Strategyproof and Fair Matching Mechanism for Ratio Constraints

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ABSTRACT

We introduce a new type of distributional constraints called ratio constraints, which explicitly specify the required balance among schools in two-sided matching. Since ratio constraints do not belong to the known well-behaved class of constraints called M-convex set, developing a fair and strategyproof mechanism that can handle them is challenging. We develop a novel mechanism called Quota Reduction Deferred Acceptance (QRDA), which repeatedly applies the standard DA by sequentially reducing artificially introduced maximum quotas. As well as being fair and strategyproof, QRDA always obtains a weakly better matching for students compared to a baseline mechanism called Artificial Cap Deferred Acceptance (ACDA), which uses predetermined artificial maximum quotas. Experimentally, QRDA performs better in terms of student welfare and nonwastefulness than ACDA and another fair and strategyproof mechanism called Extended Seat Deferred Acceptance (ESDA), in which ratio constraints are transformed into minimum/maximum quotas.

KEYWORDS

two-sided matching; strategyproofness; distributional constraints; deferred acceptance mechanism; M-convex set

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1 INTRODUCTION

The theory of matching has been extensively developed for markets in which two types of agents (e.g., students/schools, hospitals/residents) are matched [28]. Recently, this topic has been attracting considerable attention from AI researchers [1, 17, 19, 21, 24]. A standard market deals with maximum quotas, which are capacity limits that cannot be exceeded. However, many real-world matching markets are subject to a variety of distributional constraints, including regional maximum quotas, which restrict the total number of students assigned to a set of schools [20], minimum quotas, which guarantee that a certain number of students are assigned to each school [2, 9, 13, 16, 29, 30], and diversity constraints, which enforce that a school satisfies a balance between different types of students, typically in terms of socioeconomic status [5, 15, 22, 24].

Policymakers often hope for a well-balanced matching outcome, i.e., where the number of students (or doctors) assigned to each school (or hospital) is not too diverse. For example, the Japanese government does not want the number of doctors assigned to rural hospitals to be drastically fewer than the number to urban hospitals [20]. The United States Military Academy solicits cadet preferences over assignments to various army branches, while simultaneously trying to keep a good balance among the branches [29, 30]. In China, there are two types of master's degrees: professional and academic. Since academic master programs are much more popular than professional ones, the Chinese government seeks a good balance between these two programs [20]. One way to obtain a balanced outcome is to impose artificially low maximum quotas to guarantee that students/doctors are not overly concentrated in popular schools/hospitals. Another way is to introduce minimum quotas to guarantee that a certain number of students/doctors are allocated to unpopular schools/hospitals.

In this paper, we introduce a new type of constraints called *ratio constraints* that can explicitly specify the required balance among schools/hospitals, where parameter α specifies the acceptable minimum ratio between the least/most popular schools. Such ratio constraints are used in practice. For example, in many universities (including the authors' university), a department is divided into several courses. When assigning undergraduate students to courses, ratio constraints are imposed to maintain the balance among courses.

To the best of our knowledge, we are the first to formally examine ratio constraints even though a similar concept called "proportionality constraints" is introduced [26]. However, that model is fundamentally different from ours since it assumes students are partitioned into different types (e.g., minority/majority) and deals with the ratio between different types of students within a school; our model considers the ratio among schools. Also, in their model, proportionality constraints are *soft*, which can be violated to some extent; in our model, constraints are *hard* and cannot be violated. Furthermore, they do not consider strategyproofness.

In this paper, we develop a novel mechanism called Quota Reduction Deferred Acceptance (QRDA), which repeatedly applies the

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well-known Deferred Acceptance (DA) mechanism [12] by sequentially reducing artificially introduced maximum quotas. Fragiadakis and Troyan [10] use the idea of sequentially reducing maximum quotas for a different goal. In their model, students are partitioned into different types and the goal is to satisfy type-specific minimum/maximum quotas.

