relevant sequence is *BAB*, which is analogous to the sequence *ABA* analyzed in the previous section. Therefore, Bob too does not want any dominated pair. The proof and its SPE implementation are summarized in Algorithm 2.

5.2 Three agents

In this section there are four items and three agents — Alice Bob and Carl — with incomes $a > b > c$. This case was left open in [5]. Our new technique using pixeps allows us to prove that in this case there exists a CEFAI. The proof is summarized in Algorithm 3.

First, it is easy to check that all price-sequences are decreasing, no agent wants a dominated bundle, and Carl has no unrelated bundles. So it only remains to check that Alice and Bob do not want any unrelated bundle that they can afford.

Algorithm 3 Implementing Competitive Equilibrium with $m = 4$ items and $n = 3$ agents with $a > b > c$.

(1) If $a > 2b + c$ then	**A** $a-b-c^{--}$	**A** b^{+}	**B** b	**A** c^{+}
(2) If $2b + c > a > 2b$ then	**A** $a-b^{-}$	**A** b^{+}	**B** b	**C** c
(3) If $2b > a > b+c$ & $a+c > 2b$ then	**A** b^{+}	**B** b	**A** $a-b^{-}$	**C** c

(4) If $2b > a > b+c$ and $2b > a+c$ (implies $b > 2c$, $a > 3c$) then:

Alice may choose:	**A** $a-c^{--}$	**B** $b-c^{-}$	**A** c^{++}	**B** c^{+}
Else, Bob may choose:	**B** b	**A** $a-2p^{--}$	**A** p^{+}	**A** p^{+}
	where $p := \max(c, \ (a-b)/2)$			
Else:	**A** $a/2$	**A** $a/2$	**B** $b/2$	**B** $b/2$

(5) If $b + c > a > 2c$ and $2c > b$ then play:

Alice may choose:	**A** a	**B** b^{-}	**C** c	**B** 0^{+}
Else:	**B** b	**A** $a-c^{-}$	**A** c^{+}	**C** c

(6) If $b + c > a > 2c$ and $b > 2c$ then play:

Bob may choose:	**A** $a-c^{--}$	**B** $b-c^{-}$	**A** c^{++}	**B** c^{+}
Else, Alice may choose:	**A** a	**B** $b/2$	**B** $b/2$	**C** c
Else:	**B** b	**A** $a-c^{-}$	**A** c^{+}	**C** c

(7) If $2c > a$ then play the sequential game below:

Alice may choose:	**A** a	**B** b^{-}	**C** c	**B** 0^{+}
Else:	**B** b	**A** c^{+}	**C** c	**A** $a-c^{-}$

Ranges 1 and 2 and 3 are straightforward: no agent can afford any unrelated bundle.

Range 4 is analyzed similarly to range 2 in Algorithm 2. The picking-sequences are the same — only the prices are different. Recall that in the Fisher market model, money is used only to purchase items in the market, and has no value outside the market. Therefore, an agent who gets an item, does not care whether the item was cheap or expensive. The agents care only about the final bundle that they receive. Hence their strategic behavior is the same.

In Range 5, in both steps, Bob cannot afford any unrelated bundle. To analyze Alice's behavior, rename the items such that Bob's best item is w and Alice's best pair without w is xy. Then Alice chooses *ABCB* iff she prefers w to xy.

If she chooses *ABCB*, she gets w, does not want xy (or any other pair without w), and cannot afford a triplet.

If she chooses *BAAC*, she gets xy, does not want w, and cannot afford any unrelated pair.

To analyze **Range 6**, rename the items such that Bob's best item is w, Alice's best pair without w is xy (hence for Alice $xy > z$), and for Bob $xz > yz$.

