A Closed-Form Characterization of Buyer Signaling Schemes in Monopoly Pricing*

Weiran Shen
IIIS, Tsinghua University
Beijing, China
emersonswr@gmail.com

Pingzhong Tang
IIIS, Tsinghua University
Beijing, China
kenshinping@gmail.com

Yulong Zeng
IIIS, Tsinghua University
Beijing, China
freeof123@qq.com

ABSTRACT

We consider a setting where a revenue maximizing monopolist sells a single item to a buyer. A mediator first collects the buyer's value and can reveal extra information about the buyer's value by sending signals. Mathematically, a signal scheme can be thought of as a decomposition of the prior value distribution into a linear combination of posterior value distributions, and based on each of them, the monopolist separately posts a price. According to the theory of Bayesian persuasion, a well-designed signal scheme can lead to utility improvements for both the monopolist and the buyer.

We put forward a novel technique to analyze the effects of signal schemes of the mediator. Using this technique, we are able to construct explicitly a closed-form solution, and thus characterize the set of seller-buyer utility pairs achievable by any signal scheme, for any prior type distribution. Our result generalizes a well-known result by Bergemann et. al., who derive a characterization for the same problem but only restricted to the discrete distribution case.

Similar to the result derived by Bergermann et. al., we show that the set of seller and buyer utility pairs achievable form a triangle: any point within the triangle can be achieved by an explicitly constructed signal scheme and any point outside the triangle cannot be achievable by any such scheme. Our result is obtained by establishing the endpoints of the triangle: one corresponds to the point where the buyer obtains the highest utility among all schemes, another corresponds to the point where the buyer obtains zero utility and the seller has the lowest possible revenue, and the third corresponds to the point where the buyer has zero utility while the seller extracts full social surplus. We then prove that the triangle described fully characterizes all possible signal schemes.

KEYWORDS

signal; closed-form characterization; monopoly pricing;

ACM Reference Format:
Weiran Shen, Pingzhong Tang, and Yulong Zeng. 2018. A Closed-Form Characterization of Buyer Signaling Schemes in Monopoly Pricing. In *Proc. of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018), Stockholm, Sweden, July 10–15, 2018,* IFAAMAS, 9 pages.

1 INTRODUCTION

We study a setting where a seller sells an item to a buyer to maximize revenue. The buyer's value for the item is drawn from a commonly known distribution. It is known that the optimal strategy for the seller is to set a posted price for the item [16].

Now suppose there is a trustworthy mediator between the seller and the buyer. The mediator collects the buyer's value and can reveal extra information to the seller about the buyer's value by sending signals. The seller can then post a price conditional on the revealed signal. Such mediators exist in many real-world markets. For example, in the online advertising industry, there are bidding agencies that bid on behalf of the advertisers in ad-auction platforms. Typically, these agencies create multiple accounts and use different accounts to send different signals.

Equivalently, one can view the Bayesian buyer as a population of buyers with publicly known valuations. Such signaling strategies are often interpreted, from the seller's point of view, as market segmentation strategies that partition the population of buyers into several markets based on their external characteristics (say geographic information such as gender and age) and the seller then posts market-specific prices as a means of price discrimination.

Consider a example where the buyer's value is uniformly distributed on $[0, 1]$. It is clear that the seller would set a posted price at 0.5, which yields an expected revenue of 0.25 for the seller and an (*ex ante*) expected utility of 0.25 for the buyer. Now define the signal set to be $\{high, low\}$, and the buyer is said to have *low* value if his value is in $[0, 0.5]$ and *high* otherwise. The mediator sends a signal to the seller after collecting the buyer's value: he sends signal *low* if the buyer has a *low* value and *high* otherwise. The seller has equal probability of observing each signal, and is now able to set the price conditional on the signal: to maximize revenue, he will set price at 0.25 when she sees *low* and 0.5 when *high*.

As a result, such extra information benefits both parties, since intuitively the extra information revealed to the seller can increase the probability of sale. In the above example, the seller increases the revenue by $1/16$ (25% increase) and the buyer increases the utility by $1/32$ (12.5% increase). Therefore it is of great importance for the mediator to understand what can be achieved by sending signals and how to design such signals, and the goal of this paper is to investigate the effects of different signaling schemes when facing a revenue maximizing monopolist [16].

1.1 A brief review of related works

The example described so far is at the intersection of two streams of important research. The first investigates the power and limit of price discrimination [3, 18], where the buyer is interpreted as a population, within which each individual has a deterministic type. The seller can segment the population into different markets (subsets of the population) and price differently (aka. *the third degree of price discrimination*). The impact of different segmentation strategies has been investigated and the set of (seller, buyer) utility profiles have been characterized under various scenarios. We refer readers to [3] for a comprehensive survey on price discrimination.

*This paper is supported in part by the National Natural Science Foundation of China Grant 61561146398, a China Youth 1000-talent Program and an Alibaba Innovative Research Program.

Proc. of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018), M. Dastani, G. Sukthankar, E. André, S. Koenig (eds.), July 10–15, 2018, Stockholm, Sweden.

The second strand of research concerns the power of signaling in the so-called persuasion model, studied in a series of economics papers [2, 15, 17, 19], where they study the general problem of a sender strategically revealing information based on external signals and give a method to find the optimal signaling scheme for the sender in a number of realistic scenarios. The basic model has been extended to a number of scenarios in the past five years: [9] considers the situation where sender's payoff also depends on the signal cost. [11] studies the simultaneous-move game where multiple senders simultaneously send signals. [10] proposes new approaches to the Bayesian persuasion problem. [7] studies the hardness of designing optimal information structures in zero-sum game, while [22] obtains hardness results of designing signal structures in Stackelberg Games. [5] consider the problem of designing the optimal information structure when the designer has control over the information environment. [8] gives a $(1 - 1/e)$-approximation of the optimal private signaling scheme and proves the NP-hardness to constantly approximate the optimal public scheme. Our problem, described in their terminology, is to characterize the buyer signaling schemes in the monopolist pricing problem.

The concept of signaling has also been studied in the auction scenario by Daskalakis et al. [6] and Bro Miltersen and Sheffet [4]. Both works consider the case where the seller has additional information than the buyer and how the seller can strategically reveal this additional information (together with designing the auction format itself in [6]) to maximize revenue. In contrast, in our model, both parties share the same information and the buyer is the one that designs the signal.

1.2 Our contributions

To best describe our contributions, let us start by reviewing a related work of Bergemann et al. [3], which aims to understand the impact of signaling (or in their terminology, market segmentation), in the same setting as ours. They characterize, for any discrete distribution, the set of (seller, buyer) utility profiles achievable by some buyer signal scheme. It is not hard to see that, for any signal scheme, the utility profile must necessarily satisfy the following three bounds: 1) the buyer's utility must be nonnegative, following from individual rationality; 2) the seller's utility must be no less than the case where he does not receive any signal at all; and 3) the sum of both parties' utilities must be no higher than the value of the item. The main effort and result of the paper is to show that these three bounds are actually sufficient, in that they fully characterize all possible profiles achievable by any signaling scheme.

To establish the main result, one of the main difficulties is to establish the utility profile yielded by the buyer optimal signaling, i.e, the point where the item is sold efficiently while the seller is held to the lowest revenue, the same as in the case of no signaling at all. The authors use an iterative decomposition method to show that such utility profile (and in fact all the utility profiles where the seller has the lowest revenue) can be achieved by a convex combination of equal-revenue distributions (segmentations). They also use a limit argument to show that, such iterative method can be extended to the continuous case and there exists a decomposition that achieves the buyer optimal point in the continuous case as well. However, to the best of our knowledge, there is no explicit closed-form decomposition for the continuous case.