Note that developing a non-trivial strategyproof and fair mechanism that can handle ratio constraints is theoretically interesting/challenging. In existing works, it is shown that if constraints belong to a well-behaved class (which is called M-convex set), then a mechanism called *generalized DA*, which is based on DA, is strategyproof and fair [14, 23]. As we discuss later, ratio constraints do not belong to this class. Our result is a first step toward identifying a class beyond an M-convex set, such that we can develop a non-trivial strategyproof and fair mechanism. As well as being fair and strategyproof, we show that in terms of student welfare, QRDA outperforms a baseline mechanism called Artificial Cap Deferred Acceptance (ACDA), which uses predetermined artificial maximum quotas, both theoretically and experimentally. In terms of another desirable property called nonwastefulness (i.e., no student claims an empty seat of a more desirable school), we experimentally show that QRDA outperforms ACDA. We extend these experiments by comparing QRDA with an additional mechanism, Extended Seat Deferred Acceptance (ESDA), with similar conclusions. Finally, we propose an extended model in Section 5.

2 MODEL

A student-school matching market with ratio constraints is defined by a tuple $(S, C, X, \succ_S, \succ_C, \alpha)$.

- $S = \{s_1, \ldots, s_n\}$ is a finite set of n students.
- $C = \{c_1, \ldots, c_m\}$ is a finite set of m schools.
- $X = S \times C$ is a finite set of all possible contracts. Contract $(s, c) \in X$ means that student s is matched to school c. For $\dot{X} \subseteq X$, $\dot{X}_s$ denotes $\{(s, c) \in \dot{X} \mid c \in C\}$, and $\dot{X}_c$ denotes $\{(s, c) \in \dot{X} \mid s \in S\}$. In other words, $\dot{X}_s$ (resp. $\dot{X}_c$) denotes all contracts in $\dot{X}$ related to s (resp. c)
- $\succ_S = (\succ_{s_1}, \ldots, \succ_{s_n})$ is the profile of the student preferences, where each $\succ_s$ is a strict preference over all contracts that are related to s. For example, if s strictly prefers c over c', it is denoted by $(s, c) \succ_s (s, c')$. We sometimes write $c \succ_s c'$ instead of $(s, c) \succ_s (s, c')$.
- $\succ_C = (\succ_{c_1}, \ldots, \succ_{c_m})$ is the profile of the school preferences, where each $\succ_c$ is a strict preference over all contracts that are related to c. For example, if c strictly prefers s over s', it is denoted by $(s, c) \succ_c (s', c)$. We sometimes write $s \succ_c s'$ instead of $(s, c) \succ_c (s', c)$.
- $0 \leq \alpha \leq 1$ defines the acceptable minimum ratio between the least/most popular schools.

For $\dot{X} \subseteq X$, we define $r(\dot{X})$ as follows:

$$r(\dot{X}) = \frac{\min_{c \in C} |\dot{X}_c|}{\max_{c \in C} |\dot{X}_c|}.$$

In other words, $r(\dot{X})$ is the ratio between the numbers of students in the least/most popular schools in $\dot{X}$.

Definition 2.1 (Feasibility). For $\dot{X} \subseteq X$, $\dot{X}$ is student-feasible if $|\dot{X}_s| = 1$ for all $s \in S$. We call a student-feasible set of contracts a *matching*. $\dot{X}$ is school-feasible if $r(\dot{X}) \geq \alpha$. $\dot{X}$ is feasible if it is both student/school-feasible.

In this market, we assume all schools are acceptable to all students and vice versa.[1] To guarantee the existence of a feasible matching, ratio α must be at most $\lfloor n/m \rfloor / \lceil n/m \rceil$, since even in the most balanced matching, the most popular school has $\lceil n/m \rceil$ students and the least popular school has $\lfloor n/m \rfloor$ students. For matching $\dot{X}$, we say school c is *strictly minimum* if for all $c' \neq c$, $|\dot{X}_c| < |\dot{X}_{c'}|$ holds. Also, school c is *strictly maximum* if for all $c' \neq c$, $|\dot{X}_c| > |\dot{X}_{c'}|$ holds.

