Fairly Dividing a Cake after Some Parts Were Burnt in the Oven

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ABSTRACT

There is a heterogeneous resource that contains both good parts and bad parts, for example, a cake with some parts burnt, a land-estate with some parts heavily taxed, or a chore with some parts fun to do. The resource has to be divided fairly among n agents with different preferences, each of whom has a personal value-density function on the resource. The value-density functions can accept any real value — positive, negative or zero. Each agent should receive a connected piece and no agent should envy another agent. We prove that such a division exists for 3 agents.

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1 INTRODUCTION

Most research works on fair division assume that the *manna* (the resource to divide) is good, e.g., tasty cakes, precious jewels or fertile land-estates. A substantial minority of the works assume that the manna is bad, e.g., house-chores or night-shifts. Recently, Bogomolnaia et al. [2] introduced the more general setting of *mixed manna* — every resource can be good for some agents and bad for others. Here are some illustrative examples.

(1) A cake with some parts burnt has to be divided among children. Some (like this author as a child) find the burnt parts tasty, but most children consider them bad (but still must eat what they get in order not to insult the host).
(2) A land-estate has to be divided among heirs, where landowners are subject to taxation. The value of a land-plot to an heir may be either positive or negative, depending on his/her valuation of the land and tax status.
(3) A house-chore such as washing the dishes has to be divided among family members. Most of them consider this bad, but some of them may view dish-washing, in some parts of the day, as a perfect relaxation after spending hours in solving mathematical problems.

While Bogomolnaia et al. [2] focused on dividing homogeneous resources, we study the classic problem of *cake-cutting* [18] — dividing a single heterogeneous resource. The cake-cutting problem comes in many flavors: the cake can be one-dimensional or multi-dimensional [17]; the fairness criterion can be *proportionality* (each agent receives a piece he values as at least $1/n$ of the total) or *envy-freeness* (each agent receives a piece he values at least as much as the piece of any other agent); the pieces can be connected or disconnected; and more. See [3, 14, 16] for recent surveys. All variants were studied in the good-cake setting (all agents consider every piece of cake good). Some variants were also studied in the bad-cake setting (all agents consider every piece of cake bad). So far, no variants were studied in the general mixed-cake setting.

While all variants of the cake-cutting problem are interesting, this paper focuses on a specific variant in which (a) the cake is one-dimensional, (b) the fairness criterion is envy-freeness, (c) the pieces must be connected (see **Section 2** for the formal model).

The main question of interest in this paper is:

Does there exist a connected envy-free division of a mixed cake?

It is known that the answer is "yes" both for good cakes and for bad cakes [22]. Moreover, there are procedures for approximating such a division for any number of agents. However, the proofs are based on a specific combinatorial structure, based on the well-known *Sperner's lemma*; this structure breaks down in the mixed-cake setting, so the existing proofs are inapplicable (see **Section 3**).

Working with mixed cakes requires a new, more general combinatorial structure. This structure is based on a generalization of Sperner's lemma. Based on this structure, it is possible to prove the main result (**Section 4**):

A connected envy-free division always exists for three agents.[1]

The existence of a connected envy-free division implies that an existing approximation algorithm can be adapted to approximate such a division to any desired accuracy (**Section 5**).

Most parts of the proof are valid for any number of agents. However, there is one part which we do not know how to generalize to an arbitrary number of agents. Recently, Meunier and Zerbib [11] presented a proof that an envy-free division exists when the number of agents is 4 or prime. A sketch of a proof for the case of prime n, using more elementary arguments, appears in the full version.

2 MODEL

A cake — modeled as the interval $[0, 1]$ — has to be divided among n agents. The agents are called $A_1, \ldots, A_n$ or Alice, Bob, Carl, etc. The cake should be partitioned into n pairwise-disjoint intervals, $X_1, \ldots, X_n$ (some possibly empty), whose union equals the entire cake. Interval X_i should be given to A_i such that the division is *envy-free* — each agent weakly prefers his piece over any other agent's piece. Two models for the agents' preferences are considered.

(A) Additive agents: each agent A_i has an integrable value-density function v_i. The value of a piece is the integral of the value-density

[1] Division problems with 3 agents are quite common in practice. For example, according to www.pewsocialtrends.org/2015/05/07/family-size-among-mothers, about 25% of mothers have 3 children. Hence, about 25% of inheritance cases involve division among 3 agents. As another example, in the spliddit.org website [8], about 62% of all requests for fair division of items involve 3 agents. We thank Nisarg Shah for this information.

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Cake	Additive agents	Selective agents
Good	$v_i(x) \geq 0$ for every $x \in [0,1]$.	s_i always contains a non-empty piece.
Bad	$v_i(x) \leq 0$ for every $x \in [0,1]$.	s_i always contains an empty piece, if one exists.
Mixed	v_i is any integrable function.	s_i is any continuous selection function.

Table 1: Assumptions in different cake-cutting models.

on that piece: $V_i(X_j) = \int_{x \in X_j} v_i(x)dx$. Note, the value of any single point is 0, so it is irrelevant who receives the endpoints of pieces. A division is *envy-free* if each agent believes his piece's value is above the value of any other agent's piece: $\forall i, j : V_i(X_i) \geq V_i(X_j)$.

(B) Selective agents: each agent A_i has a function s_i that accepts a nonempty set of pieces X and returns a nonempty subset of X. The interpretation is that the agent "prefers" each of the pieces in $s_i(X)$ over all other pieces in X (this implies that the agent is indifferent between the pieces in $s_i(X)$). A division X is *envy-free* if each agent receives one of his preferred pieces: $\forall i : X_i \in s_i(X)$. The preference functions should be *continuous* — any piece that is preferred for a convergent sequence of partitions is preferred for the limit partition (equivalently: for each i, j, the set of partitions X in which $X_j \in s_i(X)$ is a closed set. See Su [22]). This, again, implies that it is irrelevant who receives the endpoints of pieces.

Model (A) is more common in the cake-cutting world, while model (B) is much more general. Every additive agent is also a selective agent. But selective agents may have non-additive valuations and even some externalities: the preference of an agent may depend on the entire set of pieces in the partition rather than just his own piece (however, the preference may not depend on which agent receives what piece; see [5] for a discussion of such externalities). In the good-cake and bad-cake settings, additional assumptions are made on the agents' preferences besides continuity, as shown in Table 1. The present paper removes these assumptions.