In the last step *BAAC*, Bob gets w and Alice gets xy. Therefore we will get to the last step if both (1) Alice prefers xy to the singleton she can get in *ABBC*, and (2) Bob prefers w to the pair he can get in *ABAB*. Alice can afford only one unrelated bundle — the singleton w — but by (1) she prefers xy even to her best singleton, hence also to w. Bob can afford only one unrelated bundle — the cheaper of xz, yz. There exists a SPE where Alice picks x before y; then, Bob can afford only yz. But by (2) he prefers w to the pair he can get in *ABAB*, which implies that this pair must be one of xy, xz, yz, and not the worst of them. So by choosing *ABAB* Bob could get at least yz. But he did not choose so, therefore he prefers w.

In the middle step *ABBC*, Bob cannot afford even his cheapest unrelated bundle (singleton). Alice has to choose between her best singleton and xy. If her best singleton is x, y, z she surely prefers xy, so we play *ABBC* only if Alice's best singleton is w and she prefers it to xy. Hence, Alice does not want any unrelated pair. Additionally she cannot afford a triplet.

In the first step *ABAB*, Alice cannot afford even her cheapest unrelated bundle (triplet) since $b + c > a$. Bob can afford only one unrelated bundle — a singleton. Denote the pair that Bob is going to get by X. Bob chooses *ABAB* iff he prefers X to the bundle he can get by not choosing. This bundle depends on Alice's preferences: (a) If Alice prefers xy to w, then she will choose *BAAC*, so Bob chooses *ABAB* iff he prefers X to his best singleton w. (b) If Alice prefers w to xy, then she will choose *ABBC*, so Bob chooses *ABAB* iff he prefers X to the best pair that does not contain w. This means that X contains w, so it is better than w. In both cases Bob does not want any unrelated singleton.

In Range 7, in both steps, Bob cannot afford any unrelated bundle. To analyze Alice's behavior, rename the items such that Bob's best item is w, and Carl's worst item besides w is z.

In the last step, Bob picks w and Carl picks x or y, so Alice can get the best of xz, yz. She can afford only one unrelated singleton (w), but if she wants it she can choose *ABCB* in the first step.

In the first step necessarily Alice prefers w to $best(xz, yz)$, so w is her best singleton and she picks it. Since $b + c > a$, Alice can afford only two unrelated pairs — the ones containing the last item. There exists a SPE in which this last item is z. So Alice can afford only xz, yz. But if Alice wanted one of these, she could have waited to the last step. Finally, it is easy to check that the seven ranges handled by Algorithm 3 cover all the income space except a finite number of hyperplanes (corresponding to the equalities $a = b, b = c, a = 2b + c, a = 2b, a = b + c, a + c = 2b, a = 2c, 2c = b$). Therefore, there exists a CEFAI. □

5.3 Four agents

In this section there are four items and four agents. This case was left open in [5]. We show that there may exist a subset of the income-space with a positive volume in which no CE exists.

Consider the income subspace defined by:

$$2b > 2c > b + d > a > c + d > d + d > b > c > d$$

There are four items denoted by: w, x, y, z. The agents' preferences contain the following relations:

- Alice: $xy > w > xz > yz > x > y > z$
- Bob: $w > z > x > y$
- Carl: $x > y > w > z$

Suppose by contradiction that a CE exists. There are several cases:

Case #1: Dana gets an item. So each agent gets one item. So Alice gets w (her best item), Bob gets z (his best remaining item), Carl gets x and Dana gets y. But, $a > c + d$ so Alice can afford xy, which she prefers over w — contradiction.

Case #2: Dana gets no item and Alice gets one item. So either Bob or Carl gets two items. But, $d > b/2 > c/2$ so Dana can afford one of the two items of Bob/Carl — contradiction.

Case #3: Dana gets no item and Alice gets w plus one or more other items. As in Case #2, Bob cannot get two items since then Dana will be able to afford one of them. So Bob gets at most one item which is not w. Bob must not be able to afford w, so the price of w must be more than b. So the price of Alice's remaining item/s must be less than $a - b < d$. So Dana can afford one of Alice's items — contradiction.

