Complexity of Shift Bribery in Iterative Elections

Cynthia Maushagen, Marc Neveling, Jörg Rothe, and Ann-Kathrin Selker
Heinrich-Heine-Universität Düsseldorf
Düsseldorf, Germany
{maushagen,neveling,rothe,selker}@cs.uni-duesseldorf.de

ABSTRACT

In iterative voting systems, candidates are eliminated in consecutive rounds until either a fixed number of rounds is reached or the set of remaining candidates does not change anymore. We focus on iterative voting systems based on the positional scoring rules plurality, veto, and Borda and study their resistance against shift bribery attacks. In constructive shift bribery, an attacker seeks to make a designated candidate win the election by bribing voters to shift this candidate in their preferences; in destructive shift bribery, the briber's goal is to prevent this candidate's victory. We show that many iterative voting systems, including those due to Hare (a.k.a. single transferable vote, instant-runoff voting, or alternative vote), Coombs, Baldwin, and Nanson, are resistant to these types of attack, i.e., the corresponding problems are NP-hard.

KEYWORDS

computational social choice; voting; bribery; shift bribery; Hare election; Coombs election; Baldwin election; Nanson election

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1 INTRODUCTION

One of the main themes in computational social choice is to study the complexity of manipulative attacks on voting systems, in the hope that proving computational hardness of such attacks may provide some sort of protection against them. Besides manipulation (also referred to as strategic voting) and electoral control, much work has been done to study bribery attacks. For a comprehensive overview, we refer to the book chapters by Conitzer and Walsh [6] for manipulation, by Faliszewski and Rothe [13] for control and bribery, and by Baumeister and Rothe [3] for all three topics.

Bribery in voting was introduced by Faliszewski et al. [10] (see also [11]). We will focus on shift bribery, a special case of swap bribery, which was introduced by Faliszewski et al. [11] in the context of so-called irrational voters for Copeland elections and was then studied in detail by Elkind et al. [9] for the constructive variant and studied by Kaczmarczyk and Faliszewski [15] in the destructive variant. In swap bribery, the briber has to pay for each swap of any two candidates in the votes. Shift bribery additionally requires that swaps always involve the designated candidate that the briber wants to see win.

Swap bribery generalizes the possible winner problem [16, 24], which itself is a generalization of unweighted coalitional manipulation. Therefore, each of the many hardness results known for the possible winner problem is directly inherited by the swap bribery problem. This was the motivation for Elkind et al. [9] to look at restricted variants of swap bribery such as shift bribery.

Even though shift bribery possesses a number of hardness results [9], it has also been shown to allow exact and approximate polynomial-time algorithms in a number of cases [8, 9, 21]. For example, Elkind et al. [9] provided a 2-approximation algorithm for shift bribery when using Borda voting. This result was extended by Elkind and Faliszewski [8] to all positional scoring rules; they also obtained somewhat weaker approximations for Copeland and maximin voting. For Bucklin and fallback voting, the shift bribery problem is even exactly solvable in polynomial time [21].[1] In addition, Bredereck et al. [5] analyzed shift bribery in terms of *parameterized* complexity.

We study shift bribery for six iterative voting systems that each proceed in rounds, eliminating after each round the candidates performing worst: The *system of Baldwin* [1] eliminates the candidates with lowest Borda score and the *system of Nanson* [18] eliminates the candidates whose scores are lower than the average Borda score, while the *system of Hare* eliminates the candidates with lowest plurality score, the system called *iterated plurality* eliminates the candidates that do not have the highest plurality score (both Hare voting and iterated plurality are defined, e.g., in the book by Taylor [22]), and the *system of Coombs* (defined, e.g., in the paper by Levin and Nalebuff [17]) eliminates the candidates with lowest veto score. The last system that we consider differs from the rest because it always uses exactly two rounds: *Plurality with runoff* eliminates the candidates that do not have the highest plurality score (except in the case where there is a unique plurality winner, it then eliminates all candidates that do not have the highest or second-highest plurality score, see the book by Taylor [22]). Among the systems we consider, Hare voting and variants thereof (some of which are called single transferable vote, instant-runoff voting, or alternative vote) are most widely used, for example in Australia, India, Ireland, New Zealand, Pakistan, the UK, and the USA.

We show NP-completeness of the shift bribery problem for each of these iterative voting systems for both the constructive and the destructive case.[2] Our results complement results by Davies et al. [7] who have shown unweighted coalitional manipulation to be NP-complete for Baldwin and Nanson voting (even with just a single

[1] Faliszewski et al. [12] have complemented these results on Bucklin and fallback voting. In particular, they studied a number of bribery problems for these rules, including a variant called "extension bribery," which was previously introduced by Baumeister et al. [2] in the context of campaign management when the voters' ballots are truncated.

[2] As shown by Xia [23], destructive bribery is closely related to the margin of victory, a critical robustness measure for voting systems. Reisch et al. [20] add to this connection by showing that the former problem can be easy while the latter is hard.

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Table 1: Summary of complexity results

	Hare	Coombs	Baldwin
C	NP-c (Thm. 3.1)	NP-c (Thm. 3.4)	NP-c (Thm. 4.1)
D	NP-c (Thm. 3.3)	NP-c (Thm. 3.5)	NP-c (Thm. 4.2)
	Nanson	Iterated Plurality	Plurality with Runoff
C	NP-c (Thm. 4.3)	NP-c (Thm. 5.1)	NP-c (Thm. 5.1)
D	NP-c (Thm. 4.4)	NP-c (Thm. 5.2)	NP-c (Thm. 5.2)

manipulator)—and also for the underlying Borda system (with two manipulators; for the latter result, see also the paper by Betzler et al. [4]). Davies et al. [7] also list various appealing features of the systems by Baldwin and Nanson, including that they have been applied in practice (namely, in the State of Michigan in the 1920s, in the University of Melbourne from 1926 through 1982, and in the University of Adelaide since 1968) and that (unlike Borda) they both are Condorcet-consistent.[3] Table 1 gives an overview of our complexity results for constructive ("C") and destructive ("D") shift bribery in our six voting systems ("NP-c" stands for NP-complete).

2 PRELIMINARIES

Below, we provide the needed notions and notation.

Elections and voting systems. An *election* is specified as a pair (C, V) with C being a set of candidates and V a profile of the voters' preferences over C, typically given by a list of linear orders of the candidates. A *voting system* is a function that maps each election (C, V) to a subset of C, the *winner(s) of the election*. An important class of voting systems is the family of positional scoring rules whose most prominent members are plurality, veto, and Borda count, see, e.g., the book chapters by Zwicker [25] and Baumeister and Rothe [3]. In *plurality*, each voter gives her top-ranked candidate one point; in *veto* (a.k.a. *antiplurality*), each voter gives all except the bottom-ranked candidate one point; in *Borda* with m candidates, each candidate in position i of the voters' rankings scores $m - i$ points; and the winners in each case are those candidates scoring the most points.