Approximately-envy-free division. There are two ways to define an approximately envy-free division. (A) With additive agents, the approximation is measured in units of value: an *ϵ-envy-free division* is a division in which each agent believes that his piece's value is at most ϵ less than the value of any other piece: $\forall i, j : V_i(X_i) \geq V_i(X_j) - \epsilon$. The valuations are usually normalized such that the value of the entire cake is 1 for all agents, so ϵ is a fraction (e.g., 1% of the cake value). (B) With selective agents there are no numeric values, so the approximation is measured in units of length: a *δ-envy-free division* is a division in which, for every agent A_i, movement of the borders by at most δ results in a division in which A_i prefers his piece over any other piece.

Unless stated otherwise, all results in this paper are valid for selective agents, therefore also for additive agents.

3 EXISTING PROCEDURES

With $n = 2$ agents, the classic "I cut, you choose" protocol produces an envy-free division whether the cake is good, bad or mixed. The fun begins at $n = 3$.

3.1 Reduction to all-goods and all-bads

One might think that mixed-manna problems could be reduced to good-manna and bad-manna ones in the following way. For each part of the resource: (a) if there is one or more agents who think it is good, then divide it among them using any known procedure for dividing goods; (b) otherwise, all agents think it is bad — divide it among them using any known procedure for dividing bads.

However, this simple reduction does not work when there are additional requirements besides fairness, such as economic efficiency or connectivity. In Bogomolnaia et al. [2] the requirements are envy-freeness and Pareto-efficiency; in this paper the requirements are envy-freeness and connectivity. It is impossible to guarantee all three properties simultaneously [20]. Hence the techniques and results are quite different, and no one implies the other.

3.2 Moving-knives and approximations

Three procedures for connected envy-free division for three *additive* agents are known: Stromquist [19], Robertson and Webb [15, pages 77-78] and Barbanel and Brams [1]. All of them use one or more knives moving continuously. They were originally designed for good cakes and later adapted to bad cakes. All of them crucially rely on a *monotonicity* assumption: all agents weakly prefer a piece to all its subsets (in a good cake), or all agents weakly prefer a piece to all its supersets (in a bad cake). However, monotonicity does not hold with a mixed cake, so these procedures cannot be used.

Another algorithm that does not work, but for a different reason, is the generic approximation algorithm recently presented by Brânzei and Nisan [4] for additive agents. Their algorithm can approximate any division that is described by linear conditions; in particular, it can approximate an envy-free division, whenever such a division exists. Since an envy-free division of a mixed cake among three agents always exists (as will be proved in this paper), their algorithm can be used to find an approximation of it. The problem is the runtime complexity: while with good cakes and bad cakes their algorithm runs in time $O(n/\epsilon)$ when ϵ is the additive approximation factor, with mixed cakes the runtime complexity might be unbounded. Details are in an appendix in the full version.

3.3 Simplex of partitions

With four or more agents, or even with three *selective* agents, no moving-knives procedures are known. A different approach, which works for any number of selective agents, was suggested by Stromquist [19] and further developed by Su [22]. It is based on the *simplex of partitions*. To present it we introduce some notation that will also be used in the rest of the paper.

Δ^{n-1} is the $(n-1)$-dimensional standard simplex — the points $(l_1, \ldots, l_n)$ with $l_1 + \cdots + l_n = 1$. Each such point represents a cake-partition where the piece lengths are $l_1, \ldots, l_n$; see Figure 1.

$[n]$ denotes the set $\{1, \ldots, n\}$. The n vertices of Δ^{n-1} are called its *main vertices* and denoted by F_j, for $j \in [n]$. Each face of Δ^{n-1} is the convex hull of some subset of its main vertices, $\text{conv}_{j \in J}(F_j)$ for some $J \subseteq [n]$; this face is denoted by F_J. E.g, the face connecting F_1 and F_2 is denoted $F_{\{1,2\}}$, or just F_{12} for short. For each $j \in [n]$, we denote $F_{-j} := F_{[n] \setminus \{j\}}$ = the face opposite to F_j. In all points on F_{-j}, the j-th coordinate is 0, so they represent partitions in which piece number j is empty.

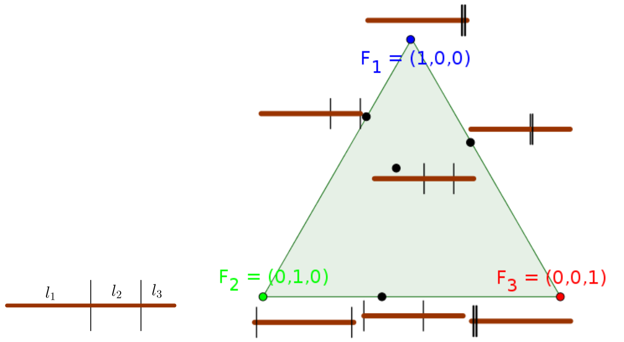


Figure 1: Left: a generic partition of the cake among $n = 3$ agents. $l_1 + l_2 + l_3 = 1$. Right: The simplex of partitions for $n = 3$ agents. Each point represents a partition. Seven points are marked, and the corresponding partitions are shown.

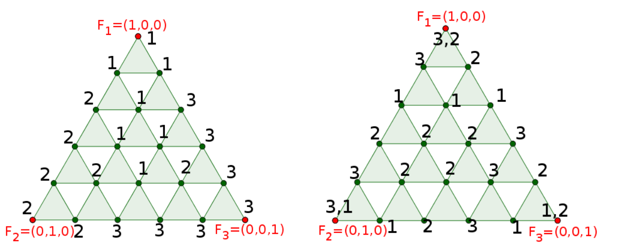


Figure 2: Possible labeling L_i of a single agent. Left: the value of the entire cake is positive. Hence, in each main vertex F_j, the agent prefers only piece j, since it is the only non-empty piece. In the edges between two main vertices F_j, F_k, the agent prefers either j or k.
Right: the value of the entire cake is negative, but it contains some positive parts. In each main vertex F_j, the agent prefers the two empty pieces — the two pieces that are NOT j. In the edges between two main vertices, all three labels may appear.