Such closed-form solutions cannot be derived by directly applying the characterization of [3], since their iterative construction steps depend on previous ones. In this paper, we not only obtain the closed-form of buyer optimal signaling for any continuous type distribution, but also introduce original techniques that can provide more insight of the problem and may be helpful for further studies on this topic. We mainly focus on the decomposition of two extreme points. Both of the points correspond to the lowest seller revenue, but one has the highest buyer utility and the other has the lowest (which is 0). We introduce a tool called "revenue function" and transform the original problem to the decomposition of such revenue functions. We are able to solve the problem under some technical conditions with such a tool. To solve the more challenging case without those conditions, we incorporate the "ironing" trick to this setting and define different "ironing" methods. Furthermore, we introduce a powerful "scaling" technique that produces the desired decomposition by modifying another known decomposition. By aggregating all these tools and techniques, we are able to solve the general case with arbitrary value distributions.

For the point with the highest buyer utility, the value distribution for each signal is a piecewise function consisting of two parts: one is an equal-revenue function and the other is a "scaled function" of the prior distribution.[1] For the point with the lowest buyer utility, the value distribution for each signal consists of three parts: the first part is simply the same as the prior distribution, the second part is an equal-revenue distribution and the third part is a point mass. Our construction uses the equal-revenue distribution as an extreme distribution. Similar constructions can also be seen in the recent literature[12, 13, 21].

Given the closed-form decompositions of the two extreme points, it is straight forward to construct the closed-form decomposition for any point inside the triangle area by [3], by taking the convex combination of the decompositions of the extreme points.

2 SETTING

Suppose the seller has a single item to sell to a buyer. The buyer's value for the item is drawn from a distribution F, which is called the *prior distribution*, with the density function f. Define the closed support (called support hereafter for simplicity) of the density function f to be the closure of the subset of $\mathbb{R}$ where f is non-zero: $\mathbf{supp}(f) = \mathbf{cl}(\{v | f(v) > 0\})$. We use $\underline{v}$ and $\bar{v}$ to denote the smallest and largest value in $\mathbf{supp}(f)$ respectively: $\underline{v} = \min_x \mathbf{supp}(f)$, $\bar{v} = \max_x \mathbf{supp}(f)$.

A signal scheme $\Omega = (S, \pi)$ consists of a set of signals S, and a function that maps the buyer's value to a distribution over signals $\pi : \mathbf{supp}(f) \mapsto \Delta(S)$, where $\Delta(S)$ denotes the set of all probability distributions over S. After collecting the buyer's value, the mediator chooses a signal from the S according to the distribution $\pi(v)$ [15].

Upon receiving a signal t, the seller updates the prior belief $F(v)$ and gets a posterior belief $F(v|t)$. Using the Bayes rule, one can easily verify that, designing a signal scheme is equivalent to designing posterior value distributions [3, 6], such that:

$$\int_S F(v|t)\,\mathrm{d}\mu = F(v),\ \forall v, \tag{1}$$

where the left side is a Lebesgue integral with respect to the probability measure μ with respect to signal t. Thus we can use an alternative notation of signal schemes in terms of $F(v|t)$ and $\mu(t)$.

[1] It is notable that [3] gives another kind of segmentation for discrete case called "direct segmentations", such that each segment consists of some amount of a "direct value" and a "scaled function" of the prior distribution truncated from the "direct value". So our closed-form decomposition can be regarded a combination of the two segmentations provided in [3].

We use $f(v|t)$ to denote the density function of $F(v|t)$. Similarly, we also define $\underline{v}_t$ and $\bar{v}_t$ to be the smallest and the largest value in the support of $f(v|t)$ respectively. We sometimes abuse notation and denote the support of $f(v|t)$ by $\mathbf{supp}(t)$. [2]

2.1 Signal Representation

Given any signal t, if the seller posts a price r, the expected seller's revenue is $r(1 - F(r|t))$. We assume that the seller always set the monopoly reserve price r_t based on the posterior distribution, i.e. $r_t = \arg\max_r r(1 - F(r|t))$ to maximize his expected revenue.

We use a mapping $\eta : S \mapsto \mathbb{R}$ that maps a signal to its monopoly reserve price of $F(v|t)$ to represent a signal t:

Definition 2.1 (Signal Representation). For each signal $t \in S$, let the monopoly reserve price to be $r_t = \arg\max_r r(1-F(r|t))$. Define the function η to be $\eta(t) = r_t$.

If two posterior distributions have the same reserve price r, we can always combine the two distributions to one distribution and use that r to represent the signal. One can easily verify that the monopoly reserve price for the combined distribution is still r and all the outcomes remain the same. For the case that there are multiple reserve prices that maximize the seller's revenue, we can arbitrary choose one and use that price to represent the signal.

With the above representation, we can define $P(t)$ to be the cumulative distribution function on the signal space: $P(t) = \mu(T \leq t)$, where T is a random variable drawn from S according to μ.

Throughout this paper, we focus on the case where both the prior and posterior density functions are differentiable. Since we consider continuous distributions, we also assume that the cumulative distribution function $P(t)$ is differentiable, with density function $p(t)$. However, our main results and constructions also apply to much more general cases, even when the prior and posterior distribution contains point masses[3].

To sum up, we use $\Omega = (S, F(v|t), p(t))$ to represent a decomposition (signal scheme), where

- $S \in \mathbb{R}$ is the signal space;
- $F(v|t)$ is the posterior distribution given signal $t \in S$;
- $p(t)$ is the probability (density) of signal $t \in S$;

subject to Equation (1) and $t \in \arg\max_r r(1 - F(r|t))$.

3 REVENUE FUNCTION

In this section, we develop a tool that will be used throughout the paper. And our main techniques to construct signaling schemes relies crucially on this tool.

Definition 3.1 (Revenue Function). For any cumulative distribution $F(v)$, define the corresponding revenue function[4] to be:

$$R(v) = v(1 - F(v)),$$

which is the seller revenue when setting the reserve price v.

[2]Note that it is not necessarily true that $\mathbf{supp}(t) = [\underline{v}_t, \bar{v}_t]$ since the support may not be a single interval. If a certain signal with probability measure 0 is added to or removed from a signal scheme, both the buyer utility and the seller revenue do not change. Thus for ease of presentation, we ignore such cases and use some notations in a probabilistic sense (for example, "$\forall t \in S$" means "for almost all signals t in S").

[3]In this case, the density functions are called *general functions* or simply *distributions*, and the derivatives used in later analysis become *distributional derivatives*. We will not discuss this in detail, but refer interested readers to [20, Chapter 6].

[4]The revenue function is well-defined even if $v \notin \mathbf{supp}(f)$. Therefore, the domain of $R(v)$ is $\mathbb{R}$. However, we only need to consider $R(v)$ in $\mathbb{R}_+$ in this paper, since the value v is always non-negative. When v is not in the support, we have either $F(v) = 0$ or $F(v) = 1$. And $R(v) = v$ if $F(v) = 0$ and $R(v) = 0$ if $F(v) = 1$.

Remark 1. *Note that $R(v)$ is not the same as the* revenue curve *well known in the literature [1, 14], since revenue curve is normally represented in* quantile $q = 1 - F(v)$.

Note that a revenue function must satisfy certain conditions. Clearly, $1 - \frac{R(v)}{v}$ needs to be a cumulative distribution function. We call such functions *feasible* revenue functions.

Definition 3.2 (Feasible Revenue Function). A function $R(v)$ is a feasible revenue function, if:

- $\lim_{v\to 0} \frac{R(v)}{v} = 1$;
- $\lim_{v\to\infty} \frac{R(v)}{v} = 0$;
- $\frac{R(v)}{v}$ is a decreasing function.