With a slight abuse of notation, for two matchings $\dot{X}$ and $\ddot{X}$, we denote $\dot{X}_s \succ_s \ddot{X}_s$ if $\dot{X}_s = \{x'\}$, $\ddot{X}_s = \{x''\}$, and $x' \succ_s x''$ (i.e., if student s prefers the school she obtained in $\dot{X}$ to the one in $\ddot{X}$). Furthermore, we denote $\dot{X}_s \succeq_s \ddot{X}_s$ if either $\dot{X}_s \succ_s \ddot{X}_s$ or $\dot{X}_s = \ddot{X}_s$.

A mechanism φ is a function that takes a profile of student preferences $\succ_S$ as input[2] and returns the set of contracts. Let $\varphi_s(\succ_S)$ denote $\dot{X}_s$, where $\varphi(\succ_S) = \dot{X}$. Let $\succ_{S \setminus \{s\}}$ denote a profile of the preferences of all students except s, and let $(\succ_s, \succ_{S \setminus \{s\}})$ denote a profile of the preferences of all students, where s's preference is $\succ_s$ and the profile of the preferences of the other students is $\succ_{S \setminus \{s\}}$.

Definition 2.2 (Strategyproofness). Mechanism φ is strategyproof if for all $s, \succ_s, \succ_{S \setminus \{s\}}$, and $\succ'_s$ (where $\succ'_s$ is an arbitrary preference of student s), $\varphi_s((\succ_s, \succ_{S \setminus \{s\}})) \succeq_s \varphi_s((\succ'_s, \succ_{S \setminus \{s\}}))$ holds.

Definition 2.3 (Fairness). In matching $\dot{X}$, where $(s, c) \in \dot{X}$, student s has justified envy toward another student s' if for some $c' \in C$, $(s, c') \succ_s (s, c)$, $(s', c') \in \dot{X}$, and $(s, c') \succ_{c'} (s', c')$ hold. Matching $\dot{X}$ is fair if no student has justified envy in $\dot{X}$. Furthermore, a mechanism is fair if it always produces a fair matching.

In other words, student s has justified envy toward student s' if she is assigned to school c', which is better for s than her current school, even though c' prefers s over s'.

Definition 2.4 (Nonwastefulness). In matching $\dot{X}$, where $(s, c) \in \dot{X}$, student s claims an empty seat of c', if $(s, c') \succ_s (s, c)$ and $(\dot{X} \setminus \{(s, c)\}) \cup \{(s, c')\}$ is school-feasible. Matching $\dot{X}$ is nonwasteful if no student claims an empty seat in $\dot{X}$. Furthermore, a mechanism is nonwasteful if it always produces a nonwasteful matching.

In other words, s claims an empty seat of c', which is better than her current school c, if moving her from c to c' does not violate ratio constraints.

In standard matching terminology, fairness and nonwastefulness are combined to form a notion called *stability* [10, 13, 14]. However, in our setting, fairness and nonwastefulness are incompatible as Theorem 2.6 shows. Thus, in this paper, we divide the notion into fairness and nonwastefulness, and focus on finding a fair outcome, while reducing wastefulness as much as possible. Dividing stability into fairness and nonwastefulness is commonly used when dealing with distributional constraints [9, 14, 20, 23].

We use the following example to show that, when considering ratio constraints, fairness and nonwastefulness are incompatible in general.

[1] Even though this is a strong assumption, we require it to guarantee the existence of a feasible matching. The same assumption is widely used in existing works [9, 13, 14].

[2] We assume the profile of school preferences $\succ_C$ is publicly known and concentrate on strategyproofness for students (the proposing side). Thus, we do not specify it as an input of a mechanism.

Example 2.5.
$S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, $\alpha = 1/2$,
s_1, s_3: $c_2 \succ_s c_3 \succ_s c_1$,
s_2, s_4: $c_3 \succ_s c_2 \succ_s c_1$,

c_1: $s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \succ_{c_1} s_4$,
c_2: $s_3 \succ_{c_2} s_2 \succ_{c_2} s_1 \succ_{c_2} s_4$, and
c_3: $s_4 \succ_{c_3} s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$.