Agent labelings. Given a partition of the cake into n intervals, each agent has one or more *preferred pieces*. The preferences of agent A_i can be represented by a function $L_i : \Delta^{n-1} \to 2^{[n]}$. The function L_i maps each cake-partition (= a point in the standard simplex) to the set of pieces that A_i prefers in this partition (= a set of labels from $[n]$). The set of preferred pieces always contains at least one label; it may contain more than one label if the agent is indifferent between two or more best pieces. This is particularly relevant in case the agent prefers an empty piece, since there are partitions in which there is more than one empty piece. If x is such a partition then $L_i(x)$ contains the set of all empty pieces. See Figure 2. An *envy-free division* corresponds to a point x in the partition-simplex where it is possible to select, for each i, a single label from $L_i(x)$, such that the n labels are distinct.

Triangulations. A triangulation of a simplex is a partition into sub-simplices satisfying some technical conditions. An example is shown in Figure 2. We denote a triangulation by T, and the set of vertices in the triangulation by $\textsc{Vert}(T)$.

*Definition 3.1 (**Envy-free simplex**).* Suppose we let n agents label the vertices of T, so we have n labelings $L_i : \textsc{Vert}(T) \to 2^{[n]}$ for $i \in \{1, \ldots, n\}$. An *envy-free simplex* is a sub-simplex in T with vertices $(t_1, \ldots, t_n)$, such that, for each $i \in [n]$, it is possible to select a single label from $L_i(t_i)$ such that the n labels are distinct.

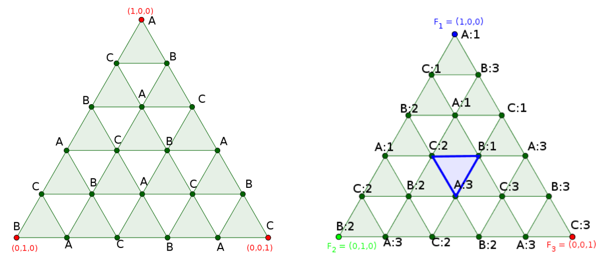


Figure 3: Left: Assignment of vertices to agents such that, in each sub-triangle, each vertex is owned by a different agent.
Right: A combined labeling based on this ownership-assignment. The emphasized triangle at the center is an envy-free simplex.

If the diameter of each sub-simplex in T is at most δ, then each envy-free simplex corresponds to a δ-envy-free division. If, for every δ, there is an envy-free simplex with diameter at most δ, then the continuity of the preference functions s_i implies the existence of an envy-free division; see Su [22].

Good Cakes. In a partition of a good cake, there always exists a non-empty piece with a weakly-positive value, so it is always possible to assume that each agent prefers a non-empty piece. Therefore, every labeling L_i satisfies *Sperner's boundary condition*: every triangulation-vertex in the face F_J is labeled with a label from the set J (see Figure 2/Left). By Sperner's lemma, for every i there is a *fully-labeled simplex* — a simplex whose n vertices are labeled by L_i with n distinct labels.

In order to get an *envy-free simplex*, we combine the n agent-labelings $L_1, \ldots, L_n$ to a single labeling $L^W : \textsc{Vert}(T) \to 2^{[n]}$ in the following way. Each triangulation-vertex is assigned to one of the n agents, such that in each sub-simplex, each of its vertices is owned by a unique agent. See Figure 3/Left. Now, each vertex is labeled with the corresponding labels of its owner: if a vertex x is owned by agent A_i, then $L^W(x) := L_i(x)$. See Figure 3/Right. If all the L_i satisfy Sperner's boundary condition, then the combined labeling L^W also satisfies Sperner's boundary condition. Therefore, by Sperner's lemma, L^W has a fully-labeled simplex. By definition of L^W, this simplex is an envy-free simplex [22].

Bad Cakes. In a partition of a bad cake, the values of all non-empty pieces are weakly negative, so it is always possible to assume that each agent prefers an empty piece. In the main vertices, there are $n-1$ empty pieces; the agent is indifferent between them, so we may label each main vertex with an arbitrary empty piece. We can always do this such that the resulting labeling satisfies Sperner's boundary condition [22]. Hence, an envy-free simplex exists.

Mixed Cakes. When the value of the entire cake is negative, but the cake may contain positive pieces, each agent may prefer in each point either an empty piece or a non-empty piece. Hence, the agent labelings no longer satisfy Sperner's boundary condition; see Figure 2/Right. Here, our work begins.

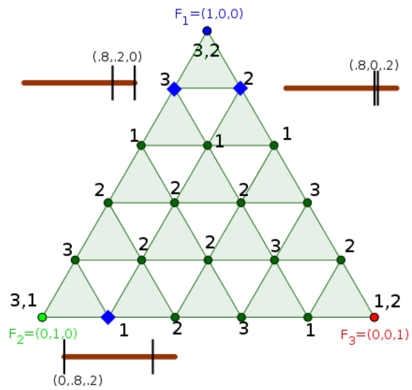


Figure 4: Three points representing the same physical partition.

4 CUTTING MIXED CAKES

4.1 The Consistency Condition

The first step in handling a mixed cake is to find boundary conditions that are satisfied for *all* agent labelings, regardless of whether the cake is good, bad or mixed. Our boundary condition is based on the observation that *different points on the boundary of the partition-simplex may represent the same physical cake-partition.* For example, consider the three diamond-shaped points in Figure 4. In each of these points, the set of pieces is the same: $\{[0,.8],[.8,1],\emptyset\}$. Therefore, a consistent agent will select the same piece in all three partitions, even though this piece might have a different index in each point. This means that the agent's label in each of these points uniquely determines the agent's labels in the other two points. For example, if the agent labels the top-left diamond point by "3", this means that he prefers the empty piece, so he must label the top-right diamond point by "2" and the bottom-left diamond point by "1" (as in the figure).

To formalize this boundary condition we need several definitions.

Definition 4.1. Two points in Δ^{n-1} are called *friends* if they have the same ordered sequence of nonzero coordinates.

For example, on Δ^{3-1}, the points $(0,.2,.8) \in F_{-1}$ and $(.2,0,.8) \in F_{-2}$ and $(.2,.8,0) \in F_{-3}$ are friends, since their ordered sequence of nonzero coordinates is $(.2,.8)$.

Since our boundary conditions have a bite only for friends, we will consider from now on only triangulations that are "friendly":

Definition 4.2. A triangulation T is called *friendly* if, for every vertex $x \in \textsc{Vert}(T)$, all the friends of x are in $\textsc{Vert}(T)$.