The revenue functions with respect to the prior and posterior distributions are called the prior and posterior revenue functions, respectively. The following lemma shows that decomposing the prior value distribution is equivalent to decomposing the prior revenue function.

Lemma 3.3. *Let $F(v)$ and $F(v|t)$ be the prior and posterior value distribution. Let $p(t)$ be the density function for signal t. Define $R(v)$ and $R(v|t)$ to be the prior revenue function and the posterior revenue function, respectively. Then*

$$F(v) = \int_{t\in S} F(v|t)p(t)\,\mathrm{d}t, \ \forall v \neq 0, \tag{2}$$

if and only if

$$R(v) = \int_{t\in S} R(v|t)p(t)\,\mathrm{d}t, \ \forall v \neq 0. \tag{3}$$

Proof.

$$\begin{aligned} & & F(v) &= \int_{t\in S} F(v|t)p(t)\,\mathrm{d}t \\ &\iff & 1 - F(v) &= \int_{t\in S} p(t)\,\mathrm{d}t - \int_{t\in S} F(v|t)p(t)\,\mathrm{d}t \\ & & &= \int_{t\in S} (1 - F(v|t))p(t)\,\mathrm{d}t \\ &\iff & v(1 - F(v)) &= \int_{t\in S} v(1 - F(v|t))p(t)\,\mathrm{d}t \\ &\iff & R(v) &= \int_{t\in S} R(v|t)p(t)\,\mathrm{d}t. \end{aligned}$$

□

Therefore, for $v \neq 0$, we can construct $R(v)$ instead of $F(v)$. As for $v = 0$, we always have $R(0) = 0$ and $F(0)$ cannot be obtained from $R(0)$. However, if we already have $F(v)$ for $v \neq 0$, we can derive $F(0)$ by $F(0) = \lim_{v\to 0} F(v)$. Again, the posterior revenue functions must be feasible. And we call such signaling schemes *feasible decompositions* or *feasible signaling schemes*.

4 CLOSED FORM SOLUTIONS

Before we consider any signaling scheme, let's first examine the possible revenue and utility pairs of the signaling problem. Let $\mathrm{REV}(S)$ and $\mathrm{UTL}(S)$ be the seller revenue and the buyer utility of signaling scheme S. Define REV^* to be the revenue of the prior distribution when Myerson auction is applied. The following result is already given by [3]. However, we provide an alternative proof, which gives us with more insights and is helpful for later analysis.

Theorem 4.1 (Bergemann et al. [3]). *Let $\mathbf{E}[v]$ be the expected value of the buyer. A pair of seller revenue and buyer utility is attainable by a signaling scheme if and only if it is inside the triangle in Figure 1, where point A corresponds to seller revenue $\mathbf{E}[v]$ and buyer*

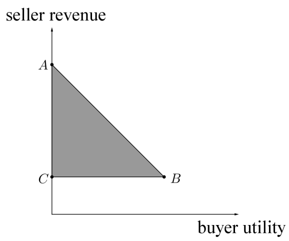

Figure 1: Range of seller revenue and buyer utility.

utility 0, point B seller revenue REV^* and buyer utility $\mathbf{E}[v] - \mathrm{REV}^*$ and point C seller revenue REV^* and buyer utility 0.

Here, we only prove the "only if" direction, as the "if" direction will be immediate after we present our construction.

Proof. For any signaling scheme S, the seller revenue and buyer utility must satisfy the following conditions:

(1) $\mathrm{UTL}(S) \geq 0$;
(2) $\mathrm{REV}(S) + \mathrm{UTL}(S) \leq \mathbf{E}[v]$;
(3) $\mathrm{REV}(S) \geq \mathrm{REV}^*$.

The intersection area of the above 3 conditions are exactly the triangle in Figure 1. The first condition is equivalent to individual rationality and the second condition comes directly from the definition of seller revenue and buyer utility. Now it suffices to show that the third condition must hold. Let r^* be the monopoly reserve of the prior distribution. Consider the mechanism $\mathcal{M}$ that ignores all signals in T and always uses r^* as the reserve price. Let $\mathrm{REV}(\mathcal{M})$ be its revenue. Clearly, $\mathrm{REV}^* = \mathrm{REV}(\mathcal{M})$. For each $t \in S$, setting t as the reserve price can extract at least the same revenue as setting r^*, since t is the optimal reserve for this signal. Integrating over t yields $\mathrm{REV}(S) \geq \mathrm{REV}(\mathcal{M})$. It follows that $\mathrm{REV}(S) \geq \mathrm{REV}^*$. □

Theorem 4.1 states that the pair of seller revenue and buyer utility is always inside the triangle. In fact, all points inside the triangle can be obtained by some signaling scheme, and this result is also confirmed by [3]. However they only give a characterization of such signaling schemes, while we aim to construct explicitly the closed form of the signals.

4.1 Signaling scheme for Point A

According to Theorem 4.1, point A has seller revenue $\mathbf{E}[v]$, which is already the maximum possible revenue. This indicates that the item is always sold, and the price $t = \underline{v}_t, \forall t \in S$. Furthermore, the buyer utility for point A is always 0, which implies that the price $t = \bar{v}_t, \forall t \in S$. It follows that the support of each signal contains only a single value t, with probability 1.

4.2 Signaling Scheme for Point B

According to Theorem 4.1, point B satisfies: $\mathrm{REV}(S)+\mathrm{UTL}(S) = \mathbf{E}[v]$ and $\mathrm{REV}(S) = \mathrm{REV}^*$. For now, let's first focus on the case where the prior revenue function $R(v)$ is concave in the interval $[\underline{v}, r^*]$, where r^* is the monopoly reserve of the prior distribution. As our main result for this section, we give the following theorem:

Theorem 4.2. *Let $F(v)$ be the prior value distribution, and $R(v) = v(1 - F(v))$ be the corresponding revenue function. Let r^* be the monopoly reserve for $F(v)$ ($r^* \in \arg\max_r R(v)$). If $R(v)$ is concave in the interval $[\underline{v}, r^*]$, then the signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point B, where $S = [\underline{v}, r^*]$, $p(t) = -R''(t)$, and*

$$F(v|t) = \begin{cases} 0 & v \leq t \\ 1 - \frac{t}{v} & t < v \leq r^* \\ 1 - \frac{R(v)}{R(r^*)}\frac{t}{v} & r^* < v \leq \bar{v} \end{cases}$$

One can easily verify that the above signaling scheme satisfies the two conditions that defines point B. We will not prove the above theorem directly. Instead, we provide a three-step construction that can give us more insight about the structure of the problem.

According to Lemma 3.3, we can design $R(v|t)$ instead of $F(v|t)$. In the first step we show that $S = [\underline{v}, r^*]$ is sufficient to represent all signals (Lemma 4.4), i.e., no signal with $t \in \mathbb{R} \setminus S$ is needed. In the second step, for each signal t, we construct the part in $[\underline{v}_t, r^*]$. Then according to the $R(v)$ and the first part of $R(v|t)$, we compute the density function $p(t)$ (Lemma 4.5). In the third step, with the "scaling technique" (Lemma 4.6), we design the part in $(r^*, \bar{v}_t]$, and finally get the complete construction of point B (Theorem 4.2).