Iterative voting systems. The voting systems we study are based on plurality, veto, and Borda, but their election winner(s) are determined in consecutive rounds. In each round, all candidates with the lowest score are eliminated.[4] If in a round all remaining candidates have the same score (there may be only one candidate left), those candidates are proclaimed winners of the election (with the exception of plurality with runoff). We use six different scoring methods: The voting systems due to *Hare*, *Coombs*, and *Baldwin* use, respectively, plurality, veto, and Borda scores. The *Nanson* system eliminates in every round all candidates that have less than the average Borda score, and *iterated plurality* eliminates all that do not have the highest plurality score. As an exception, *plurality with runoff* always uses two rounds and in the first round eliminates all candidates that do not have the highest plurality score, except when there is a unique plurality winner, it then eliminates all candidates except those with the highest or second-highest scores.

Shift bribery. For any given voting system $\mathcal{E}$, we now define the problem $\mathcal{E}$-Shift-Bribery, which is a special case of $\mathcal{E}$-Swap-Bribery, introduced by Faliszewski et al. [11] in the context of so-called irrational voters for Copeland and then comprehensively studied by Elkind et al. [9]. Informally stated, given a profile of votes, a swap-bribery price function exacts a price for each swap of any two candidates in the votes, and in shift bribery only swaps involving the designated candidate are allowed.

$\mathcal{E}$-Constructive-Shift-Bribery	
Given:	An election (C, V) with n votes, a designated candidate $p \in C$, a budget B, and a list of price functions $\rho = (\rho_1, \ldots, \rho_n)$.
Question:	Is it possible to make p the unique $\mathcal{E}$ winner of the election by shifting p in the votes such that the total price does not exceed B?

In the corresponding problems $\mathcal{E}$-Destructive-Shift-Bribery, we ask whether it is possible to prevent p from being a unique winner. Membership in NP is obvious for all considered problems.

Regarding the list of price functions $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i : \mathbb{N} \to \mathbb{N}$, in the constructive case $\rho_i(k)$ indicates the price the briber has to pay in order to move p in vote i by k positions to the top (respectively, to the bottom in the destructive case). For all i, we require that ρ_i is nondecreasing ($\rho_i(\ell) \le \rho_i(\ell+1)$), $\rho_i(0) = 0$, and if p is at position r in vote i then $\rho_i(\ell) = \rho_i(\ell - 1)$ whenever $\ell \ge r$ in the constructive case (respectively, $\ell \ge |C| - r + 1$ in the destructive case). The latter condition ensures that p can be shifted upward no farther than to the top (respectively, the bottom).[5] When the voter i in ρ_i is clear from the context, we omit the subscript and simply write ρ.

Our proofs use the following notation: A vote of the form $a\ b\ c$ indicates that the voter ranks candidate a on top position, then candidate b, and last candidate c. If a set $S \subseteq C$ of candidates appears in a vote as $\overrightarrow{S}$, its candidates are placed in this position in lexicographical order. By $\overleftarrow{S}$ we mean the reverse of the lexicographical order of the candidates in S. If S occurs in a vote without an arrow on top, the order in which the candidates from S are placed here does not matter for our argument. We use $\cdots$ in a vote to indicate that the remaining candidates may occur in any order.

Computational complexity. We assume familiarity with the standard concepts of complexity theory, including the classes P and NP, polynomial-time many-one reducibility, and NP-hardness and -completeness. We will use the following NP-complete problem:

Exact-Cover-by-3-Sets (X3C)	
Given:	Sets $X = \{x_1, \ldots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$ such that $S_i \subseteq X$ and $\|S_i\| = 3$ for all $S_i \in \mathcal{S}$.
Question:	Does there exist an exact cover of X, i.e., a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\|\mathcal{S}'\| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$?

[3] A *Condorcet winner* is a candidate who defeats every other candidate in a pairwise comparison. Such a candidate does not always exist. A voting rule is *Condorcet-consistent* if it chooses only the Condorcet winner whenever there exists one.

[4] In the original sources stated in the Introduction, certain tie-breaking schemes are used if more than one candidate has the lowest score in some round. For the sake of convenience and uniformity, however, we prefer eliminating them all and disregarding tie-breaking issues in such a case.

[5] If p is in the first (respectively, the last) position of a vote, this voter cannot be bribed and we tacitly assume a price function of $\rho(t) = 0$ for each $t \ge 0$. We will disregard these voters when setting price functions for the other voters in our proofs.

We assume that each $x_j \in X$ is contained in exactly three sets $S_i \in \mathcal{S}$; thus $|X| = |\mathcal{S}|$. Gonzalez [14] showed that X3C under this restriction remains NP-hard. Note that if not stated otherwise, we will use $(X, \mathcal{S})$ to denote an X3C instance, where $X = \{x_1, \ldots, x_{3m}\}$, $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$, and $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$. Also note that we assume $x_{i,1}$ to be the $x_j \in S_i$ with the smallest subscript and $x_{i,3}$ to be the $x_j \in S_i$ with the largest subscript.

3 HARE AND COOMBS

We start by showing hardness of shift bribery for Hare elections.

Theorem 3.1. *Hare-Constructive-Shift-Bribery is* NP*-hard.*

Proof. Hardness follows from a reduction from X3C. Construct from $(X, \mathcal{S})$ an instance $((C, V), p, B, \rho)$ of Hare-Constructive-Shift-Bribery with candidate set $C = X \cup \mathcal{S} \cup \{p\}$, designated candidate p, budget B, list ρ of price functions, and the following list V of votes, with # denoting their number:

#	vote	for
1	$S_i\ x_{i,1}\ \overrightarrow{X \setminus \{x_{i,1}\}}\ \cdots$	$1 \le i \le 3m$
1	$S_i\ x_{i,2}\ \overrightarrow{X \setminus \{x_{i,2}\}}\ \cdots$	$1 \le i \le 3m$
1	$S_i\ x_{i,3}\ \overrightarrow{X \setminus \{x_{i,3}\}}\ \cdots$	$1 \le i \le 3m$
4	$x_i\ \overrightarrow{X \setminus \{x_i\}} \cdots$	$1 \le i \le 3m$
1	$S_i\ p\ \cdots$	$1 \le i \le 3m$
4	$p\ \cdots$	

For voters with votes of the form $S_i\ p\ \cdots$, we use the price function $\rho(0) = 0$, $\rho(1) = 1$, and $\rho(t) = 1$ for all $t \ge 2$; and for every other voter, we use the price function ρ with $\rho(0) = 0$ and $\rho(t) = B + 1$ for all $t \ge 1$. Finally, set the budget $B = m$.

Note that all candidates score exactly four points, so p is not a unique winner without bribing voters.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Hare-Constructive-Shift-Bribery.