Our boundary condition is that the label of a vertex in F_{-1} uniquely determines the labels of all its friends on the other faces. Specifically, consider a vertex $x_k \in F_{-k}$. By definition of F_{-k}, the k-th coordinate of x_k iz zero. If we move the k-th coordinate of x_k to position 1 and push coordinates $1,\ldots,k-1$ one position rightwards, we get a vertex on F_{-1} that is a friend of x_k; denote it by $f_k(x_k)$. Since the triangulation is friendly, it contains $f_k(x_k)$.

Suppose that the label of $f_k(x_k)$ is l. Then the label on x_k is:

$$\pi_{-k}(l) := \begin{cases} k & l = 1 \text{ [agent prefers empty piece]} \\ l-1 & 1 < l \le k \\ l & l > k \end{cases} \tag{1}$$

For every k, the function π_{-k} is a permutation (a bijection from $[n]$ to $[n]$). π_{-1} is the identity permutation.

The table below shows the three permutations for $n = 3$: π_{-1}, π_{-2} and π_{-3}. The rightmost column is an illustration corresponding to the sequence of labels on each face in Figure 4:

Preferred piece:	Empty	Left	Right	{ER}ELRE{EL}
Label on F_{-1}:	1	2	3	{13}1231{12}
Label on F_{-2}:	2	1	3	{23}2132{21}
Label on F_{-3}:	3	1	2	{32}3123{31}

Definition 4.3. A labeling $L : \textsc{Vert}(T) \to 2^{[n]}$ is **consistent** if, for every $k \in [n]$ and vertex $x_k \in F_{-k}$:

$$L(x_k) = \pi_{-k}(L(f_k(x_k)))$$

where π_{-k} is defined by (1), and $f_k(x_k)$ is a friend of x_k on F_1, derived from x_k by moving its k-th coordinate to position 1.

Note that $L(x_1)$ may be a set of more than one label, and in this case, consistency implies that $L(x_k)$ is a set with the same number of labels. For example, if $x_1 \in F_{-1}$ and $L(x_1) = \{1,2\}$ and $x_3 \in F_{-3}$ then $L(x_3) = \pi_{-3}(\{1,2\}) = \{\pi_{-3}(1), \pi_{-3}(2)\} = \{3,1\}$.

Figures 2, 4 show examples of consistent labelings.

We present two lemmas about consistency. The first relates to the *parity* of permutations. Recall that a permutation is *even/odd* if it can be implemented by an even/odd number of swaps. The following lemma is simple and its proof is omitted:

Lemma 4.4. *The permutation π_{-k} is even/odd iff k is odd/even.*

The case $n = 3$ is shown in the above table: π_{-1} is even (the identity permutation), π_{-2} is odd (maps 123 to 213) and π_{-3} is even (maps 123 to 312).

The second lemma relates to labels on faces F_J where $|J| \le n-2$. Such faces are intersections of two or more $n-1$-dimensional faces. For example, let x be the main vertex $F_3 = (0,0,1)$. Then, x is a friend of itself, with $f_2(x) = x$. Therefore, consistency implies that $L(x) = \pi_{-2}(L(x))$. Hence, $L(x)$ contains 2 if-and-only-if it contains 1. This makes sense: since all empty pieces are identical, the agent prefers an empty piece if and only if it prefers all empty pieces. This is generalized in the following lemma, whose proof appears in the full version :

Lemma 4.5. *Let L be a consistent labeling. Then, for every vertex $x \in F_{[n]\setminus J}$, either $L(x) \cap J = J$ or $L(x) \cap J = \emptyset$.*

Our goal now is to prove that, if all n agent-labelings are consistent, then an envy-free simplex exists. We will proceed in two steps.

- If all labelings $L_1,\ldots,L_n$ are consistent, then there exists a single consistent combined labeling L^W (Subsection 4.2).
- If a labeling is consistent, then it has a fully-labeled simplex (Subsections 4.3-4.4).

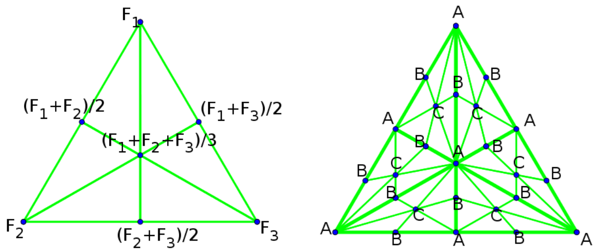


Figure 5: Left: barycentric subdivision of a triangle. Right: Barycentric triangulation of a triangle, with a friendly and diverse ownership assignment (here A, B, C are agents A_1, A_2, A_3).

4.2 Combining n labelings to a single labeling

The consistency condition is valid for a single agent. We have to find a way to combine n different consistent labelings into a single consistent labeling. For this we need several definitions.

Definition 4.6. An *ownership-assignment* of a triangulation T is a function from the vertices of the triangulation to the set of n agents, $W : \text{Vert}(T) \to \{A_1, \dots, A_n\}$.

Definition 4.7. Given a triangulation T, n labelings $L_1, \dots, L_n$, and an ownership assignment W, the *combined labeling* L^W is the labeling that assigns to each vertex in $\text{Vert}(T)$ the label/s assigned to it by its owner. I.e., if $W(x) = A_i$, then $L^W(x) := L_i(x)$.

Definition 4.8. An ownership-assignment W is called:
(a) *Diverse* — if in each sub-simplex in T, each vertex of the sub-simplex has a different owner;
(b) *Friendly* — if it assigns friends to the same owner. I.e., for every pair x, y of friends (see Definition 4.1), $W(x) = W(y)$.

The diversity condition was introduced by Su [22]. As an example, the ownership-assignment of Figure 3 is diverse. However, it is not friendly. For example, the two vertices near $(1, 0, 0)$ are friends since their coordinates are $(.8, .2, 0)$ and $(.8, 0, .2)$, but they have different owners (B, C). This means that the combined labeling is not necessarily consistent. Fortunately, there always exists an ownership-assignment that is both friendly and diverse.

Lemma 4.9. *For any $n \geq 3$ and $\delta > 0$, there exists a friendly triangulation T of Δ^{n-1} where the diameter of each sub-simplex is $\leq \delta$, and an ownership-assignment of T that is friendly and diverse.*

Proof. The construction is based on the *barycentric subdivision.*[2] The barycentric subdivision of a simplex with main vertices $F_1, \dots, F_n$ is constructed as follows.