Lemma 4.3. *Signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point B, if and only if the following three conditions are satisfied:*

- *$t = \underline{v}_t$;*
- *$\{t, r^*\} \subset \arg\max_v R(v|t), \forall t \in S$;*
- *$p(t) \geq 0$, $\int_{t\in S} p(t)\,\mathrm{d}t = 1$ and $F(v) = \int_{t\in S} F(v|t)p(t)\,\mathrm{d}t,\ \forall v$.*

Proof. We first prove the necessity of the conditions. Recall that point B must satisfy two conditions:

$$\mathrm{REV}(S) + \mathrm{UTL}(S) = \mathbf{E}[v] \quad \text{and} \quad \mathrm{REV}(S) = \mathrm{REV}^*.$$

The first equation requires the seller to always sell the item. Therefore we have $t = \underline{v}_t, \forall t \in S$. And according to the proof of Theorem 4.1, the second equation requires that, for each signal t, setting reserve r^* extracts the same amount of revenue as setting t as the reserve price, i.e., $R(t|t) = R(r^*|t)$. And since we use the monopoly reserve price to represent the signal, we have that t maximizes $R(v|t)$. It follows that $\{t, r^*\} \subset \arg\max_v R(v|t)$. The third condition is natural since $P(t)$ is a distribution function.

Now we prove the sufficiency of the conditions. Suppose that the three conditions are satisfied. Then according to the second condition, we can choose t as the reserve price for signal t for the seller, since t maximizes the revenue function $R(v|t)$. Thus we can indeed use t to represent the signal t. Furthermore, the second condition implies $R(t|t) = R(r^*|t)$. Therefore we have $\mathrm{REV}(S) = \mathrm{REV}^*$. The first condition indicates that the seller always sells the item. Then we have $\mathrm{REV}(S) + \mathrm{UTL}(S) = \mathbf{E}[v]$. The third condition shows that S is indeed a signaling scheme. □

Lemma 4.4. *The signal space $S = [\underline{v}, r^*]$ is sufficient to represent all signals for point B.*

Proof. We show that there is no signal with $t < \underline{v}$ or $t > r^*$. It is clear that $\mathbf{supp}(t) \subseteq [\underline{v}, \bar{v}], \forall t$. Thus the monopoly reserve t for each signal cannot be smaller than $\underline{v}$.

On the one hand, according to Lemma 4.3, we have $R(t|t) = R(r^*|t)$ and $t = \underline{v}_t$. On the other hand, $R(\underline{v}_t|t) = \underline{v}_t(1 - F(\underline{v}_t|t)) = \underline{v}_t$. Combining the above arguments, we get $R(r^*|t) = R(t|t) = R(\underline{v}_t|t) = \underline{v}_t$. This equation implies that there is no signal with

$t > r^*$, since otherwise, we will have $R(t|t) = \underline{v}_t = t > r^* \geq r^*(1 - F(r^*|t)) = R(r^*|t)$, contradicting to the above equation. □

As the second step, we construct $R(v|t)$ in the interval $[\underline{v}_t, r^*]$, and compute the density function $p(t)$. Note that when $v \leq \underline{v}_t$, $R(v|t)$ is already defined, since we have $F(v|t) = 0$ and $R(v|t) = v$. In particular, $R(\underline{v}_t|t) = \underline{v}_t$. According to Lemma 4.3, we know that $R(r^*|t) = R(t|t) = R(\underline{v}_t|t) = \underline{v}_t = t$, and that both t and r^* maximize $R(v|t)$. A natural and simple choice is to let the function $R(v|t)$ be a constant t in the interval $[t, r^*]$ (see Figure 2).

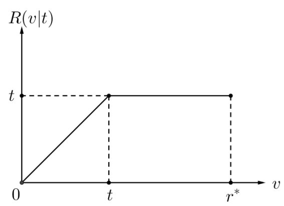

Figure 2: Revenue function for signal t

Now we compute the density function $p(t)$. We need to guarantee that $p(t) \geq 0$. The following lemma states that this condition can be satisfied if $R(v)$ is concave in the interval $[0, r^*]$.

Lemma 4.5. *Given that $R(v)$ is concave in $[\underline{v}, r^*]$, we have $p(t) = -R''(t), \forall t \in S = [\underline{v}, t^*]$ if $R(v|t)$ has the following form in the interval $[\underline{v}_t, r^*]$:*

$$R(v|t) = \begin{cases} v & v \leq t \\ t & t < v \leq r^* \end{cases}$$

Proof. By Lemma 3.3, $p(t)$ satisfies $R(v) = \int_{t \in S} R(v|t)p(t)\, dt$. Replacing $R(v|t)$ yields:

$$R(v) = \int_{\underline{v}}^{v} tp(t)\, dt + \int_{v}^{r^*} vp(t)\, dt.$$

Taking derivative on both sides with respect to v, we get:

$$R'(v) = vp(v) + \int_{v}^{r^*} p(t)\, dt - vp(v) = \int_{v}^{r^*} p(t)\, dt.$$

Taking derivative again, we have $R''(v) = -p(v)$. Therefore, $p(t) = -R''(v)$, and $p(t) \geq 0$ since $R(v)$ is concave in $[\underline{v}, r^*]$. □

Remark 2. *The function $p(t) = -R''(t)$ given by Lemma 4.5 is indeed a density function. Note that $R(v) = v$ when $v \leq \underline{v}$, and $R'(\underline{v}) = 1$. Since r^* maximizes $R(v)$, we have $R'(r^*) = 0$. Therefore,*

$$\int_{\underline{v}}^{r^*} p(t)\, dt = \int_{\underline{v}}^{r^*} -R''(t)\, dt = R'(\underline{v}) - R'(r^*) = 1.$$

In order to construct the rest part of $R(v|t)$, we now introduce a powerful tool called the "scaling technique", which will be intensively used in later analysis. Formally,

Lemma 4.6 (Scaling Technique). *Consider feasible revenue functions $R_1(v)$ and $R_2(v)$. $R_1(v)$ has a feasible signaling scheme $\Omega_1 = (S_1, F_1(v|t), p_1(t))$ with corresponding posterior revenue function $R_1(v|t)$. Suppose $R_2(v) \leq R_1(v)$ and there exists an open interval X (may be unbounded), such that:*

- *$R_2(v) < R_1(v), \forall v \in X$ and $R_2(v) = R_1(v), \forall v \notin X$;*
- *$X \subset \mathbf{supp}(t), \forall t \in S_1$;*
- *$R_1(v|t) = g(t), \forall v \in X, \forall t \in S_1$, i.e., given t, $R_1(v|t)$ is constant in the interval X.*

Then $\Omega_2 = (S_2, F_2(v|t), p_2(t))$ is feasible[5] *for $R_2(v)$, where $S_2 = S_1$, $p_2(t) = p_1(t)$ and*

$$R_2(v|t) = \begin{cases} R_1(v|t) & v \notin X \\ \frac{R_2(v)}{R_1(v)} R_1(v|t) & v \in X \end{cases}$$

Proof. We show that $R_2(v|t)$ is a decomposition. When $v \notin X$,

$$\int_{t \in S_2} R_2(v|t)p_2(t)\, dt = \int_{t \in S_1} R_1(v|t)p_1(t)\, dt = R_1(v) = R_2(v).$$

When $v \in X$,

$$\int_{t \in S_2} R_2(v|t)p_2(t)\, dt = \int_{t \in S_1} \frac{R_2(v)}{R_1(v)} R_1(v|t)p_1(t)\, dt = R_2(v).$$

Next we show that the decomposition is feasible. Clearly, $0 \notin X$ since X is an open interval. Therefore, we have $R_2(v|t) = R_1(v|t)$ in the neighborhood of 0 and $\lim_{v \to 0} \frac{R_2(v|t)}{v} = \lim_{v \to 0} \frac{R_1(v|t)}{v} = 0$. The last equation holds since $R_1(v|t)$ is feasible. For $v > 0$, we have $0 \leq \lim_{v \to \infty} \frac{R_2(v|t)}{v} \leq \lim_{v \to \infty} \frac{R_1(v|t)}{v} = 0$, which indicates $\lim_{v \to \infty} \frac{R_2(v|t)}{v} = 0$.