($\Rightarrow$) Suppose $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique Hare winner of an election obtained by shifting p in the votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $S_i\ p\ \cdots$ by shifting p once, so her new vote is of the form $p\ S_i\ \cdots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. In the first round, p now has $m + 4$ points, every candidate from $\mathcal{S}'$ has 3 points, and every other candidate has 4 points. Therefore, all candidates in $\mathcal{S}'$ are eliminated. In the second round, all candidates in X now gain one point from the elimination of $\mathcal{S}'$, since it is an exact cover. Therefore, p and all candidates in X proceed to the next round and the remaining candidates $\mathcal{S} \setminus \mathcal{S}'$ are eliminated. In the next round with only p and the candidates from X remaining, p has $3m + 4$ points, while every candidate in X scores 7 points (recall that every $x_i \in X$ is contained in exactly three members of $\mathcal{S}$). Since all candidates from X have been eliminated now, p is the only remaining candidate and thus the unique Hare winner.

($\Leftarrow$) Suppose $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \le m$ covers X. We now show that we cannot make p become a unique Hare winner of an election obtained by bribing voters without exceeding budget B. Note that we can only bribe at most m voters with votes of the form $S_i\ p\ \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote $S_i\ p\ \cdots$ has been bribed. Clearly, $|\mathcal{S}'| \le m$ and in all those votes p has been shifted once to the left, so she is now ranked first in these votes. Therefore, p now has $4 + |\mathcal{S}'|$ points and every $S_i \in \mathcal{S}'$ scores 3 points. Since every other candidate scores as many points as before the bribery (namely, 4 points), the candidates in $\mathcal{S}'$ are eliminated in the first round. Let $X' = \{x_i \in X \mid x_i \notin \bigcup_{S_j \in \mathcal{S}'} S_j\}$ be the subset of candidates $x_i \in X$ that are not covered by $\mathcal{S}'$. We have $X' \neq \emptyset$ (otherwise, $\mathcal{S}'$ would be an exact cover of X). In the second round, unlike the candidates from $X \setminus X'$, the candidates in X' will not gain additional points from the elimination the candidates in $\mathcal{S}'$. Thus, in the current situation, the candidates from X' and $\mathcal{S} \setminus \mathcal{S}'$ are trailing behind with 4 points each and are eliminated in this round. Therefore, in the next round, only p and the candidates from $X \setminus X'$ are remaining in the election. Let $x_\ell \in X \setminus X'$ be the candidate from $X \setminus X'$ with the smallest subscript. Since all candidates from $\mathcal{S}$ are eliminated, p has $3m + 4$ points and every candidate from $X \setminus X'$ except x_ℓ has 7 points. On the other hand, x_ℓ gains additional points from eliminating the candidates from X'; therefore, x_ℓ survives this round by scoring more than 7 points. In the final round with only p and x_ℓ remaining, p is eliminated, since $3m \cdot 7 > 3m + 4$ for $m > 1$. □

Example 3.2. Let $(X, \mathcal{S})$ be a yes-instance of X3C defined by $X = \{x_1, \ldots, x_6\}$ and $\mathcal{S} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 6\}, \{2, 4, 5\}, \{1, 3, 4\}, \{1, 5, 6\}\}$. Construct $((C, V), p, B, \rho)$ from $(X, \mathcal{S})$ as in the proof of Theorem 3.1; in particular, the budget is $B = 2$. If we bribe the voters with $S_1\ p\ \cdots$ and $S_2\ p\ \cdots$ to shift p to the top of their vote, p will be the unique winner of the election that proceeds as follows:

Round	p	$x \in X$	S_1, S_2	S_3, S_4, S_5, S_6
1	6	4	3	4
2	6	5	out	4
3	10	7	out	out

Now consider a no-instance $(X, \mathcal{S})$ with $X = \{x_1, \ldots, x_6\}$ and $\mathcal{S} = \{\{1, 2, 4\}, \{4, 5, 6\}, \{2, 3, 6\}, \{2, 3, 5\}, \{1, 3, 4\}, \{1, 5, 6\}\}$. If we bribe no voter, every candidate has four points and wins, so p does not win alone. If we bribe one voter, say the one with vote $S_1\ p\ \cdots$, p loses the direct comparison in the last round of the election proceeding as follows:

Round	p	x_1	$x_2,\ x_4$	$x_3,\ x_5,\ x_6$	S_1	$S_i \in \mathcal{S} \setminus \{S_1\}$
1	5	4	4	4	3	4
2	5	5	5	4	out	4
3	10	28	7	out	out	out
4	10	42	out	out	out	out

Since $(X, \mathcal{S})$ is a no-instance of X3C, no matter which two subsets $S_i,\ S_j$ we choose, at least one x_k is in both subsets, so p again loses the direct comparison in the last round. For example, if we bribe the voters with $S_1\ p\ \cdots$ and $S_2\ p\ \cdots$, the election proceeds as follows:

Round	p	x_1	x_3	x_4	x_2, x_5, x_6	$S_1,\ S_2$	S_3, S_4, S_5, S_6
1	6	4	4	4	4	3	4
2	6	5	4	6	5	out	4
3	10	14	out	7	7	out	out
4	10	42	out	out	out	out	out

Theorem 3.3. *Hare-Destructive-Shift-Bribery is* NP*-hard.*

Proof. Again, we use a reduction from X3C. Construct from $(X, \mathcal{S})$ a Hare-Destructive-Shift-Bribery instance $((C, V), p, B, \rho)$ as follows. Let $D = \{d_1, \dots, d_{3m}\}$ be a set of $3m$ dummy candidates. The candidate set is $C = X \cup \mathcal{S} \cup D \cup \{w, p\}$ with designated candidate p, budget B, and list ρ of price functions. The list V of votes is constructed as follows:

#	vote	for
2	$S_i\ x_{i,1}\ \overrightarrow{X \setminus \{x_{i,1}\}}\ w\ p\ \cdots$	$1 \le i \le 3m$
2	$S_i\ x_{i,2}\ \overrightarrow{X \setminus \{x_{i,2}\}}\ w\ p\ \cdots$	$1 \le i \le 3m$
2	$S_i\ x_{i,3}\ \overrightarrow{X \setminus \{x_{i,3}\}}\ w\ p\ \cdots$	$1 \le i \le 3m$
7	$x_i\ \overrightarrow{X \setminus \{x_i\}}\ w\ p\ \cdots$	$1 \le i \le 3m$
1	$p\ S_i\ \cdots$	$1 \le i \le 3m$
12	$w\ p\ \cdots$	
$18m$	$p\ \cdots$	
6	$d_i\ S_i\ p\ \cdots$	$1 \le i \le 3m$

For voters with votes of the form $p\ S_i\ \cdots$, we use the price function $\rho(0) = 0$, $\rho(1) = 1$, and $\rho(t) = B + 1$ for all $t \ge 2$; and for every other voter, we use the price function ρ with $\rho(0) = 0$ and $\rho(t) = B + 1$ for all $t \ge 1$. Finally, set the budget $B = m$.