Case #4: Dana gets no item and Alice does not get w. So someone else gets w. So Alice must get a bundle better than w. This bundle must contain xy. Carl receives at most one item and it is worse than x and y. But, $c > a/2$ so Carl can afford one of Alice's items — contradiction.

Hence, no allocation can satisfy the CE conditions. □

6 FIVE ITEMS

In this section there are five items and two agents. [5] already showed that there may exist a subset of the income-space with positive measure in which no CE exists. They used cardinal preferences for ease of presentation. For completeness, we give here an example based on ordinal preferences.

Consider the income subspace defined by: $a > b > 3a/4$.

There are five items: v, w, x, y, z. The agents' preferences contain the following relations (where a comma between two bundles implies that the preference between them is irrelevant to the proof):

- Alice: quartets $> vwx, vwy, vwz > vw > xyz >$ $vxy, vxz, vyz, wxy, wxz, wyz >$ pairs-except-$vw >$ singletons
- Bob: quartets $>$ triplets-except-$xyz > vx, vy, vz, wx, wy, wz >$ $xyz > vw > v > w > xy, xz, yz > x, y, z$

Suppose by contradiction that a CE allocation exists. There are several cases depending on the number of items given to Alice:

Alice gets **1 item**: she obviously envies Bob — contradiction.

Alice gets **2 items**: these must be vw, otherwise Alice envies Bob. So Bob gets xyz. But, because $b > b/3 + a/2$, Bob can afford the pair made of his cheapest item and Alice's cheapest item, which is one of vx, vy, vz, wx, wy, wz. Bob prefers all these to xyz — contradiction.

Alice gets **3 items**: they can't be worse than xyz, since Alice can afford either vw or xyz (one of these costs at most $(a+b)/2 < a$). They can't be better than xyz, since then Bob gets $yz/xz/xy$, but he can afford either xyz or v or w (one of these costs at most $(a+b)/3 < b$). So Alice gets xyz. But then she envies Bob — contradiction.

Alice gets **4 items**: because $b > 3a/4$, Bob can afford a triplet. But Bob prefers all triplets to all singletons — contradiction. □

7 FAIRNESS PROPERTIES OF CE

It is known that a CE is always Pareto-efficient, and when incomes are equal it is also envy-free [8]. We now present a generalized fairness guarantee that holds even when incomes are different.

Throughout this section, we focus on a specific agent, Alice, with income a, bundle A and preference-relation $\succ$. We define:

- For every bundle X and integer d, $\text{Partition}(X, d)$ is the set of all partitions of X to d sub-bundles (some possibly empty).
- For every vector $Y = (Y_1, Y_2, \dots)$ and integer l, $\text{Union}(Y, l)$ is the set of all unions of l bundles from Y, $Y_{j1} \cup Y_{j2} \cup \dots \cup Y_{jl}$.
- For every bundle X and integers l, d, the *l-out-of-d-maximin-bundle* of X is denoted $\begin{bmatrix} l \\ d \end{bmatrix} X$ and defined as: [4]

$$\begin{bmatrix} l \\ d \end{bmatrix} X := \max_{Y \in \text{Partition}(X,d)} \min_{Z \in \text{Union}(Y,l)} Z$$

where max, min are based on Alice's preference-relation $\succ$.

In other words, $\begin{bmatrix} l \\ d \end{bmatrix} X$ is the best bundle that Alice can guarantee to herself by dividing X to d parts, letting an adversary pick $d - l$ parts, and taking the remaining l parts. This is a generalization of the *maximin share* of Alice, defined by Budish [14] as "the most preferred bundle she could guarantee herself as divider in divide-and-choose against adversarial opponents". In our notation, Alice's maximin share is denoted $\begin{bmatrix} 1 \\ n \end{bmatrix} \mathcal{M}$, where $\mathcal{M}$ is the set of all items.

We are now ready to state the generalized fairness guarantee.