To show that $\frac{R_2(v|t)}{v}$ is decreasing in v, observe that $\forall v \in X$

$$R_1(v) = \int_{t \in S_1} R_1(v|t)p_1(t)\, dt = \int_{t \in S_1} g(t)p_1(t)\, dt$$

is independent of v. Let $c = R_1(v), \forall v \in X$. Therefore, $\forall v \in X$,

$$\frac{R_2(v|t)}{v} = \frac{R_2(v)R_1(v|t)}{vR_1(v)} = \frac{g(t)}{c} \frac{R_2(v)}{v}$$

is decreasing in X, for any signal t. Also, $\frac{R_2(v|t)}{v}$ is decreasing when $v \notin X$. For any boundary point a of X, we know that $a \notin X$ since X is open. Thus $R_1(a) = R_2(a)$ and $\frac{R_2(v|t)}{v}$ is decreasing at point a. Therefore, $\frac{R_2(v|t)}{v}$ is a decreasing function. □

Remark 3. *The above lemma also applies when there are multiple such intervals, since $R_1(v|t) \neq R_2(v|t)$ only in the interval X, and we can scale $R_1(v|t)$ for all such intervals to get $R_2(v|t)$.*

A major challenge in constructing a feasible revenue function is how to satisfy the third condition in Definition 3.1. Intuitively, when two feasible revenue functions are similar (differ only in an interval), the corresponding decompositions should also be similar. Lemma 4.6 shows that a simple "scaling" trick maintains the feasibility property.

With the scaling technique, we can now easily construct the rest part of $R(v|t)$. Consider the following revenue function $R^*(v)$:

$$R^*(v) = \begin{cases} R(v) & v \leq r^* \\ R(r^*) & v > r^* \end{cases}$$

It is clear that $R^*(v)$ is feasible. The first part of $R^*(v)$ is identical to that of $R(v)$, and we can use the same decomposition for these two revenue functions. Since $R^*(v) = R(r^*)$ is constant for $v > r^*$, a straightforward decomposition for the rest part of $R^*(v)$ is $R^*(v|t) = R^*(r^*|t) = t$. Now compare $R^*(v)$ and $R(v)$ and apply Lemma 4.6. We can set

$$R(v|t) = \frac{R(v)}{R^*(v)} R^*(v|t) = \frac{R(v)}{R(r^*)} t, \forall v \in (r^*, \bar{v}].$$

[5] Although the signaling scheme in Lemma 4.6 is feasible, it may not accord with the signal representation described in Section 2.1, unless $S_1 \cap X = \emptyset$. This is because $R_1(v|t) \neq R_2(v|t)$ only in the interval X, and any $t \notin X$ that maximizes $R_1(v|t)$ also maximizes $R_2(v|t)$.

Therefore, we get the complete posterior revenue function $R(v|t)$:

$$R(v|t) = \begin{cases} v & v \le t \\ t & t < v \le r^* \\ \frac{R(v)}{R(r^*)}t & r^* < v \le \bar{v} \end{cases}$$

$R(v|t)$ is clearly feasible and the posterior distributions $F(v|t)$ can be computed accordingly.

Example. Consider the case where the buyer's value v is uniformly distributed among the interval $[0,1]$. We have that for $0 \le v \le 1$, $F(v) = v$, $R(v) = v(1-v)$, and $r^* = \frac{1}{2}$. Note that $R(v)$ is concave in the interval $\left[0, \frac{1}{2}\right]$. The signaling scheme for point B is $\Omega = (S, F(v|t), p(t))$, where $S = \left[0, \frac{1}{2}\right]$, $p(t) = 2$ and

$$F(v|t) = \begin{cases} 0 & v \le t \\ 1 - \frac{t}{v} & t < v \le \frac{1}{2} \\ 1 - 4t(1-v) & \frac{1}{2} < v \le 1 \end{cases}$$

The posterior value distribution for signal $t = 0$ is a single point mass at $v = 0$, with probability 1.

4.2.1 The non-concave case. The above analysis is based on the assumption that the prior revenue function $R(v)$ is concave in the interval $[\underline{v}, r^*]$. In this section, we relax the assumption and consider the non-concave case. We start from "ironing" the original revenue function (we call it the "concavity ironing" to distinguish it from the "monotonicity ironing" used in the construction of point C). We first construct the signaling scheme for the ironed revenue function and then we modify it with the scaling technique.

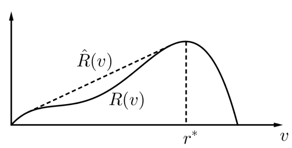

Figure 3: $\hat{R}$ is the ironed revenue function

Let the ironed revenue function $\hat{R}(v)$ be the smallest function upper-bounding $R(v)$, such that $\hat{R}(v)$ is concave in the interval $(\underline{v}, r^*)$ (slightly different from the notation "concave closure" since we only need $\hat{R}(v)$ to be concave in an interval, see Figure 3). The intervals where $\hat{R}(v) \neq R(v)$ are called ironed intervals. Formally, we give the following theorem:

Theorem 4.7. *Let $R(v) = v(1 - F(v))$ be the prior revenue function, and r^* be the monopoly reserve for the prior value distribution ($r^* \in \arg\max_r R(v)$). Let $\hat{R}(v)$ be the resulting function if we apply concavity ironing to $R(v)$ in the interval $[\underline{v}, r^*]$. Denote by I and K the set of all ironed intervals and its index set. For any $i \in K$, let $I_i = (a_i, b_i)$. Then the signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point B, where $S = [\underline{v}, r^*] \setminus U$, $p(t) = -R''(t)$ and*

$$F(v|t) = \begin{cases} 0 & 0 \le v \le t \\ 1 - \frac{t}{v} & t < v \le r^* \text{ and } v \notin U \\ 1 - \frac{R(v) - s_i v}{R(a_i) - s_i a_i}\frac{t}{v} & t < v \le r^* \text{ and } \exists i \in K \text{ such that } v \in I_i \\ 1 - \frac{R(v)}{R(r^*)}\frac{t}{v} & v > r^* \end{cases}$$

where $U = \cup_{i \in K} I_i$ and $s_i = \hat{R}'(v), \forall v \in I_i$.

Proof. Clearly, we have $\hat{R}(a_i) = R(a_i), \hat{R}(b_i) = R(b_i), \forall i \in K$ and $I_i \cap I_j = \emptyset, \forall i, j \in K, i \neq j$.