Without bribing, the election (C, V) proceeds as follows:

Round	p	w	$x_i \in X$	$S_i \in \mathcal{S}$	$d_i \in D$
1	$21m$	12	7	6	6
2	$39m$	12	13	out	out
3	$39m + 12$	out	13	out	out

It follows that p has won the election after three rounds.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Hare-Destructive-Shift-Bribery.

($\Rightarrow$) Suppose $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible to eliminate p from an election obtained by shifting p in the votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $p\ S_i\ \cdots$ by shifting p once, so her new vote is of the form $S_i\ p\ \dots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. Now the election proceeds as follows:

Round	p	w	$x_i \in X$	$S_i \in \mathcal{S}'$	$S_i \in \mathcal{S} \setminus \mathcal{S}'$	$d_i \in D$
1	$20m$	12	7	7	6	6
2	$32m$	12	11	13	out	out
3	$32m$	$33m + 12$	out	13	out	out
4	$39m$	$39m + 12$	out	out	out	out

We see that p is eliminated in the fourth round.

($\Leftarrow$) Suppose $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \le m$ covers X. We now show that p will not be eliminated in an election obtained by bribing voters without exceeding budget B. Note that we can only bribe at most m voters with votes of the form $p\ S_i\ \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that for every $S_i \in \mathcal{S}'$ we have bribed the voter whose vote is $p\ S_i\ \cdots$. We can assume that $|\mathcal{S}'| > 0$. Every candidate in $\mathcal{S}'$ will gain an additional point and therefore survives the first round. All candidates from D and $\mathcal{S} \setminus \mathcal{S}'$ will be eliminated, since p only loses at most m points.

In the second round, the remaining candidates from $\mathcal{S}$ will gain additional six points from the elimination of candidates in D and score 13 points in this round (and in all subsequent rounds with p still standing). If a candidate $S_i \in \mathcal{S}$ was eliminated in the last round, every $x_i \in S_i$ gains two additional points in this round. Partition X into sets X_0, X_1, X_2, and X_3 so that $x_i \in X_l \Leftrightarrow |\{S_j \in \mathcal{S}' \mid x_i \in S_j\}| = l$. Note that X_0, X_1, X_2, and X_3 are disjoint and $|X_0| > 0$, but one or two of X_1, X_2, and X_3 may be empty. Then $x_i \in X_j$ scores $7 + (6 - 2j) \in \{7, 9, 11, 13\}$ points depending on how many times x_i is covered by $\mathcal{S}'$.

Therefore, every $x_i \in X_0$ scores more points than w who has 12 points. So, there are candidates from X that survive this round and other candidates from X (i.e., candidates from X_1, X_2, or X_3), who are eliminated. In the third round, the candidate $x_\ell \in X$ with the smallest subscript who is still standing gains at least 7 points from the eliminated candidates so that she scores at least 14 points. All other candidates still score the same points as in the last round. Therefore, p scores at least $20m$ points, w scores 12 points, every $S_i \in \mathcal{S}'$ scores 13 points, and every still standing candidate from X except x_ℓ scores at most 13 points. Since w can only gain additional points when all candidates from X are eliminated and only x_ℓ gains points from the elimination of candidates from $X \setminus \{x_\ell\}$ in the subsequent rounds, all candidates $X \setminus (\{x_\ell\} \cup X_0)$ and w are eliminated. Then all still standing candidates from $X_0 \setminus \{x_\ell\}$ and candidates from $\mathcal{S}'$ who each score 13 points are eliminated, which leaves p and x_ℓ in the last round. In this round, p scores $39m + 12$ points and x_ℓ scores $39m$ points, so p wins the election. ◻

Next, we turn to shift bribery for Coombs elections.

Theorem 3.4. *Coombs-Constructive-Shift-Bribery is* NP*-hard.*

Proof. To prove NP-hardness, we now describe a reduction from X3C to Coombs-Constructive-Shift-Bribery. We construct an election (C, V) with the set $C = \{p, w, d_1, d_2, d_3\} \cup X \cup Y$ of candidates, where p is the designated candidate and $Y = \{y_i \mid x_i \in X\}$. We construct the following list V of votes:

#	vote	for
1	$\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$	$1 \le i \le 3m$
$2m$	$\cdots\ p\ \overrightarrow{Y \setminus \{y_i\}}\ y_i\ x_i$	$1 \le i \le 3m$
$2m$	$\cdots\ p\ \overrightarrow{Y}\ w\ d_1\ d_2\ d_3$	
1	$\cdots\ p\ \overrightarrow{Y}\ w\ X\ d_1\ d_2\ d_3$	
m	$\cdots\ p\ \overrightarrow{Y}\ w$	

For the voters with a vote of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$, we use the price function $\rho(0) = 0$, $\rho(1) = \rho(2) = \rho(3) = 1$, and $\rho(t) = B + 1$ for all $t \ge 4$ and for all the remaining voters, we use the price function $\rho(0) = 0$ and $\rho(t) = B + 1$ for all $t \ge 1$. The candidates have the following number of vetoes: p has $3m$, each $x_i \in X$ has $2m$, w has m, d_3 has $2m+1$, and the remaining candidates each have 0 vetoes. Furthermore, our budget is $B = m$.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Coombs-Constructive-Shift-Bribery.

($\Rightarrow$) Assume that (X, S) is in X3C. This means that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$. So we have a partition of X into three sets, $X = X_1 \cup X_2 \cup X_3$, such that:

$$X_1 = \{x_i \in S_i \mid x_i \text{ has the lowest subscript in } S_i \in \mathcal{S}'\},$$
$$X_3 = \{x_i \in S_i \mid x_i \text{ has the highest subscript in } S_i \in \mathcal{S}'\},$$
$$X_2 = X \setminus (X_1 \cup X_3).$$

Let $Y = Y_1 \cup Y_2 \cup Y_3$ be the corresponding partition of Y. We bribe the voters with votes of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ and $S_i \in \mathcal{S}'$ so that they change their votes to $\cdots\ p\ x_{i,1}\ x_{i,2}\ x_{i,3}$. It follows that p now has a total of $2m$ vetoes, whereas each $x_{i,3} \in X$ receives an additional veto for a total of $2m + 1$. The number of vetoes for the remaining candidates remain unchanged.

If a candidate has the highest number of vetoes then she has the fewest number of points and cannot proceed to the next round. Here, the candidates in X_3 and d_3 have the fewest number of points and are eliminated in this round.