Proposition 7.1. *Let $(X_1, \dots, X_n)$ be a CE allocation. Let K be a subset of the agents, $K \subseteq \mathcal{N}$. For every two integers l, d with $1 \le l \le d$:*

$$a \ge \frac{l}{d} \sum_{i \in K} t_i \quad \Longrightarrow \quad A \succeq \begin{bmatrix} l \\ d \end{bmatrix} \bigcup_{i \in K} X_i$$

Proof. Let P be the price of the union in the right-hand side, $P := p\left(\bigcup_{i \in K} X_i\right)$. By CE Condition 1, for every i, $t_i \ge p(X_i)$. Therefore, $\sum_{i \in K} t_i \ge P$. By the proposition assumption, $a \ge \frac{l}{d} \cdot P$. Consider a partition $Y \in \text{Partition}(\bigcup_{i \in K} X_i, d)$. Order the d parts in Y by increasing price, i.e, $p(Y_1) \le \dots \le p(Y_d)$. Then, $p(Y_1) + \dots + p(Y_l) \le \frac{l}{d} \cdot P$. Define $Z := Y_1 \cup \dots \cup Y_l$. Then, $p(Z) \le \frac{l}{d} \cdot P \le a$, so Alice can afford Z. By the CE Condition 2, Alice's bundle must be at least as good as Z: $A \succeq Z$. Since $Z \in \text{Union}(Y, l)$, by definition $Z \succeq \begin{bmatrix} l \\ d \end{bmatrix} \bigcup_{i \in K} X_i$. By transitivity, $A \succeq \begin{bmatrix} l \\ d \end{bmatrix} \bigcup_{i \in K} X_i$. □

To appreciate the generality of Proposition 7.1, we show that several known facts are special cases of it.

(1) When all incomes are equal, a CE allocation is envy-free. *Proof*: take $K = \{i\}$ (i.e, a single agent) and $l = d = 1$. The left-hand side is true since $a = t_i$. In the right-hand side, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} X_i = X_i$ so it becomes $A \succeq X_i$, which means that Alice does not envy agent i.

(2) When all incomes are equal, a CE allocation guarantees each agent his maximin share. *Proof*: take $K = \mathcal{N}$, $l = 1$ and $d = n$. The left-hand side is true since $a = \frac{1}{n}$ of the sum of all incomes; In the right-hand side, the union equals the set $\mathcal{M}$ of all items, and $\begin{bmatrix} 1 \\ n \end{bmatrix} \mathcal{M}$ is exactly Alice's maximin share.

(3) When the incomes are "almost" equal, i.e, the income of each agent is at least $1/(n+1)$ of the income sum, a CE allocation guarantees each agent the "approximate maximin share" of Budish [14], which is defined as $\begin{bmatrix} 1 \\ n+1 \end{bmatrix} \mathcal{M}$. The proof is similar to (2).

(4) The following fact was proved directly by [5]. When the sum of all incomes is 1, a CE allocation guarantees Alice her l-out-of-d-maximin share, for every integers l, d such that $l/d \le a$. *Proof*: take $K = \mathcal{N}$. Then the right-hand side becomes $\begin{bmatrix} l \\ d \end{bmatrix} \mathcal{M}$, which is the l-out-of-d-maximin-share.

(5) The following fact was proved directly by Reijnierse and Potters [22], where it was called *α-envy-freeness.* Suppose all goods are divisible, and the preferences of Alice are represented by a linear value-function v_A (so the value of each bundle is a linear function of the quantities of the goods in the bundle). Then, in a CE allocation, for every $i \in \mathcal{N}$, $v_A(A) \ge \frac{a}{t_i} v_A(X_i)$. *Proof:* Take $K = \{i\}$. Since the goods are divisible, for every integer d, Alice can partition X_i to d parts whose value is exactly $v_A(X_i)/d$. So for every integer l, $v_A(\begin{bmatrix} l \\ d \end{bmatrix} X_i) \ge \frac{l}{d} v_A(X_i)$. So Proposition 7.1 implies that, for every l, d such that $\frac{l}{d} \le \frac{a}{t_i}$ and for all $i \in \mathcal{N}$, $v_A(A) \ge \frac{l}{d} v_A(X_i)$. We can take l, d such that $\frac{l}{d}$ is arbitrarily close to $\frac{a}{t_i}$. Therefore, $v_A(A) \ge \frac{a}{t_i} v_A(X_i)$.