Let $\Omega = (\hat{S}, \hat{F}(v|t), \hat{p}(t))$ be the signaling scheme for $\hat{R}(v)$ and $\hat{R}(v|t)$ be the corresponding posterior revenue function. Clearly, after the "ironing" step, r^* still maximizes the function $\hat{R}(v)$ and $\hat{S} = [\underline{v}, r^*]$. Also, $\hat{R}(v)$ is linear in each of the ironed intervals and let $s_i = \frac{\hat{R}(b_i) - \hat{R}(a_i)}{b_i - a_i} = \frac{R(b_i) - R(a_i)}{b_i - a_i}$ be its slope. Assume there is only one ironed interval $I_i = (a_i, b_i)$. Our goal is to scale the function $\hat{R}(v|t)$ to find a construction for $R(v)$. According to the analysis for the concave case, $\hat{p}(t) = -\hat{R}''(t) = 0, \forall t \in I_i$. Thus we can remove the signals $t \in I_i$ and split the signal space into two parts $\hat{S}_a = [\underline{v}, a_i]$ and $\hat{S}_b = [b_i, r^*]$. We also decompose $\hat{R}(v)$ into two new functions $\hat{R}_a(v)$ and $\hat{R}_b(v)$, For any $v \in I_i$, we have

$$\hat{R}_a(v) = \frac{\int_{\underline{v}}^{a_i} \hat{R}(v|t)\hat{p}(t)\,\mathrm{d}t}{\int_{\underline{v}}^{a_i} \hat{p}(t)\,\mathrm{d}t} = \frac{\int_{\underline{v}}^{a_i} t\hat{p}(t)\,\mathrm{d}t}{1 - R'(a_i)} = \frac{\int_{\underline{v}}^{a_i} t\hat{p}(t)\,\mathrm{d}t}{1 - s_i}$$
$$= \frac{\hat{R}(a_i) - s_i a_i}{1 - s_i} = \frac{R(a_i) - s_i a_i}{1 - s_i},$$

$$\hat{R}_b(v) = \frac{\int_{b_i}^{r^*} \hat{R}(v|t)\hat{p}(t)\,\mathrm{d}t}{\int_{b_i}^{r^*} \hat{p}(t)\,\mathrm{d}t} = \frac{\int_{b_i}^{r^*} v\hat{p}(t)\,\mathrm{d}t}{\int_{b_i}^{r^*} \hat{p}(t)\,\mathrm{d}t} = v$$
$$\hat{R}(v) = (1 - s_i)\hat{R}_a(v) + s_i\hat{R}_b(v).$$

Assume, after scaling, the two parts become $R_a(v)$ and $R_b(v)$, and the corresponding signals are $R_a(v|t)$ and $R_b(v|t)$, respectively. Since any $v \in I_i$ is not in $\mathbf{supp}(t), \forall t \in \hat{S}_b$, we cannot scale the signals in $\hat{S}_b$. So $\hat{S}_b$ is left untouched, i.e., $R_b(v|t) = \hat{R}_b(v|t)$. Therefore, $R_b(v) = \hat{R}_b(v)$, and

$$R_a(v) = \frac{R(v) - s_i R_b(v)}{1 - s_i} = \frac{R(v) - s_i v}{1 - s_i}.$$

One can easily verify that $R_a(v)$ is a feasible revenue function. Notice that the functions $R_a(v)$ and $\hat{R}_a(v)$ satisfy the conditions of Lemma 4.6, and therefore we have a feasible decomposition for $R_a(v)$: $R_a(v|t) = \frac{R_a(v)}{\hat{R}_a(a_i)}\hat{R}_a(v|t)$, where denominator is $\hat{R}_a(a_i)$ since the signal space for $\hat{R}_a(v)$ is $\hat{S}_a = [\underline{v}, a_i]$. Simplify the above equation and we get $R_a(v|t) = \frac{R(v) - s_i v}{R(a_i) - s_i a_i}t$. Therefore the complete signal space is given by $R_a(v|t)$ and $R_b(v|t)$. The posterior distribution $F(v|t)$ can be computed accordingly.

Note that such decomposition only scales $\hat{R}(v|t)$ in the interval I_i. If there are multiple such intervals, we can scale each interval independently and still get a feasible decomposition. □

The revenue function $\hat{R}(v)$ can be decomposed using the analysis of the previous section. However, it is impossible to directly apply Lemma 4.6 since for some signal t, there exists v such that $v \notin \mathbf{supp}(t)$ ($v < \underline{v}_t$). For there signals, $\hat{R}(v|t) = v$ is already fixed and cannot be scaled. The solution is to remove these signals and scale the posterior distributions of other signals.

4.3 Signaling Scheme for Point C

The signaling scheme S for point C satisfies $\mathrm{UTL}(S) = 0$ and $\mathrm{REV}(S) = \mathrm{REV}^*$. We first consider the case where $R(v)$ is decreasing in the interval $[r^*, \bar{v}]$, but defer the more challenging non-monotone case to Section 4.3.1.

The signaling scheme for the decreasing case is given by the following theorem:

Theorem 4.8. *Let $F(v)$ be the prior value distribution, and $R(v) = v(1-F(v))$ be its corresponding revenue function. Let r^* be the monopoly reserve for $F(v)$ ($r^* \in \arg\max_r v(1-F(v))$). If $R(v)$ is decreasing in the interval $[r^*, \bar{v}]$, then the signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point C, where $S = [r^*, \bar{v}]$, $p(t) = -\frac{R'(t)}{R(r^*)}$ and*

$$F(v|t) = \begin{cases} F(v) & 0 \le v < r^* \\ 1 - \frac{R(r^*)}{v} & r^* \le v < t \\ 1 & v \ge t \end{cases}$$

We do not prove this theorem but provide a three-step construction instead. In the first step, we show that the signal space $S = [r^*, \bar{v}]$ is sufficient to represent all signals (Lemma 4.10). In the second step, we construct the posterior revenue function $R(v|t)$ in the interval $[r^*, \bar{v}]$ and compute the density function $p(t)$ (Lemma 4.11). Then, as a final step, we construct the part in $[\underline{v}, r^*]$ and obtain the complete posterior revenue function for each signal (Theorem 4.8).

We first give characterize the signaling scheme for point C:

Lemma 4.9. *Signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point C, if and only if:*

- $t = \bar{v}_t$;
- $\{t, r^*\} \subset \arg\max_v R(v|t), \forall t \in S$;
- $p(t) \ge 0$, $\int_{t\in S} p(t)\,\mathrm{d}t = 1$ *and* $F(v) = \int_{t\in S} F(v|t)p(t)\,\mathrm{d}t,\ \forall v$.

Proof. The second condition corresponds to the equation $\mathrm{REV}(S) = \mathrm{REV}^*$ and the third condition is natural for any probability distributions. These two conditions follow similar analysis in the proof of Lemma 4.3. As for the first condition, $\mathrm{UTL}(S) = 0$ implies that when the item is sold, it is sold at a price equal to the buyer's valuation of the item, which is equivalent to setting a price $t = \bar{v}_t$. □

According to Lemma 4.9, each posterior distribution must contain a point mass at $v = \bar{v}_t$, otherwise the seller revenue must be 0. In this case we define $R(v) = \lim_{x\to v^-} x(1-F(x))$, which is consistent with Definition 3.1 when $R(v)$ is continuous.

Lemma 4.10. *The signal space $S = [r^*, \bar{v}]$ is sufficient to represent all signals for point C.*

Proof. Clearly, for any signal t, $\bar{v}_t \le \bar{v}$. Therefore the reserve price $t \le \bar{v}_t \le \bar{v}$. Now it suffices to show that there is no signal with $t < r^*$. Assume on the contrary that such a signal exists. According to Lemma 4.9, we have $t = \bar{v} < r^*$. Thus setting a price as high as r^* extracts no revenue, since r^* is even larger than $\bar{v} = \max \mathbf{supp}(t)$. Again, by Lemma 4.9, setting t as the reserve price also extracts no revenue. The only possibility is that the support of this signal contains only a single value 0, with probability 1. In this case, we have $t = 0$ and $p(0) < +\infty$ (otherwise the prior distribution f has a point mass at 0, contradicting to our assumption that $f(v)$ is differentiable). Clearly, $0 \in \mathbf{supp}(f)$. Then instead of dealing with a signal containing only a point mass at point 0, we can distribute the density $f(0)$ to all other signals by setting $f(0|t) = f(0), \forall t \ne 0$. For any $t \ne 0$, add some density at a single point make no difference in its reserve price and thus does not affect the seller revenue and the buyer utility. Therefore, the new signaling scheme contains only signals with $t \in [r^*, \bar{v}]$. □