Theorem 2.6. *With ratio constraints, fairness and nonwastefulness are incompatible in general.*

Proof. Assume the situation in Example 2.5. For satisfying the ratio constraints, two students must be assigned to exactly one school, and each of the other two schools must be given one student. If feasible matching is fair, it must contain (s_3, c_2) and (s_4, c_3); otherwise, either s_3 or s_4 has justified envy. Here, c_1 is the least popular school for everybody, but at least one student must be assigned to it. Assigning both s_1 and s_2 to c_1 is wasteful. Assume we assign s_1 to c_1. If we assign s_2 to c_2, s_2 claims an empty seat of c_3. If we assign s_2 to c_3, s_1 has justified envy toward s_2. Next, assume we assign s_2 to c_1. If we assign s_1 to c_3, s_1 claims an empty seat of c_2. If we assign s_1 to c_2, s_2 has justified envy toward s_1. □

For $\dot{X}$, let $\zeta(\dot{X})$ denote m-element vector $(|\dot{X}_{c_1}|, |\dot{X}_{c_2}|, \ldots, |\dot{X}_{c_m}|)$. Assume distributional constraints are defined by a set of vectors $\mathcal{V}$, i.e., $\dot{X}$ is school-feasible if $\zeta(\dot{X}) \in \mathcal{V}$. If $\mathcal{V}$ is an M-convex set (which is a discrete analogue of maximum elements of a convex set in a continuous domain),[3] then a mechanism called *generalized DA*, based on DA, is strategyproof and fair [14, 23].

Definition 2.7 (M-convex set). Let χ_i denote an m-element unit vector, where its i-th element is 1 and all other elements are 0. A set of m-element vectors $\mathcal{V} \subseteq \mathbb{N}_0^m$ forms an M-convex set, if for all $\zeta, \zeta' \in \mathcal{V}$, for all i such that $\zeta_i > \zeta'_i$, there exists $j \in \{k \in \{1, \ldots, m\} \mid \zeta_k < \zeta'_k\}$ such that $\zeta - \chi_i + \chi_j \in \mathcal{V}$ and $\zeta' + \chi_i - \chi_j \in \mathcal{V}$ hold.

The following theorem shows we cannot apply the generalized DA for ratio constraints.

Theorem 2.8. *In general, ratio constraints cannot be represented as an M-convex set.*

Proof. Assume $n = 10$, $m = 4$, and $\alpha = 1/3$. Consider two school-feasible vectors: $\zeta = (1, 3, 3, 3)$ and $\zeta' = (2, 2, 2, 4)$. For $i = 2$, we can choose either $j = 1$ or $j = 4$. For $j = 1$, $\zeta' + \chi_2 - \chi_1 = (1, 3, 2, 4)$ is not school-feasible, and for $j = 4$, $\zeta - \chi_2 + \chi_4 = (1, 2, 3, 4)$ is not school-feasible. □

3 QUOTA REDUCTION DEFERRED ACCEPTANCE MECHANISM (QRDA)

3.1 Mechanism Description

Let us first introduce the standard DA, which is a component of QRDA. A standard market is a tuple $(S, C, X, \succ_S, \succ_C, q_C)$, whose definition resembles a market with ratio constraints. The only difference is that its constraints are given as a profile of maximum quotas: $q_C = (q_c)_{c \in C}$. Matching $\dot{X}$ is school-feasible if for all $c \in C$, $|\dot{X}_c| \leq q_c$ holds. The standard DA is defined as follows:

Mechanism 1 (standard DA).