Pick a permutation σ of the main vertices. For every prefix of the permutation, $\sigma_1, \dots, \sigma_m$ (for $m \in \{1, \dots, n\}$), define v_m as their barycenter (arithmetic mean): $v_m := (\sigma_1 + \cdots + \sigma_m)/m$. We call v_m a *level-m vertex*. The vertices $v_1, \dots, v_n$ define a subsimplex.

Each permutation yields a different subsimplex, so all in all, the barycentric subdivision of an $(n-1)$-dimensional simplex contains $n!$ subsimplices. Note that each sub-simplex has exactly one vertex of each level $m \in [n]$.

By recursively applying a barycentric subdivision to each subsimplex (as in Figure 5/Right), we get iterated *barycentric triangulations*. The ownership assignment is determined by the levels of vertices in the last subdivision step: each vertex with level i is assigned to agent A_i (see Figure 5/Right). This ownership assignment is:

- diverse — since for every i, each subsimplex has exactly one vertex of level i.
- friendly — since, by the symmetry of the barycentric subdivision, every two friend-vertices have the same level. □

Lemma 4.10. *Let $L_1, \dots, L_n$ be consistent labelings of a friendly triangulation T. If W is a friendly ownership-assignment, then the combined labeling L^W is consistent.*

Proof. Consistency restricts only the labels of friends. Since all friends are labeled by the same owner, and the labeling of each owner is consistent, the combined labeling is consistent too. □

Lemmas 4.9 and 4.10 reduce the problem of finding an envy-free simplex with n labelings, to the problem of finding a fully-labeled simplex with a single labeling. This is our next task.

4.3 The Degree Lemma

We want to prove that any consistent labeling has a fully-labeled simplex. For this we develop a generalization of Sperner's lemma.

In this subsection we will consider single-valued labelings. To differentiate them from the multi-valued labelings denoted by $L : \text{Vert}(T) \to 2^{[n]}$, we will denote them by $\ell : \text{Vert}(T) \to [n]$.

We will use the following claim that we call the *Degree Lemma*:

> *Let $\ell : \text{Vert}(T) \to [n]$ a labeling of a triangulation T. The* interior degree *of ℓ equals its* boundary degree.

To explain this lemma we have to explain what are "interior degree" and "boundary degree" of a labeling.[3]

Throughout this subsection, Q denotes a fixed $n-1$-dimensional simplex in $\mathbb{R}^{n-1}$ whose vertices are denoted by $Q_1, \dots, Q_n$. Q' denotes a fixed face of Q of co-dimension 1 (so Q' is an $n-2$-dimensional simplex). Most illustrations are for the case $n = 4$.

4.3.1 ***Interior degree****.* Let P be an $n-1$ dimensional simplex in $\mathbb{R}^{n-1}$. Let $g : P \to Q$ be a mapping that maps each of the n vertices of P to a vertex of Q. By basic linear algebra, there is a unique way to extend g to an affine transformation from P to Q. Define $\deg(g)$ as the sign of the determinant of this transformation:

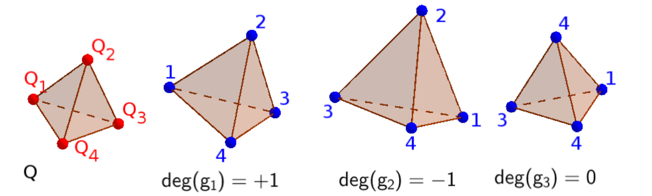
- $\deg(g) = +1$ means g is onto Q and can be implemented by translations, rotations and scalings (but no reflections);
- $\deg(g) = -1$ means g is onto Q and can be implemented by translations, rotations, scalings and a single reflection;
- $\deg(g) = 0$ means g is not onto Q (i.e., it maps the entire P into a single face of Q with dimension $n-2$ or less).

Every labeling $\ell : \text{Vert}(P) \to [n]$ defines a mapping g_ℓ where for each vertex $v \in \text{Vert}(P)$ whose label is j, we let $g_\ell(v) = Q_j$. The pictures below show three such mappings with different degrees from different source simplices in $\mathbb{R}^3$ to the same target Q: [4]

[2]the explanation follows the Wikipedia page "barycentric subdivision".

[3] The Degree Lemma can be proved as a corollary of much more general theorems in algebraic topology. See Corollary 3 in Meunier [10] and Corollary 3.1 in Musin [13]. For simplicity and self-containment we present it here using stand-alone geometric arguments. Some of the definitions follow Matveev [9].

[4] To visualize the degree, imagine that you transform the source simplex until it overlaps the target simplex Q, such that each vertex labeled with j overlaps Q_j. If you manage to do that without reflections then the degree is $+1$; otherwise it is -1.



We make several observations that relate the labeling to the degree.

(1) If P is *fully-labeled* (each vertex has a unique label), then g_ℓ is onto Q, so $\deg(g_\ell)$ is either $+1$ or -1 (examples g_1 and g_2 above). If P is not fully-labeled (two or more vertices have the same label), then g_ℓ is not onto Q so $\deg(g_\ell) = 0$ (ex. g_3).

(2) Swapping two labels on P corresponds to a reflection. Therefore, an odd permutation of the labels inverts the sign of $\deg(g_\ell)$, while an even permutation keeps $\deg(g_\ell)$ unchanged.

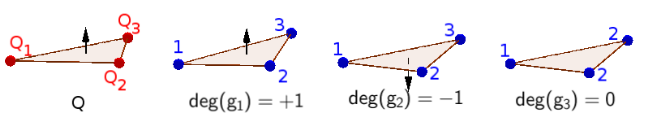
Let T be a triangulation of some simplex and $\ell : \text{VERT}(T) \to [n]$ a labeling. In each $n-1$ dimensional sub-simplex t of the triangulation T, the labeling ℓ defines an affine transformation $g_{\ell,t} : t \to Q$. The *interior degree* of ℓ is defined as the sum of the degrees of all these transformations:

$$\text{ideg}(\ell) := \sum_{t \in T} \deg(g_{\ell,t})$$

Note that each fully-labeled sub-simplex of T contributes either $+1$ or -1 to this sum and each non-fully-labeled sub-simplex contributes 0. So if $\text{ideg}(\ell) \neq 0$, there is at least 1 fully-labeled simplex.