While previous results only consider the cases in which $|K| = 1$ or $|K| = n$, Proposition 7.1 is more general. For example, it implies that if Alice's income is at least 1/2 of the sum of incomes of Bob and Carl, then a CE allocation gives her a bundle worth at least as much as the 1-out-of-2 maximin-share of the union of Bob's and Carl's bundles. Thus, the algorithms for finding a CE allocation, presented in the previous sections, can be seen as algorithms for fair allocation: each of these algorithms guarantees, to each agent, a multitude of fairness properties that naturally generalize the properties of both envy-freeness and maximin-share-guarantee.

8 RELATED WORK

1. CE with indivisible items. Recently there has been a lot of interest in the computational complexity of finding a CE in markets with indivisibilities.

Deng et al. [16] studied a market to which each agent comes with an initial endowment (rather than an initial income) and all valuations are additive. They proved that deciding whether CE exists is NP-hard even if there are 3 agents. They presented an approximation algorithm which relaxes the CE conditions in two ways: (1) The bundle allocated to each agent is valued at least $(1-\epsilon)$ of the optimum given the prices, and (2) the demand is at least $(1-\epsilon)$ times the supply. Both these relaxations are unrelated to our setting, in which the preferences are ordinal and all items must be allocated.

Bouveret and Lemaître [8] studied CE-from-equal-incomes (CEEI) as a rule for fair allocation of items. They related it to four other fairness criteria assuming all agents have additive valuation functions. They asked what is the computational complexity of deciding whether CEEI exists. This question was answered soon afterwards

[4] Babaioff et al. [5] introduced the l-out-of-d-maximin-bundle. We introduced the notation "l-above-d" to make Proposition 7.1 symmetric and easy to visualize.

by Aziz [3], who proved that the problem is weakly NP-hard when there are two agents and m items, and strongly NP-hard when there are n agents and $3n$ items. Brânzei et al. [12] further proved that even verifying whether a given allocation is CEEI is co-NP-hard.

Brânzei et al. [12] studied CEEI also for single-minded agents. In this case, verifying whether a given allocation is CEEI is polynomial but checking if CEEI exists is co-NP-complete. Single-minded agents were further studied by Brânzei et al. [13]. In contrast to our setting, they assume that each item can come in multiple units, all of which must have the same price. They show an example in which (1) a CE where all agents exhaust their income does not exist, (2) a CE where some agents spend less than their income does exist. They call this solution CAEI — Competitive Allocation from Equal Incomes. Interestingly, in contrast to CEEI, it is possible to find a CAEI (if one exists) in polynomial time.

Heinen et al. [18] extended [8] from additive to *k-additive utility functions*, in which each agent reports a value for bundles containing at most k items, and the values of larger bundles are determined by adding and subtracting the values of the basic bundles.

Budish [14] studied the most general setting in which agents can have arbitrary preference relations over bundles. He invented a beautiful and practical *approximate CEEI* mechanism, which relaxes the CEEI conditions in two ways: (1) The agents' incomes are not exactly equal, and (2) a small number of items may remain unallocated. He proved that an approximate-CEEI always exists (although Othman et al. [21] recently proved that the computation of approximate-CEEI is PPAD-complete). The first relaxation (1) is closely related to our setting where incomes must not be exactly equal. However, the second relaxation (2) make his solution less useful when the initial number of items is small.