Step 1 *Each student s applies to her most preferred school according to $\succ_s$ from the schools that did not reject her so far.*

Step 2 *Each school c tentatively accepts the top q_c students from the applying students based on $\succ_c$ and rejects the rest of them (no distinction between newly applying students and already tentatively accepted students).*

Step 3 *If no student is rejected, return the current matching. Otherwise, go to **Step 1**.*

Let σ denote the sequence of schools[4] based on the round-robin order $c_1, c_2, \ldots, c_m$. Let $\sigma(k)$ denote the k-th school in σ, i.e., $\sigma(k) = c_j$, where $j = 1 + (k - 1 \mod m)$.

Let q_{max} be the largest value that satisfies the following equation:

$$\alpha \cdot q_{max} \leq \left\lfloor \frac{n - q_{max}}{m - 1} \right\rfloor. \qquad (1)$$

If t ($t > q_{max}$) students are assigned to c, there exists a school that is assigned at most $t' = \lfloor (n - t)/(m - 1) \rfloor$ students. Since q_{max} is the largest value satisfying Eq. (1), $t'/t < \alpha$ holds. Thus, any matching is infeasible where t students are assigned to c; a school accepts at most q_{max} students in a feasible matching.

The Quota Reduction Deferred Acceptance (QRDA) mechanism is defined as follows.

Mechanism 2 (QRDA).

Initialization:
For all $c \in C$, $q_c^1 \leftarrow q_{max}$, $k \leftarrow 1$.

Stage k ($k \geq 1$):

Step 1 *Run the standard DA in market $(S, C, X, \succ_S, \succ_C, q_C^k)$ and obtain matching $\dot{X}^k$.*

Step 2 *If $\dot{X}^k$ is school-feasible, then return $\dot{X}^k$.*

Step 3 *Otherwise, for school $c' = \sigma(k)$, $q_{c'}^{k+1} \leftarrow q_{c'}^k - 1$, and for $c \neq c'$, $q_c^{k+1} \leftarrow q_c^k$. Go to **Stage** $k + 1$.*

Let us illustrate the execution of QRDA in Example 2.5. We choose $q_{max} = 2$ such that Eq. (1) is satisfied. In Stage 1, s_1 and s_3 are assigned to c_2, and s_2 and s_4 are assigned to c_3. This matching is infeasible. Thus, in Stage 2, the quota of c_1 is decreased but the obtained matching is identical. In Stage 3, the quota of c_2 is decreased. Then s_2 is assigned to c_1, s_3 is assigned to c_2, and s_1 and s_4 are assigned to c_3. This matching is feasible and fair.

3.2 Mechanism Properties

Theorem 3.1. *QRDA returns a feasible and fair matching.*

Proof. QRDA terminates when it obtains a feasible matching. Assume QRDA continues to reduce the maximum quotas of the schools without obtaining a feasible matching. Eventually, there will be stage k such that the following conditions hold: $\sum_{c \in C} q_c^k = n$

[3] To be precise, this condition [14, 23] is an M$^\natural$-convex set, which is a generalization of an M-convex set. When all students must be assigned to schools, it becomes equivalent to an M-convex set. Their results are built upon various earlier works [7, 8, 11, 18].

[4] For simplicity, we assume σ is based on a fixed round-robin order, but the results in this paper hold for any balanced sequence σ, i.e., for each $\ell \in \mathbb{N}_0$, $\sigma(m\ell + 1)$, $\sigma(m\ell + 2)$, ..., $\sigma(m\ell + m)$ is a permutation of c_1, c_2, ..., c_m. This requirement is necessary to guarantee the strategyproofness.

and for all $c \in C$, $\lfloor n/m \rfloor \le q_c \le \lceil n/m \rceil$. In this stage k, for obtained matching $\dot{X}$, $r(\dot{X}) = \lfloor n/m \rfloor / \lceil n/m \rceil \ge \alpha$ holds. Thus, $\dot{X}$ is feasible. Therefore, QRDA must terminate at stage k' ($k' \le k$). Also, $\dot{X}^{k'}$ is identical to the matching obtained by the standard DA for the market $(S, C, X, \succ_S, \succ_C, q_C^{k'})$. Since DA is fair [12], $\dot{X}^{k'}$ must be fair. □

From the proof of Theorem 3.1, we immediately obtain the following lemma.