Without the candidates in X_3 each candidate in X_2 gets an additional veto and the candidates in Y_3 each take the vetoes of the eliminated candidates X_3. As a consequence, in this round the candidates in X_2 and d_2 have the fewest number of points and are eliminated. Similarly to the first round, vetoes from candidates in X_2 and d_2 are passed on to candidates in X_1, Y_1, and d_1. Thus the candidates receive the following number of vetoes in the third round: p and each $y \in Y_2 \cup Y_3$ receive $2m$ vetoes, w receives m, each $y \in Y_1$ receives 0, and d_1 and each $x_i \in X_1$ receive $2m + 1$ vetoes. Consequently, all the candidates $x_i \in X_3$ and d_1 are eliminated in this third round, so in the next round there are no candidates from X and no d_i with $1 \le i \le 3$. It follows that w receives $2m + 1$ additional vetoes in this round, so w has the most vetoes in the fourth round and is eliminated. We need $3m$ further rounds until p ends up as the last remaining candidate and sole winner of the election. In each of these rounds, the candidate in Y that is still alive and has the highest subscript receives at least $2m + 2m + 1 + m$ vetoes, while p always gets only $3m$ vetoes.

($\Leftarrow$) Suppose that (X, S) is a no-instance for X3C. Observe that if we want to make p a unique winner of the election, we have to bribe at least m voters with a vote of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$. If we do not bribe these voters, p has at least $2m + 1$ vetoes and would be eliminated in the first round. Due to our budget we have to bribe exactly m such voters and cannot bribe further voters. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote of the form $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ has been bribed. The best case for p is that those voters change their vote from $\cdots\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ to $\cdots\ p\ x_{i,1}\ x_{i,2}\ x_{i,3}$, where $S_i \in \mathcal{S}'$. Note that $|\mathcal{S}'| = m$ and $\mathcal{S}'$ does not cover X because we have a no-instance of X3C. Now p has only $2m$ vetoes and will not be eliminated in the first round.

Let X_1 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the smallest subscript in S_i, let X_2 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the second smallest subscript in S_i, and let X_3 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the highest subscript in S_i. Note that $X_1 \cup X_2 \cup X_3 \neq X$, since $\mathcal{S}'$ does not cover X.

For w to have more vetoes than p, the candidates d_1, d_2, and d_3 need to be eliminated. For that to happen, there must be three rounds in which no other candidate has more than $2m + 1$ vetoes. In the round where d_i, $1 \le i \le 3$, is eliminated, all still standing candidates in X_i are eliminated as well. Assume there were three rounds in which $2m + 1$ was the maximal number of vetoes for a candidate. Then d_1, d_2, d_3, and all candidates in $X_1 \cup X_2 \cup X_3$ are eliminated. Note that those candidates that were not covered by $\mathcal{S}'$ always had only $2m$ vetoes and are still participating in the election. Therefore, in the next round, p and w have $3m$ vetoes each, the remaining candidates from X have at most $2m + 1$ vetoes, and the candidates from Y have at most $2m$ vetoes. So even if p survives the first rounds with the candidates d_1, d_2, and d_3 still present, she will then surely be eliminated in the following round. If there is at least one voter who shifts p only one or two positions upward, then p has to drop out with d_1 or even before d_1 drops out, because at the latest after two rounds (with $2m + 1$ being the maximal number of vetoes for a candidate) p receives another veto and thus has at least the same number of vetoes as d_1. □

Finally, we state the following result for the destructive case in Coombs elections. Due to space limitations, we omit the proof, which also uses a reduction from X3C.

Theorem 3.5. *Coombs-Destructive-Shift-Bribery is* NP*-hard.*

4 BALDWIN AND NANSON

We now show NP-hardness of shift bribery for Baldwin and Nanson elections. Note that similar reductions were used by Davies et al. [7] to show NP-hardness of the unweighted manipulation problem for these election systems.

For a preference profile V over a set of candidates C, let $avg(V)$ be the average Borda score of the candidates in V (i.e., $avg(V) = \frac{(|C|-1)|V|}{2}$). To conveniently construct votes, for a set of candidates C and $c_1, c_2 \in C$, let

$$W_{(c_1,c_2)} = (c_1\ c_2\ \overrightarrow{C \setminus \{c_1, c_2\}},\ \overleftarrow{C \setminus \{c_1, c_2\}}\ c_1\ c_2).$$

Under Borda, from the two votes in $W_{(c_1,c_2)}$ candidate c_1 scores $|C|$ points, c_2 scores $|C| - 2$ points, and all other candidates score $|C| - 1$ points. Also, observe that if a candidate $c^* \in C$ is eliminated in some round and $c^* \notin \{c_1, c_2\}$ then all other candidates lose one point due to the votes in $W_{(c_1,c_2)}$. If $c^* = c_1$ then c_2 loses no points but all other candidates lose one point, and if $c^* = c_2$ then c_1 loses two points and all other candidates lose one point. Therefore, if c^* is eliminated, the point difference caused by this elimination with respect to the votes in $W_{(c_1,c_2)}$ remains the same for all candidates, with two exceptions: (a) If $c^* = c_1$ then c_2 gains a point from every other candidate, and (b) if $c^* = c_2$ then c_1 loses a point to every other candidate. Furthermore let $score_{(C,V)}(x)$ denote the number of points candidate x obtains in a Borda election (C, V), and let $dist_{(C,V)}(x, y) = score_{(C,V)}(x) - score_{(C,V)}(y)$.

Theorem 4.1. *Baldwin-Constructive-Shift-Bribery is* NP*-hard.*

Proof. To prove NP-hardness, we reduce the NP-complete problem X3C to Baldwin-Constructive-Shift-Bribery.

From $(X, \mathcal{S})$, we construct an election (C, R) with the set of candidates $C = \{p, w, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate and R consists of two lists of votes, R_1 and R_2. R_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(S_j,p)}$	$1 \le j \le 3m$	2	$W_{(x_{j,3},S_j)}$	$1 \le j \le 3m$
2	$W_{(x_{j,1},S_j)}$	$1 \le j \le 3m$	2	$W_{(w,x_i)}$	$1 \le i \le 3m$
2	$W_{(x_{j,2},S_j)}$	$1 \le j \le 3m$	7	$W_{(w,p)}$	

The votes in R_1 give the following scores to the candidates in C:

$$score_{(C,R_1)}(x_i) = avg(R_1) + 4 \text{ for every } x_i \in X,$$
$$score_{(C,R_1)}(S_j) = avg(R_1) - 5 \text{ for every } S_j \in \mathcal{S},$$
$$score_{(C,R_1)}(p) = avg(R_1) - 3m - 7,$$
$$score_{(C,R_1)}(w) = avg(R_1) + 6m + 7,$$
$$score_{(C,R_1)}(d) = avg(R_1).$$

Furthermore, R_2 contains the following votes:

#	votes	for	#	votes
$2m+1$	$W_{(d,S_j)}$	$1 \le j \le 3m$	1	$W_{(p,d)}$
$2m+9$	$W_{(d,x_i)}$	$1 \le i \le 3m$	$2m+14$	$W_{(d,w)}$