Remark 4. *The above proof relies on the assumption that $f(v)$ is differentiable. In fact, this assumption can be relaxed. The only difference is that the prior value distribution now has a point mass at point $v = 0$ and that the signal distribution $p(t)$ may have a point mass at $t = 0$. Denote this signaling scheme by $S = \{0\} \cup [r^*, \bar{v}]$ and the posterior value distribution by $F(v|t)$. Assume that the signal $t = 0$ occurs with probability $0 < p_0 < 1$. Then for other signals $t \ne 0$, we must have:*

$$\int_{r^*}^{\bar{v}} p(t)\,\mathrm{d}t = 1 - p_0.$$

We can still distribute the probability of the prior value distribution at $v = 0$ to other signals. Consider the signaling scheme $(S^*, F^*(v|t), p(t))$ where $S^* = [r^*, \bar{v}]$, $F^*(v|t) = p_0 \cdot H(v) + (1-p_0)F(v|t)$ and $p^*(t) = \frac{p(t)}{1-p_0}$, where $H(v)$ is the Heaviside step function. Clearly, both $F^*(v|t)$ and $p^*(t)$ are valid probability distributions. And $R^*(v|t) = (1-p_0)R(v|t)$. Thus t still maximizes the new revenue function $R^*(v|t)$. Therefore the expected seller revenue and buyer utility is:

$$\mathrm{REV}(S^*) = \int_{t\in S^*} R^*(t|t)p^*(t)\,\mathrm{d}t = \int_{t\in S^*} R(t|t)p(t)\,\mathrm{d}t = \mathrm{REV}(S),$$

$$\begin{aligned}\mathrm{UTL}(S^*) &= \int_{t\in S^*}\left[\int_t^{\bar{v}_t}(v-t)\,\mathrm{d}F^*(v|t)\right]p^*(t)\,\mathrm{d}t \\ &= \int_{t\in S^*}\left[\int_t^{\bar{v}_t}(v-t)\,\mathrm{d}F(v|t)\right]p(t)\,\mathrm{d}t = \mathrm{UTL}(S).\end{aligned}$$

The last equation of each line holds because the signal $t = 0$ does not contribute to $\mathrm{REV}(S)$ and $\mathrm{UTL}(S)$.

Now we construct $R(v|t)$ in the interval $[r^*, \bar{v}_t]$. The corresponding revenue function $R(v|t)$ must satisfy $R(\bar{v}_t|t) = R(t|t) = R(r^*|t)$ and $\lim_{v\to t^+} R(v|t) = 0$.

We choose $R(v|t) = R(r^*), \forall r^* \le v \le t$ (see Figure 4), and this choice will greatly simplify our later analysis.

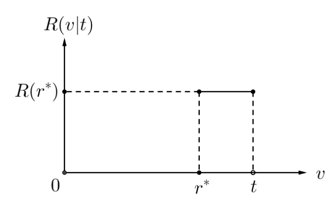

Figure 4: Revenue function for signal t

Lemma 4.11. *Given that $R(v)$ is decreasing in $[r^*, \bar{v}]$, we have $p(t) = -\frac{R'(t)}{R(r^*)}, \forall t \in S = [r^*, \bar{v}]$, if $R(v|t)$ has the following form in the interval $[r^*, \bar{v}_t]$:*

$$R(v|t) = \begin{cases} R(r^*) & r^* \le v \le t \\ 0 & v > t \end{cases}$$

Proof. For any $r^* \le v \le \bar{v}$, we have

$$R(v) = \int_{r^*}^{\bar{v}} R(v|t)p(t)\,\mathrm{d}t = \int_{v}^{\bar{v}} R(r^*)p(t)\,\mathrm{d}t.$$

Taking derivative yields $p(t) = -\frac{R'(t)}{R(r^*)}$. Note that $p(t)$ is indeed a density function:

$$\int_{r^*}^{\bar{v}} p(t)\,\mathrm{d}t = \int_{r^*}^{\bar{v}} -\frac{R'(t)}{R(r^*)}\,\mathrm{d}t = -\frac{R(\bar{v}) - R(r^*)}{R(r^*)} = 1.$$

□

So far, we have characterized $R(v|t)$ for $v \le r^*$. Now we construct $R(v|t)$ for $v > r^*$. It turns out that the simplest solution is to copy the prior revenue function: $R(v|t) = R(v), \forall v \in [\underline{v}, r^*)$. Combined with construction of the part in $[r^*, \bar{v}]$, this simple solution produces a feasible revenue function:

$$R(v|t) = \begin{cases} R(v) & \underline{v} \le v \le r^* \\ R(r^*) & r^* < v \le t \\ 0 & v > t \end{cases}$$

$F(v|t)$ can be computed accordingly, and we finally get Theorem 4.8.

Example. We still consider the case where the buyer's value v is uniformly distributed among the interval $[0, 1]$. Note that $R(v)$ is decreasing in the interval $\left[\frac{1}{2}, 1\right]$. The signaling scheme for point C is $\Omega = (S, F(v|t), p(t))$, where $S = \left[0, \frac{1}{2}\right]$, $p(t) = 2$ and

$$F(v|t) = \begin{cases} v & \underline{v} \le v \le r^* \\ 1 - \frac{1}{4v} & r^* < v \le t \\ 0 & v > t \end{cases}$$

The posterior value distribution for signal $t = 0$ has a point mass at $v = t$.

4.3.1 The non-monotone case. Theorem 4.8 requires that $R(v)$ is decreasing in the interval $[r^*, \bar{v}]$. However, this assumption can also be relaxed through a technique similar to "ironing". In contrast to the "concavity ironing" used in the construction of point B, the "monotonicity ironing" is applied in this case.

Let $\bar{R}(v)$ be the smallest function that upper bounds $R(v)$, such that $\bar{R}(v)$ is decreasing in the interval $[r^*, \bar{v}]$ (see Figure 5).

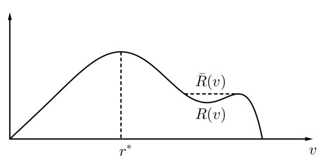

Figure 5: $\bar{R}$ is the ironed revenue function

The signaling scheme for the non-monotone case is given by the following theorem:

Theorem 4.12. *Let $F(v)$ be the prior value distribution, and $R(v) = v(1 - F(v))$ be the corresponding revenue function. Suppose r^* is the monopoly reserve for $F(v)$ ($r^* \in \arg\max_r v(1 - F(v))$). Let $\bar{R}(v)$ be the resulting function if we apply monotonicity ironing to $R(v)$ in the interval $[r^*, \bar{v}]$. Denote the ironed interval and its index set by I and K respectively. Then the signaling scheme $\Omega = (S, F(v|t), p(t))$ implements point C, where $S = [r^*, \bar{v}] \setminus U$, $p(t) = -\frac{R'(t)}{R(r^*)}$ and*

$$F(v|t) = \begin{cases} F(v) & \underline{v} \le v \le r^* \\ 1 - \frac{R(r^*)}{t} & r^* < v \le t \text{ and } v \notin U \\ \frac{R(v)}{c_i} \frac{R(r^*)}{t} & r^* < v \le t \text{ and } \exists i \in K \text{ such that } v \in I_i \\ 1 & v > t \end{cases}$$

where $U = \cup_{i \in K} I_i$ and $c_i = \bar{R}(v), \forall v \in I_i$.

Proof. For each $i \in K$, let $I_i = (a_i, b_i)$. We have $R(a_i) = \bar{R}(a_i) = \bar{R}(b_i) = R(b_i), \forall i \in K$ and $I_i \cap I_j = \emptyset, \forall i, j \in K, i \ne j$. Note that the "monotonicity ironing" does not change the part in $[\underline{v}, r^*]$, and therefore r^* still maximizes $\bar{R}(v)$.