4.3.2 ***Boundary degree***. Consider now an $n-2$-dimensional simplex in $\mathbb{R}^{n-1}$. It is contained in a hyperplane and this hyperplane divides $\mathbb{R}^{n-1}$ into two half-spaces. Define an *oriented simplex* in $\mathbb{R}^{n-1}$ as a pair of an $n-2$-dimensional simplex and one of its two half-spaces (so each such simplex has two possible orientations).

Let P', Q' be two oriented simplices in $\mathbb{R}^{n-1}$. Let g be a mapping that maps each vertex of P' to a vertex of Q', and maps the half-space attached to P' to the half-space attached to Q'. There are infinitely many ways to extend g to an affine transformation, but all of them have the same degree. Three examples are shown below; an arrow denotes the half-spaces attached to the simplex:[4]



Consider now an $n-2$-dimensional simplex P' that is a face of an $n-1$-dimensional simplex P. Since P is convex, it is entirely contained in one of the two half-spaces adjacent to P'. We orient P' by attaching to it the half-space that contains P (figuratively, we attach to P' an arrow pointing inwards, towards the interior of P).

Let Q' be a fixed $n-2$-dimensional face of Q oriented towards the interior of Q. Let $\ell : \text{VERT}(P') \to [n]$ be a labeling. If every label on P' is one of the $n-1$ labels on Q', then ℓ defines a mapping $g_\ell : P' \to Q'$ where for each vertex $v \in \text{VERT}(P')$ whose label is j, we let $g_\ell(v) = Q_j$, and the half-space attached to P' is mapped to the half-space attached to Q'. The same observations (1) and (2) above relate the labeling ℓ with the degree $\deg(g_\ell, Q')$. If some label on P' is not one of the labels on Q', then we define $\deg(g_\ell, Q') = 0$.

It is convenient to define the degree of g_ℓ w.r.t. at all n faces of Q simultaneously. We denote by $\deg(g_\ell)$ (without the extra parameter Q') the arithmetic mean of $\deg(g_\ell, Q')$ over all n faces of Q:

$$\deg(g_\ell) := \frac{1}{n} \sum_{Q' \text{ face of } Q} \deg(g_\ell, Q')$$

In this notation, if ℓ puts $n-1$ distinct labels on some face, then $\deg(g_\ell) = \pm\frac{1}{n}$ since exactly one term in the mean is ± 1 and the rest are zero. Otherwise, $\deg(g_\ell) = 0$ since all terms are zero.

Let T be a triangulation of some simplex P and let $\ell : \text{VERT}(T) \to [n]$ be a labeling of the vertices of T. Denote by ∂T the collection of $n-2$-dimensional faces of T on the boundary of P. In each such face $t' \in \partial T$, the labeling ℓ defines n affine transformation $g_{\ell,t'} : t' \to Q'$ and their average degree $\deg(g_{\ell,t'})$ can be calculated as explained above. The *boundary degree* of ℓ is defined as the sum:

$$\text{bdeg}(\ell) := \sum_{t' \in \partial T} \deg(g_{\ell,t'})$$

We now re-state the degree lemma:

LEMMA 4.11 (DEGREE LEMMA). *For every triangulation T of a simplex P and every labeling $\ell : \text{VERT}(T) \to [n]$:*

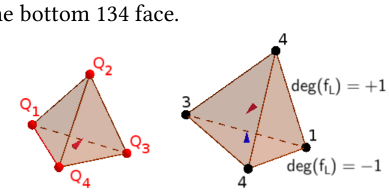
$$\text{ideg}(\ell) = \text{bdeg}(\ell)$$

PROOF. **Part 1.** We first prove the lemma for the case when the triangulation T is trivial — contains only the single $n-1$ dimensional simplex P. In this case, the sum $\text{ideg}(\ell)$ contains a single term — $\deg(g_\ell)$ — which can be either -1 or 0 or 1. The sum $\text{bdeg}(\ell)$ contains n terms — one for each face of P. We consider several cases depending on the number of distinct labels on $\text{VERT}(P)$.

If the number of distinct labels is n (i.e., P is fully-labeled), then $\text{ideg}(\ell)$ is $+1$ or -1. Each face of P is labeled with $n-1$ distinct labels so its degree is $+\frac{1}{n}$ or $-\frac{1}{n}$. The same affine mapping g_ℓ that maps P to Q, also maps each face P' to each face Q' with corresponding labels. Therefore, all terms have the same sign, and we get either $+1 = \sum \frac{+1}{n}$ or $-1 = \sum \frac{-1}{n}$, both of which are true.

If the number of distinct labels is $n-2$ or less, then P is not fully-labeled so $\text{ideg}(\ell) = 0$. No faces of P are labeled with $n-1$ distinct labels, so $\text{bdeg}(\ell) = 0$ too.

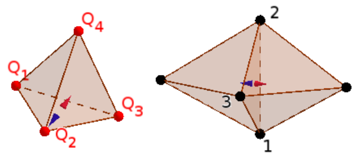
If the number of distinct labels is $n-1$, then P is not fully-labeled so $\text{ideg}(\ell) = 0$. P has exactly two faces with $n-1$ distinct labels; let's call them P'_+ and P'_-. For each $s \in \{+,-\}$ and for each face $Q' \subseteq Q$, let g'_s be the mapping from P'_s onto Q'. It can be proved that $\deg(g'_+) = -\deg(g'_-)$ (see full version). Therefore $\deg(g'_+) + \deg(g'_-) = 0$ and so $\text{bdeg}(\ell) = 0$ too. The latter case is illustrated below, where $Q' = Q_1Q_3Q_4$; the degree is $+1$ at the top 134 face and -1 at the bottom 134 face.