The votes in R_2 give the following scores to the candidates in C:

$$score_{(C,R_2)}(x_i) = avg(R_2) - (2m+9) \text{ for every } x_i \in X,$$
$$score_{(C,R_2)}(S_j) = avg(R_2) - (2m+1) \text{ for every } S_j \in \mathcal{S},$$
$$score_{(C,R_2)}(p) = avg(R_2) + 1,$$
$$score_{(C,R_2)}(w) = avg(R_2) - (2m+14),$$
$$score_{(C,R_2)}(d) = avg(R_2) + 12m^2 + 32m + 13.$$

Let $R = R_1 \cup R_2$ and $avg(R) = avg(R_1) + avg(R_2)$. Then we have the following Borda scores for the complete profile R:

$$score_{(C,R)}(x_i) = avg(R) - 2m - 5 \text{ for every } x_i \in X,$$
$$score_{(C,R)}(S_j) = avg(R) - 2m - 6 \text{ for every } S_j \in \mathcal{S},$$
$$score_{(C,R)}(p) = avg(R) - 3m - 6,$$
$$score_{(C,R)}(w) = avg(R) + 4m - 7,$$
$$score_{(C,R)}(d) = avg(R) + 12m^2 + 32m + 13.$$

Regarding the price function, for every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\ p\ C \setminus \{S_j, p\}$), let $\rho(1) = 1$ and $\rho(t) = 1$ for every $t \ge 2$. For every other vote, let $\rho(t) = B + 1$ for every $t \ge 1$. Finally, we set the budget $B = m$. It is easy to see that p is eliminated in the first round in the election (C, R).

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, R), p, B, \rho)$ is in Baldwin-Constructive-Shift-Bribery.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then we bribe the first votes of $W_{(S_j,p)}$ for every $S_j \in \mathcal{S}'$ by shifting p to the left once. Note that we won't exceed our budget, since shifting once costs 1 in those votes and $|\mathcal{S}'| = m$. Additionally, for every $S_j \in \mathcal{S}'$, the two votes in $W_{(S_j,p)}$ are now symmetric to each other and can be disregarded from now on, as all candidates gain the same number of points from those votes and all candidates lose the same number of points if a candidate is eliminated from the election. After those m votes have been bribed, only the scores of p and every $S_j \in \mathcal{S}'$ change. With $score_{(C,R)}(p) = avg(R) - 2m - 6$ and $score_{(C,R)}(S_j) = avg(R) - 2m - 7$, all candidates in $\mathcal{S}'$ are tied for the last place. If any $S_j \in \mathcal{S}'$ is eliminated in a round, the three candidates $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$ will lose two points more than the candidates from $\mathcal{S}' \setminus \{S_j\}$ that were in the last position before S_j was eliminated. Therefore, those three candidates from X will then be in the last position in the next round. This means that all candidates $\mathcal{S}'$ and every $x_i \in X$ that is covered by $\mathcal{S}'$ will be eliminated in the subsequent rounds. Since $\mathcal{S}'$ is an exact cover, now there is no candidate from X left. Thus the point difference between p and w is 1 and between p and the remaining $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ is -6. Note that p can beat d only if no candidate of $C \setminus \{p, d\}$ is still participating. So in the next round, w is eliminated. From this p gains seven points on all $S_j \in (\mathcal{S} \setminus \mathcal{S}')$, so these are tied for the last place. Therefore, the remaining candidates from $\mathcal{S}$ are eliminated, which leaves p and d for the next and final round, where d is eliminated and p wins the election alone.

($\Leftarrow$) Suppose there is no exact cover. It is obvious that at most m of the first votes of $W_{(S_j,p)}$ can be bribed without exceeding the budget. Without bribing, p is in the last place and the point difference to the second-to-last candidate(s) is $dist_{(C,R)}(p, S_j) = m$, $1 \le j \le 3m$. By bribing, as explained above, p gains $m + 1$ points on m candidates from $\mathcal{S}$, which then will be eliminated from the election. This leads to the elimination of all $x_i \in X$ that are covered by the set $\mathcal{S}' \subseteq \mathcal{S}$ of candidates that were eliminated. Since there is no exact cover, $\mathcal{S}'$ doesn't cover X. So there are candidates $X' \subseteq X$, $|X'| \ge 1$, who were not eliminated before, as for every candidate $x_i \in X'$ all three candidates $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ with $x_i \in S_j$ are still in the election. With the candidates $C_1 = \{p, w, d\} \cup (\mathcal{S} \setminus \mathcal{S}') \cup X'$ still standing, the point differences of p to the other remaining candidates are as follows:

$$dist_{(C_1,V)}(p, d) = -2m + 1 - 2m(2m + 1) - |X'|(2m + 9) - (2m + 14) < 0,$$
$$dist_{(C_1,V)}(p, w) = 1 - 2|X'| < 0,$$
$$dist_{(C_1,V)}(p, x_i) = -1 \text{ for every } x_i \in X', \text{ and}$$
$$dist_{(C_1,V)}(p, S_j) < -12 \text{ for every} S_j \in \mathcal{S} \setminus \mathcal{S}'.$$

Therefore, p is in the last place and is eliminated. □

The following theorem uses a similar proof idea as Theorem 4.1.

Theorem 4.2. *Baldwin-Destructive-Shift-Bribery is* NP*-hard.*

Proof Sketch. To prove NP-hardness, we reduce the NP-complete problem X3C to Baldwin-Destructive-Shift-Bribery.

Using $(X, \mathcal{S})$, we construct an election (C, R) with the set of candidates $C = \{p, w, b, d\} \cup X \cup \mathcal{S}$, where d is the despised candidate and R consists of two lists of votes, R_1 and R_2. R_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(d,S_j)}$	$1 \le j \le 3m$	2	$W_{(w,x_i)}$	$1 \le i \le 3m$
2	$W_{(S_j,x_{j,1})}$	$1 \le j \le 3m$	$3m+7$	$W_{(w,p)}$	
2	$W_{(S_j,x_{j,2})}$	$1 \le j \le 3m$	$m+10$	$W_{(b,S_j)}$	$1 \le j \le 3m$
2	$W_{(S_j,x_{j,3})}$	$1 \le j \le 3m$			

Furthermore, R_2 contains the following votes:

#	votes	for	#	votes
1	$W_{(p,d)}$		$6m+14$	$W_{(d,w)}$
$2m+7$	$W_{(d,S_j)}$	$1 \le j \le 3m$	$3m^2+33m+12$	$W_{(d,b)}$
$3m+3$	$W_{(d,x_i)}$	$1 \le i \le 3m$		

Let $R = R_1 \cup R_2$. Then we have the following Borda scores for the complete profile R:

$$score_{(C,R)}(x_i) = avg(R) - 3m - 11 \text{ for every } x_i \in X,$$
$$score_{(C,R)}(S_j) = avg(R) - 3m - 12 \text{ for every } S_j \in \mathcal{S},$$
$$score_{(C,R)}(p) = avg(R) - 3m - 6,$$

$$score_{(C,R)}(w) = avg(R) + 3m - 7,$$
$$score_{(C,R)}(b) = avg(R) - 3m - 12,$$
$$score_{(C,R)}(d) = avg(R) + 18m^2 + 69m + 25.$$

Regarding the price function, for every first vote of $W_{(d,S_j)}$ (i.e., a vote of the form $d\ S_j\ C \setminus \{S_j, d\}$), let $\rho(1) = 1$ and $\rho(t) = B + 1$ for every $t \geq 2$. For every other vote, let $\rho(t) = B + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$. It is easy to see that d wins the election (C, R).