Now we decompose the revenue function $\bar{R}(v)$, since it is decreasing in $[r^*, \bar{v}]$. We can remove those signals t in any of the ironed intervals, since we have $\bar{R}'(t) = 0$ and $\bar{p}(t) = 0$. Let $\bar{\Omega} = (\bar{S}, \bar{F}(v|t), \bar{p}(t))$ be the signaling scheme and $\bar{R}(v|t)$ be the corresponding posterior revenue function. Let $U = \cup_{i \in K} I_i$ and $\bar{R}(v) = c_i, \forall v \in I_i$. Then we have $barS = [r^*, \bar{v}] \setminus U$, $\bar{p}(t) = -\frac{\bar{R}'(t)}{\bar{R}(r^*)}$ and

$$\bar{R}(v|t) = \begin{cases} \bar{R}(v) & \underline{v} \le v \le r^* \\ \bar{R}(r^*) & r^* < v \le t \\ 0 & v > t \end{cases}$$

It is worth mentioning that $\bar{R}(v)$ is not differentiable at the boundaries of the ironed intervals. For those points, the $\bar{R}'(t)$ in $\bar{p}(t)$ should be replaced by $R'(t)$. As for other signals t, $\bar{R}(v) = R(v)$ in the neighborhood of t, and we have $\bar{R}'(t) = R'(t)$. And since $R(v) = \bar{R}(v), \forall v \in [\underline{v}, r^*]$, the signaling scheme $\bar{\Omega}$ becomes $\bar{S} = [r^*, \bar{v}] \setminus U$, $\bar{p}(t) = -\frac{R'(t)}{R(r^*)}$ and

$$\bar{R}(v|t) = \begin{cases} R(v) & \underline{v} \le v \le r^* \\ R(r^*) & r^* < v \le t \\ 0 & v > t \end{cases}$$

We apply Lemma 4.6 to the original revenue function $R(v)$ and the ironed revenue function $\bar{R}(v)$ and obtain

$$R(v|t) = \begin{cases} R(v) & \underline{v} \le v \le r^* \\ R(r^*) & r^* < v \le t \text{ and } v \notin U \\ \frac{R(v)}{c_i} R(r^*) & r^* < v \le t \text{ and } \exists i \in K \text{ such that } v \in I_i \\ 0 & v > t \end{cases}$$

The function $F(v|t)$ can be computed accordingly. □

4.4 Signaling Scheme for Other Points

The following theorem by Bergemann et al. [3] states that the signaling scheme for any point inside the shaded area of Figure1 can be constructed by combining the signaling schemes of the 3 extreme points of the triangle.

Theorem 4.13 (Bergemann et al. [3]). *For any prior distribution $F(v)$ and any pair of seller revenue and buyers utility inside the shaded triangle in Figure 1. There exists a closed-form signaling scheme that achieves it.*

Proof. Note that any point P inside the shaded triangle can be written as a convex combination of the three vertex A,B and C:

$$P = \lambda_1 A + \lambda_2 B + \lambda_3 C, \ \lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{4}$$

Assume signaling schemes $\Omega_i = (S_i, F_i(v|t_i), p_i(t_i))$ for $i \in \{1, 2, 3\}$ achieve the point A,B and C respectively. Define signaling scheme

$$\Omega = (S, F(v|t), p(t))$$

such that

- $S = S_1 \cup S_2 \cup S_3$
- $p(t) = \sum_{i:t \in S_i} \lambda_i p_i(t), \ \forall t \in S$
- $F(v|t) = \frac{1}{p(t)} \sum_{i:t \in S_i} \lambda_i p_i(t) F_i(v|t) \ \forall t \in S$

That is, we let Ω also to be convex combination of $\Omega_i, i = 1, 2, 3$ with coefficients according to (4), and merge the posterior distributions with the same signal (monopoly reserve) together. The signaling scheme Ω achieves point P. □

REFERENCES

[1] Saeed Alaei, Jason Hartline, Rad Niazadeh, Emmanouil Pountourakis, and Yang Yuan. 2015. Optimal auctions vs. anonymous pricing. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*. IEEE, 1446–1463.

[2] Robert J Aumann, Michael Maschler, and Richard E Stearns. 1995. *Repeated games with incomplete information*. MIT press.

[3] Dirk Bergemann, Benjamin Brooks, and Stephen Morris. 2015. The Limits of Price Discrimination. *American Economic Review* 105, 3 (2015), 921–57.

[4] Peter Bro Miltersen and Or Sheffet. 2012. Send Mixed Signals: Earn More, Work Less. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC '12)*. ACM, New York, NY, USA, 234–247.

[5] Sanmay Das and Emir Kamenica. 2015. Representations of information structures. In *Communication, Control, and Computing (Allerton), 2015 53rd Annual Allerton Conference on*. IEEE, 737–743.

[6] Constantinos Daskalakis, Christos H. Papadimitriou, and Christos Tzamos. 2016. Does Information Revelation Improve Revenue?. In *EC*. ACM, 233–250.

[7] Shaddin Dughmi. 2014. On the hardness of signaling. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*. IEEE, 354–363.

[8] Shaddin Dughmi and Haifeng Xu. 2017. Algorithmic Persuasion with No Externalities. In *Proceedings of the 2017 ACM Conference on Economics and Computation*. ACM, 351–368.

[9] Matthew Gentzkow and Emir Kamenica. 2014. Costly persuasion. *The American Economic Review* 104, 5 (2014), 457–462.

[10] Matthew Gentzkow and Emir Kamenica. 2016. A Rothschild-Stiglitz approach to Bayesian persuasion. *The American Economic Review* 106, 5 (2016), 597–601.

[11] Matthew Gentzkow and Emir Kamenica. 2017. Competition in persuasion. *The Review of Economic Studies* 84, 1 (2017), 300–322.

[12] Kira Goldner and Anna R Karlin. 2016. A prior-independent revenue-maximizing auction for multiple additive bidders. In *International Conference on Web and Internet Economics*. Springer, 160–173.

[13] Sergiu Hart and Noam Nisan. 2017. Approximate revenue maximization with multiple items. *Journal of Economic Theory* 172 (2017), 313–347.

[14] Jason D Hartline. 2012. Approximation in economic design. *Book in preparation* (2012).

[15] Emir Kamenica and Matthew Gentzkow. 2011. Bayesian Persuasion. *American Economic Review* 101, 6 (Oct. 2011), 2590–2615.

[16] Roger B Myerson. 1981. Optimal auction design. *Mathematics of operations research* 6, 1 (1981), 58–73.

[17] Michael Ostrovsky and Michael Schwarz. 2010. Information disclosure and unraveling in matching markets. *American Economic Journal: Microeconomics* 2, 2 (2010), 34–63.

[18] A.C. Pigou. 2006. *The Economics of Welfare*.

[19] Luis Rayo and Ilya Segal. 2010. Optimal information disclosure. *Journal of political Economy* 118, 5 (2010), 949–987.

[20] Walter Rudin. 1991. *Functional analysis. International series in pure and applied mathematics*. Chapter 6, 149–181.

[21] Weiran Shen, Pingzhong Tang, and Yulong Zeng. 2018. Buyer-Optimal Distribution. In *Proceedings of the 17th Conference on Autonomous Agents and MultiAgent Systems*. International Foundation for Autonomous Agents and Multiagent Systems.

[22] Haifeng Xu, Rupert Freeman, Vincent Conitzer, Shaddin Dughmi, and Milind Tambe. 2016. Signaling in Bayesian Stackelberg Games. In *Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems*. International Foundation for Autonomous Agents and Multiagent Systems, 150–158.