Part 2. We now prove the lemma for a general triangulation. For each $n-1$-dimensional sub-simplex $t \in T$, denote by ℓ_t the labeling ℓ in t, and by $\ell_{t'}$ the labeling on its $n-2$-dimensional face t'. Then:

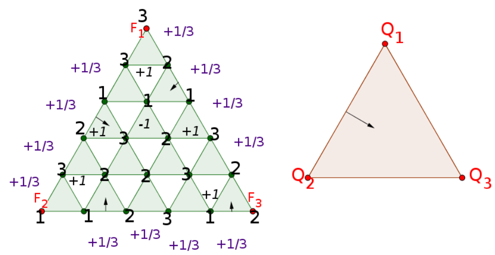
$$\text{ideg}(\ell) = \sum_{t \in T} \text{ideg}(\ell_t) \quad = \sum_{t \in T} \sum_{t' \text{ face of } t} \text{bdeg}(\ell_{t'}) \quad \text{By Part 1.}$$

The sum in the right-hand side counts all $n-2$-dimensional faces in T — both on the boundary ∂T and on the interior. Each face on the boundary is counted once since it belongs to a single sub-simplex, while each face in the interior is counted twice since it belongs to two sub-simplices. The orientations of this face in its two sub-simplices are opposite, since the interiors of these sub-simplices are in opposite directions of the face. This is illustrated below:



Therefore, the two contributions of this face to $\mathrm{bdeg}(\ell_t)$ cancel out, and the right-hand side becomes $\sum_{t' \in \partial T} \mathrm{bdeg}(\ell_{t'}) = \mathrm{bdeg}(\ell)$. □

An illustration of the Degree Lemma for $n = 3$ is shown below:



There are six fully-labeled triangles: in five of them, the transformation g_ℓ onto Q requires no reflection so its degree is $+1$. In the sixth, the transformation g_ℓ onto Q requires a single reflection so its degree is -1. Therefore: $\mathrm{ideg}(\ell) = +5 - 1 = 4$.

At the boundary of P there are four edges labeled $1, 2$. The arrow adjacent to each edge indicates its orientation. Each of these edges can be transformed onto Q' with no reflection while preserving the inwards orientation, so their degree is $+1/3$. The same is true for the four edges labeled $2, 3$ and $3, 1$. Therefore: $\mathrm{bdeg}(\ell) = 12 \cdot (1/3) = 4$.

The Degree Lemma reduces the problem of proving existence of a fully-labeled simplex, to the problem of proving that the boundary-degree is non-zero. Therefore, our next goal is to prove that every consistent labeling has a non-zero boundary degree. However, there is a technical difficulty: consistency is defined for multi-valued labelings, while the degree is only defined for single-valued labelings. For the purpose of envy-free cake-cutting, we can convert a multi-valued labeling $L : \textsc{Vert}(T) \to 2^{[n]}$ to a single-valued labeling $\ell : \textsc{Vert}(T) \to [n]$ by simply selecting, for each vertex $x \in \textsc{Vert}(T)$, a single label from the set $L(x)$. In effect, we select for the agent one of his preferred pieces in that partition; this does not harm the envy-freeness. If ℓ is created from L using such a selection, we say that ℓ is *induced by* L, and denote $\ell \sim L$.

For our purposes, it is sufficient to prove that every consistent labeling L induces at least one labeling ℓ with non-zero boundary degree. We will prove this for $n = 3$.

4.4 Consistency → Nonzero Boundary Degree

In this subsection $n = 3$, so the simplex P is the standard triangle $\Delta^{3-1} = \Delta^2$. The simplex Q is any triangle whose vertices are labeled by Q_1, Q_2, Q_3. The boundary-degree of a single-valued labeling ℓ is

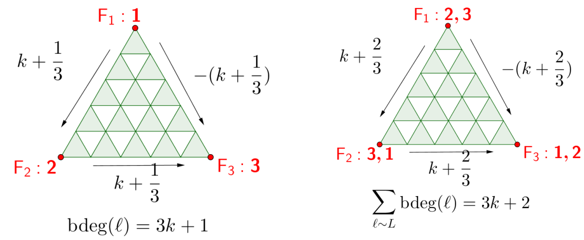


Figure 6: Boundary degrees of labelings in the positive case (left) and the negative case (right).

determined by the cyclic sequence of labels around the boundary of P (in the counterclockwise direction). For any two labels $i, j \in \{1, 2, 3\}$, denote by $\#_{ij}(\ell)$ the net number of adjacent i, j pairs in the cyclic sequence of labels going counterclockwise around the boundary of P ($i - j$ edges minus $j - i$ edges). Then:

$$\mathrm{bdeg}(\ell) = [\#_{12}(\ell) + \#_{23}(\ell) + \#_{31}(\ell)] \ / \ 3$$

Our goal now is to prove that $\mathrm{bdeg}(\ell)$ is non-zero.

Lemma 4.12. *Let $L : \textsc{Vert}(T) \to 2^{[3]}$ be a consistent labeling of a friendly triangulation of Δ^{3-1}. Then, L induces a single-valued labeling $\ell : \textsc{Vert}(T) \to [3]$ with:*

$$\mathrm{bdeg}(\ell) \not\equiv 0 \mod 3$$

Proof. First, we simplify L by removing multiple labels while keeping L consistent. This can be done arbitrarily for any interior vertex, since these vertices are not bound by consistency. For any boundary vertex x that is not a main vertex, we remove labels consistently. For example, if a label $x_1 \in F_{-1}$ is originally labeled by $\{2, 3\}$ and we remove the 2, then by consistency its friend $x_3 \in F_{-3}$ is originally labeled by $\{1, 2\}$ and we remove the 1.

For the main vertices, Lemma 4.5 implies that there are exactly two cases regarding the labels on the main vertices.

Positive case (Figure 6/Left): For each main vertex F_j, $j \in L(F_j)$ (this corresponds to the owner of the main vertices valuing the entire cake as weakly-positive). We remove all other labels from F_j. The labeling remains consistent and it is now single-valued.

We calculate the boundary degree on each face in turn. We denote by $\ell[F_{12}]$ the labeling ℓ restricted to the face F_{12}. This sequence of labels starts with 1 (the left piece) and ends with 2 (the right piece), so $\mathrm{bdeg}(\ell[F_{12}]) = k + 1/3$ for some integer k. The sequence $\ell[F_{23}]$ also starts with the left piece (2) and ends with the right piece (3); by consistency, it is exactly the same sequence as in F_{12}, up to an even permutation. Therefore its degree is the same: $k + 1/3$. The sequence $\ell[F_{13}]$ also starts with the left piece (1) and ends with the right piece (3); by consistency, it is exactly the same sequence as in F_{12}, up to an odd permutation (Lemma 4.4). Therefore $\mathrm{bdeg}(\ell[F_{13}]) = -(k + 1/3)$. When traveling around the boundary in the counter-clockwise direction, the face F_{13} is traveled backwards. Therefore, $\mathrm{bdeg}(\ell) = \mathrm{bdeg}(\ell[F_{12}]) + \mathrm{bdeg}(\ell[F_{23}]) - \mathrm{bdeg}(\ell[F_{13}]) = 3(k + 1/3) = 3k + 1$.