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, R), d, B, \rho)$ is in Baldwin-Destructive-Shift-Bribery.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then we bribe the first votes of $W_{(d,S_j)}$ for every $S_j \in \mathcal{S}'$ by shifting d to the right once. With a similar argument as in the proof of Theorem 4.1, p wins the election, i.e., d is not among the winners.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$, there is at least one $x_i \in X$ that is not covered by $\mathcal{S}'$. It is obvious that at most m of the first votes of $W_{(d,S_j)}$ can be bribed without exceeding the budget.

We can show that p will always be eliminated before w and therefore d cannot be prevented from winning the election. □

Finally, we turn to Nanson elections. The reduction below will only use pairs of votes of the form $W_{(c_1,c_2)}$. The average Borda score for those two votes is $|C| - 1$. The candidate c_1 scores one point more than the average and c_2 scores one point fewer than the average. The other candidates score exactly the average Borda score. If a candidate is eliminated in a round, the average Borda score required to survive the next round decreases by one. Regardless of which candidate is eliminated, all remaining candidates that are not c_1 or c_2 lose one point and still have exactly the average Borda score. If c_2 is eliminated, c_1 loses its advantage with respect to the average and now scores exactly the average Borda score as well. If one of the other candidates is eliminated, c_1 continues to have one point more than the average Borda score. This is analogous for c_2: If c_1 is eliminated, c_2 scores the average Borda score, and if one of the other candidates is eliminated, c_2 still has one point fewer than the average Borda score.

Theorem 4.3. *Nanson-Constructive-Shift-Bribery is* NP*-hard.*

Proof. To prove NP-hardness, we reduce the NP-complete problem X3C to Nanson-Constructive-Shift-Bribery.

Again, using $(X, \mathcal{S})$ we construct an election (C, R) with the set of candidates $C = \{p, w_1, w_2, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate. Then we construct two sets of votes, R_1 and R_2. R_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(S_j,p)}$	$1 \leq j \leq 3m$	1	$W_{(x_{j,3},S_j)}$	$1 \leq j \leq 3m$
1	$W_{(x_i,p)}$	$1 \leq i \leq 3m$	4	$W_{(S_j,w_1)}$	$1 \leq j \leq 3m$
1	$W_{(x_{j,1},S_j)}$	$1 \leq j \leq 3m$	$15m$	$W_{(w_1,w_2)}$	
1	$W_{(x_{j,2},S_j)}$	$1 \leq j \leq 3m$	$3m$	$W_{(p,w_1)}$	

Furthermore, R_2 contains the following votes:

#	votes	for
$2m$	$W_{(p,d)}$	
2	$W_{(d,S_j)}$	$1 \leq j \leq 3m$
4	$W_{(d,x_i)}$	$1 \leq i \leq 3m$

Let $R = R_1 \cup R_2$. Then we have the following Borda scores for the complete profile R:

$$score_{(C,R)}(x_i) = avg(R) \text{ for every } x_i \in X,$$
$$score_{(C,R)}(S_j) = avg(R) \text{ for every } S_j \in \mathcal{S},$$
$$score_{(C,R)}(p) = avg(R) - m,$$
$$score_{(C,R)}(w_1) = avg(R),$$
$$score_{(C,R)}(w_2) = avg(R) - 15m,$$
$$score_{(C,R)}(d) = avg(R) + 16m.$$

The price function is again defined as follows. For every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\ p\ C \setminus \{S_j, p\}$), let $\rho(1) = 1$ and $\rho(t) = 1$ for every $t \geq 2$. For every other vote, let $\rho(t) = B + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$. It is easy to see that p is eliminated in the first round in the election (C, R).

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, R), p, B, \rho)$ is in Nanson-Constructive-Shift-Bribery.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then, for every $S_j \in \mathcal{S}'$, we bribe the first vote of $W_{(S_j,p)}$ by shifting p to the left once in all those votes. Note that we won't exceed our budget, since this bribe action costs 1 per vote and $|\mathcal{S}'| = m$. With the additional m points, p reaches the average Borda score and is not eliminated in the first round. However, all candidates in $\mathcal{S}'$ lose one point and are eliminated. Additionally, w_2 will be eliminated as well.

In the next round, w_1 will be eliminated, since she has $11m$ points fewer than the average score required to survive this round. Since the candidates in $\mathcal{S}'$ were eliminated in the last round and $\mathcal{S}'$ is an exact cover, every candidate in X now has fewer points than the average and is eliminated.

In the third round, only p, d, and the candidates in $\mathcal{S} \setminus \mathcal{S}'$ are still standing. Therefore, the only pairs of votes that are not symmetric are $W_{(S_j,p)}$, twice $W_{(d,S_j)}$ for every $S_j \in (\mathcal{S} \setminus \mathcal{S}')$, and $2m$ pairs of $W_{(p,d)}$. Since $|\mathcal{S} \setminus \mathcal{S}'| = 2m$, we have that p scores exactly the average score and survives this round, just as d. Every $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ has one point fewer than the average and is eliminated. This leaves only p and d in the last round, which p wins.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$, there is at least one $x_i \in X$ that is not covered by $\mathcal{S}'$. Note that we can only bribe the first votes of any $W_{(S_j,p)}$ without exceeding the budget. For p to survive the first round, we need to bribe m of those votes by shifting p to the left once. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $\mathcal{S}'$ contains S_j exactly if the first vote of $W_{(S_j,p)}$ has been bribed. Then every $S_j \in \mathcal{S}'$ has a score of $avg(R) - 1$ and p has a score of $avg(R)$. Therefore, in the first round, every candidate from $\mathcal{S}'$ and w_2 are eliminated from the election.