2. Picking-sequences. Picking-sequences are common practical mechanisms for allocating indivisible items. They are favored due to their simplicity, privacy and low communication complexity.

Brams and Kaplan [10] and Brams [9] study picking-sequences for allocating cabinet ministries among parties. There is a coalition of parties; each party has a different number of seats in the parliament; larger parties should be allocated more ministries, or more prestigious ministries. This is an interesting use-case of fair division with different entitlements. A possible solution to this problem is to determine a picking-sequence, based on the different entitlements, and let each party pick a ministry in turn. Such a solution is used in Northern Ireland, Denmark and the European parliament [20].

Brams and Kaplan assume that each agent has a strict ordering on the items, and has responsive preferences on bundles of items. Responsive preferences are more general than additive preferences, but less general than the monotone preferences studied in this paper [5]. With responsive preferences, at each point in the picking-sequence, there is a single remaining item which is the "best item" for the agent. An agent can be "truthful" and pick the best item, or be "strategic" and pick another item based on his knowledge of the other agents' valuations. They discuss the Pareto-efficiency of the SPE of such picking-sequences. Recently, Aziz et al. [4] further discuss the strategic properties of picking-sequences when agents have additive valuations. However, they do not discuss whether the Nash equilibrium is also a competitive-equilibrium, or whether it satisfies any notion of fairness.

Another line of work related to picking-sequences is how to select a picking-sequence that maximizes some global objective. Bouveret and Lang [7] study this question under the assumption that all valuation functions are additive, and moreover, there is a *single* common scoring-function that relates the rank of an item in an agent's ranking to its monetary value. The allocator does not know the rankings of the agents, but he knows that all rankings are random draws from a given probability distribution. The allocator's goal is to maximize the expected value of some social welfare function. They show picking-sequences that maximize the expected utilitarian welfare (sum of utilities) or the expected egalitarian welfare (minimum utility) in various settings. Kalinowski et al. [19] show that, when there are two agents with a Borda scoring function, and each ranking is equally probable, the "round robin" sequence (*ABABAB*...) attains the maximal expected sum-of-utilities.

Picking-sequences fundamentally differ from *serial dictatorship* (whether deterministic or random): serial dictatorship is dominant-strategy truthful since each agent has only one chance to choose; it is usually used in matching markets where each agent is entitled to one item. In contrast, picking-sequences let each agent pick more than one time, and usually they have no dominant strategies.

3. Unequal entitlements. Fair division with unequal entitlements has been studied with respect to a divisible resource ("cake"); see Cseh and Fleiner [15], Segal-Halevi [23] for recent surveys. Recently, Farhadi et al. [17] studied fair allocation of indivisible goods to agents with unequal entitlements. This problem is closely related to competitive equilibrium with unequal incomes. Their results are mostly negative: even when all agents have additive valuations, a "fair" allocation (according to their definition of fairness) might not exist and cannot be approximated to within a factor of n. In light of these negative results, it is interesting that we can get positive results for almost all incomes.

9 FUTURE WORK

This paper assumes that agents can have any monotone preferences. An interesting topic for future work is to study more specific preference domains. In particular, in what cases does CEFAI exist when all agents have additive or responsive preferences?

Note that the preferences in Sub. 5.3 are additive (hence also responsive). For example, Alice's valuations for w, x, y, z can be $11, 7, 5, 3$ and the other agents' valuations can be arbitrary numbers consistent with their orderings. So we know that with 4 additive/responsive agents, CEFAI might not exist.

However, in Sec. 6 the preferences are not responsive (hence also not additive). Therefore, for 2 or 3 additive/responsive agents, the existence of CEFAI is still an open question. [5] provide a partial solution for the case of 2 agents with additive preferences and any number of items, but a general solution is still not known.

Another interesting question is whether the pixep technique is sufficiently general to find a CE when it exists, i.e, can any CE be supported by a pixep satisfying (R1,R2,R3)?

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