Negative case (Figure 6/Right): For each main vertex F_j, $L(F_j) = [n] \setminus \{j\}$ (this corresponds to the owner of the main vertices valuing the entire cake as strictly negative). Here, by Lemma 4.5, there is no way to remove labels while keeping L consistent. So L induces

$2^3 = 8$ single-valued labelings. We will prove that the *sum* of boundary-degrees of these eight labelings is nonzero modulo 3.

We calculate this sum for each face separately. First, consider the face F_{12} and focus on the labels on the vertices F_1, F_2. If these labels are 2, 1 then the degree is $k_1 - 1/3$ for some integer k_1; if they are 2, 3 then it is $k_2 + 1/3$; if they are 3, 1 then it is $k_3 + 1/3$; if they are 3, 3 then it is k_4. We add all these numbers, then multiply by two for the two possible labels on F_3. We get $k + 2/3$ for some integer k.

Next, consider the face F_{23}. By consistency, for every possible selection of labels on F_1, F_2, the images of these labels under the relevant permutation are a possible selection of labels on F_2, F_3 (even if the label on F_2 is not the same). Therefore, the sum of degrees for all possible selections on F_{23} is the same as on F_{12}: it is $k + 2/3$. Similarly, the sum on the face F_{13} is $-(k + 2/3)$ but it is traveled backwards. Therefore, the sum of the boundary-degree over all 2^3 selections is $3(k + 2/3) = 3k + 2$.

In all cases, $\sum_{l \sim L} \text{bdeg}(\ell) \not\equiv 0 \mod 3$. Hence, there is at least one $\ell \sim L$ with $\text{bdeg}(\ell) \not\equiv 0 \mod 3$, as claimed. □

4.5 Tying the knots

The final theorem of this section ties the knots.

Theorem 4.13. *For $n = 3$ selective agents, there always exists a connected envy-free division of a mixed cake.*

Proof. Let T be a barycentric triangulation of the partition-simplex Δ^{n-1}. Let W be a friendly and diverse ownership-assignment on T, which exists by §4.2. Ask each agent to label the vertices he owns by the indices of his preferred pieces.

All n labelings $L_1, \ldots, L_n$ are consistent (§4.1). Since W is friendly, L^W is consistent too (§4.2). Therefore, there exists a single-valued labeling, $\ell^W \sim L^W$, having $\text{bdeg}(\ell^W) \neq 0$ (§4.4). By the Degree Lemma (§4.3), $\text{ideg}(\ell^W) \neq 0$ too. Therefore ℓ^W has at least one fully-labeled simplex. Since W is diverse, a fully-labeled simplex of ℓ^W is an envy-free simplex.

All of the above can be done for finer and finer barycentric triangulations. This yields an infinite sequence of envy-free simplices. This sequence has a convergent subsequence. By continuity of preferences, the limit of this subsequence is an envy-free division. □

We could not extend Lemma 4.12 to $n > 3$; it is left as a conjecture.

Conjecture 4.14. *Let $L : \text{Vert}(T) \to 2^{[n]}$ be a consistent labeling of a friendly triangulation of Δ^{n-1}. Then, L induces a single-valued labeling $\ell : \text{Vert}(T) \to [n]$ with:*

$$\text{bdeg}(\ell) \not\equiv 0 \mod n$$

If this conjecture is true, then Theorem 4.13 is true for any n.

5 FINDING AN ENVY-FREE DIVISION

Stromquist [21] proved that connected envy-free allocations cannot be found in a finite number of queries even when all valuations are positive, so the best we can hope for is an approximation algorithm.

The following simple binary-search algorithm can be used to find a fully-labeled sub-simplex in a labeled triangulation. It is adapted from Deng et al. [6]:

(1) If the triangulation is trivial (contains one sub-simplex), stop.
(2) Divide the simplex into two halves, respecting the triangulation lines. Calculate the boundary degree in each half.
(3) Select one half in which the boundary degree is non-zero; perform the search recursively in this half.

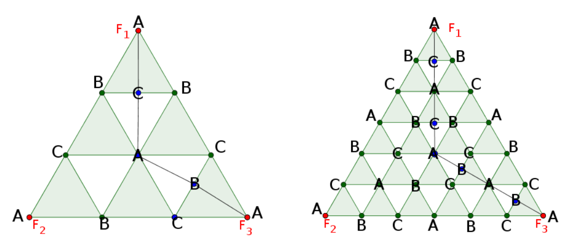
While Deng et al. [6] present this algorithm for the positive case, it works whenever the boundary degree of the original simplex is non-zero. Then, in step 3, the boundary degree of at least one of the two halves is non-zero, so the algorithm goes on until it terminates with a fully-labeled simplex. This is the case when there are $n = 3$ agents with arbitrary mixed valuations (Lemma 4.12). If Conjecture 4.14 is true, then this is also the case for any n.

To calculate the runtime of the binary search algorithm, suppose the triangulation is such that each side of the original simplex is divided into D intervals. Then, the runtime complexity of finding a fully-labeled simplex is $O(D^{n-2})$ [7].

To calculate the complexity of finding a δ-approximate envy-free allocation, we have to relate D to δ. In each barycentric subdivision, the diameter of the subsimplices is at most $n/(n + 1)$ the diameter of the original simplex [12]. Hence, to get a barycentric triangulation in which the diameter of each sub-simplex is at most δ, it is sufficient to perform $k \approx n \ln(1/\delta)$ steps of barycentric subdivision. In each step, the number of intervals in each side is doubled, so $D \in \Theta(2^k) = \Theta(1/\delta^n)$. So the total runtime complexity of finding a δ-approximate envy-free allocation using the barycentric triangulation is $O(1/\delta^{n(n-2)})$.

Deng et al. [6] note the slow convergence of the barycentric triangulation, and propose to use the *Kuhn triangulation* instead. This triangulation looks similar to the equilateral triangulations shown in Figure 3. In this triangulation, $D = 1/\delta$ so the runtime complexity of the binary search is $O(1/\delta^{n-2})$. They prove that this is the best possible for selective agents. However, their triangulation does not support a diverse and friendly ownership-assignment.

For $n = 3$, we found a variant of the equilateral triangulation that does support a diverse and friendly ownership-assignment. The first two steps of this triangulation are illustrated below:



So for $n = 3$, a δ-envy-free division can be found in time $O(1/\delta)$. Generalizing this “trick” to $n > 3$ is left for future work.

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