In the second round, w_1 will be eliminated because of the $15m$ pairs of votes $W_{(w_1,w_2)}$ and the elimination of w_2. Furthermore, a candidate $x_i \in X$ reaches the average score with p and d still standing only if all three $S_j \in \mathcal{S}$ with $x_i \in S_j$ are also not yet eliminated. Since the candidates in $\mathcal{S}'$ were eliminated in the previous round, for every $S_j \in \mathcal{S}'$, all three $x_i \in S_j$ will be eliminated in this round. Since $\mathcal{S}'$ is not an exact cover, there are candidates $X' \subseteq X$ that survive this round. d also reaches the average, as there are $2m$ candidates $\mathcal{S} \setminus \mathcal{S}'$ and those candidates $\mathcal{S} \setminus \mathcal{S}'$ survive due to w_1.

In the next round, the candidates still standing are p, d, X', and $\mathcal{S} \setminus \mathcal{S}'$. Because $|X'| \geq 1$, candidate p has $|X'|$ points fewer than the average score and is eliminated in this round. □

Our last result in this section shows that the destructive variant of shift bribery in Nanson elections is intractable as well. The proof is omitted due to space limitations.

Theorem 4.4. *Nanson-Destructive-Shift-Bribery is* NP*-hard.*

5 ITERATED PLURALITY AND PLURALITY WITH RUNOFF

In this section, we show hardness of shift bribery for iterated plurality and plurality with runoff, starting with the constructive case.

Theorem 5.1. *For iterated plurality and plurality with runoff, Constructive-Shift-Bribery is* NP*-hard.*

Proof. To prove NP-hardness, we reduce X3C to Constructive-Shift-Bribery for these two voting systems. Let $(X, \mathcal{S})$ be a given X3C instance. We construct the Constructive-Shift-Bribery instance $((C, V), p, B, \rho)$ as follows. Let $C = \{p, d\} \cup X \cup \mathcal{S} \cup D$ with $D = \{d_{i,j} | 1 \leq i \leq 3m \text{ and } 1 \leq j \leq m-4\}$. The list of voters is constructed as follows:

#	vote	for
1	$S_i \; p \; \cdots$	$1 \leq i \leq 3m$
1	$S_i \; x_{i,1} \; \overrightarrow{X \setminus \{x_{i,1}\}} \cdots$	$1 \leq i \leq 3m$
1	$S_i \; x_{i,2} \; \overrightarrow{X \setminus \{x_{i,2}\}} \cdots$	$1 \leq i \leq 3m$
1	$S_i \; x_{i,3} \; \overrightarrow{X \setminus \{x_{i,3}\}} \; \cdots$	$1 \leq i \leq 3m$
1	$S_i \; d_{i,j} \; \overrightarrow{X \setminus \{x_i\}} \; \cdots$	$1 \leq i \leq 3m$, $1 \leq j \leq m-4$
m	$x_i \; \overrightarrow{X \setminus \{x_i\}} \; \cdots$	$1 \leq i \leq 3m$
m	$d_{i,j} \; \overrightarrow{X} \; \cdots$	$1 \leq i \leq 3m$, $1 \leq j \leq m-4$
2	$d \; p \; \cdots$	

For voters with votes of the form $S_i \; p \; \cdots$, we use the price function $\rho(0) = 0$, $\rho(1) = 1$, and $\rho(t) = B + 1$ for all $t \geq 2$; and for every other voter, we use the price function $\rho(0) = 0$ and $\rho(t) = B + 1$ for $t \geq 1$. Set the budget $B = m$.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C, V), p, B, \rho)$ is in Constructive-Shift-Bribery for both voting systems.

($\Rightarrow$) Suppose $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique iterated-plurality (respectively, plurality-with-runoff) winner of an election obtained by shifting p in the votes without exceeding the budget. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $S_i \; p \; \cdots$ once, so her new vote is of the form $p \; S_i \; \cdots$. In the first round p, every $x_i \in X$, every $d_{i,j} \in D$, and every $S_i \in \mathcal{S} \setminus \mathcal{S}'$ is a plurality winner, so only these candidates participate in the next round. In the second round, p receives two further points from the two voters whose vote is $d \; p \; \cdots$. Every candidate $x_i \in X$ receives one further point from the voter with vote $S_i \; x_i \; \cdots$. Every $d_{i,j}$ with $S_i \in \mathcal{S}'$ and $1 \leq j \leq m-4$ receives one additional point from the voters with vote $S_i \; d_{i,j} \; \cdots$. It follows that p has the most points and therefore p is the unique iterated-plurality (respectively, plurality-with-runoff) winner.

($\Leftarrow$) Suppose $(X, \mathcal{S})$ is a no-instance of X3C. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$, there is at least one $x \in X$ that is not covered and therefore one $x' \in X$ which is at least in two sets $S_i, \; S_j \in \mathcal{S}'$. We show that it is not possible for p to become a winner of the election without exceeding the budget. To become a winner of the election it is necessary for p to get at least m points in the first round. Due to the budget it is necessary to bribe m voters with a vote of the form $S_i \; p \; \cdots$ with $S_i \in \mathcal{S}'$. It follows that p, each $x \in X$, each $S_i \in \mathcal{S} \setminus \mathcal{S}'$, and each $d \in D$ participate in the second round. At least one candidate $x \in X$ receives at least two further points due to the fact that $\mathcal{S}'$ is not a cover of X. If we use iterated plurality, the best case for p is that there is no candidate $x \in X$ who receives three points. Then p and some $x \in X$ participate in the third round in which the still standing $x \in X$ with the smallest subscript wins the election, so p is not a winner of the election. If we use plurality with runoff, the second round is the last round. If there is a candidate $x \in X$ who receives three points then p is not even a winner of the election; otherwise, p is not a unique winner of the election. □

We have the same result in the destructive case. Our proof, omitted due to space limitations, gives a reduction from a restricted version of the problem One-in-Three-Positive-3SAT that Porschen et al. [19] have shown to be NP-complete.

Theorem 5.2. *For iterated plurality and plurality with runoff, Destructive-Shift-Bribery is* NP*-hard.*

6 CONCLUSIONS AND OPEN QUESTIONS

We have shown that shift bribery is NP-complete for each of the iterative voting systems of Hare, Coombs, Baldwin, Nanson, iterated plurality, and plurality with runoff, both for the constructive and the destructive case. All our proofs except that of Theorem 3.1 can be adapted so as to work also in the nonunique-winner case where the bribery action is successful if the designated candidate is among the winners in the constructive case (respectively, does not win in the destructive case). While these are interesting theoretical results complementing earlier work both on shift bribery and on these voting systems, NP-hardness of course has its limitations in terms of providing protection in practice. It would be interesting to also study shift bribery for these systems in terms of approximation and parameterized complexity and to do a typical-case analysis.

In the definition of shift bribery, the designated candidate can only be shifted forward in the constructive case (respectively, backward in the destructive case). However, in nonmonotonic voting systems (such as those studied here), shifting backward (respectively, shifting forward) could also be beneficial for this candidate in the constructive case (respectively, hurtful in the destructive case). It would be interesting to find out whether the complexity of our problems changes when the nonmonotonicity of voting systems is specifically allowed to use in bribing actions, or even required.

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