Lemma 3.2. *During the execution of QRDA, the maximum quota of any school is at least* $\lfloor n/m \rfloor$.

QRDA's strategyproofness is not trivial at all. Since schools' quotas are decreasing, a student might have an incentive to terminate the mechanism early to secure the seat of a school, which might not be available in later stages. To show such manipulations are useless, we utilize several lemmas.

Lemma 3.3. *Assume in stage k of QRDA that obtained matching $\dot{X}^k$ is infeasible, and school c' is strictly maximum, i.e., for all $c \neq c'$, $|\dot{X}^k_{c'}| > |\dot{X}^k_c|$ holds. Let t denote $|\dot{X}^k_{c'}| - 1$. In stage $k+1$, if the number of students assigned to c' is decreased (due to the reduction of $q_{c'}$) to t, and the number of students assigned to another school c'' is increased, i.e., $|\dot{X}^{k+1}_{c''}| = |\dot{X}^k_{c''}| + 1$, then one of the following two cases must be true:*
(a) $|\dot{X}^{k+1}_{c''}| = t + 1$ holds, and c'' is strictly maximum.
(b) $|\dot{X}^{k+1}_{c''}| \le t$ holds, and for each school c, the number of assigned students is at most t.

Proof. If the number of students assigned to c'' in stage k is t, then the first condition of case (a) holds. Furthermore, for each school c (where $c \neq c', c''$), $|\dot{X}^{k+1}_c| = |\dot{X}^k_c| < t+1$ holds. Thus, c'' is strictly maximum. If the number of students assigned to c'' in stage k is strictly smaller than t, then the first condition of case (b) holds. Also, for each school c (where $c \neq c', c''$), $|\dot{X}^{k+1}_c| = |\dot{X}^k_c| < t + 1$ holds. □

When analyzing the effect of manipulations of student s in stage k, it is convenient to assume in stage k (and thereafter) that a matching is obtained as follows. First, all students except s are tentatively matched to schools by DA with respect to q_C^k. Continue the DA procedure by adding s to the current tentative matching. The matching obtained in this way is identical to the matching obtained by applying DA when all the students enter the market simultaneously [4]. If the matching satisfies the ratio constraints, QRDA terminates. Otherwise, the quota of school $c = \sigma(k)$ is reduced and the mechanism proceeds to stage $k + 1$. In the current tentative matching, if school c is accepting q_c^k students, the least preferred student s' is rejected. Then s' applies to the next school, and so on. Otherwise, the quota of school $c = \sigma(k + 1)$ is reduced, and the mechanism proceeds to stage $k + 2$, and so on.

In the above procedure, when s enters the market, she first applies to some school c. If c accepts all the students applying to it, then the current stage terminates. Otherwise, c rejects one student, s' (s' can be s or another student), who applies to the next school, and so on. We call such a sequence of applications and rejections a *rejection chain*. More formally, let $C_s = (c, c', \ldots, c'')$ denote a list of schools to which student s is going to apply, i.e., s applies first

Stage	Step	Action
k	1	Student s applies to school c_1.
	2	School c_1 rejects student s_1.
	3	Student s_1 applies to school c_2 (and is accepted).
$k+1$	1	School c_3 rejects student s_2 (due to its quota reduction).
	2	Student s_2 applies to school c_4.
		...

Table 1: Example of rejection chain

to c; if rejected, she applies to c', and so on. C_s is called a *scenario*, which does not need to be exhaustive. Assume s enters the market with scenario C_s. Define $\mathcal{R}(C_s)$ as the rejection chain of C_s. It starts when s applies to the first school in C_s and describes the sequence of applications and rejections until s is rejected by the last school in C_s, or the mechanism terminates.

Table 1 shows an example of a rejection chain. For rejection chains, the following property holds, inspired by the original Scenario Lemma [4], which proves the strategyproofness of the standard DA in a one-to-one matching.

Lemma 3.4 (Scenario Lemma). *Consider two scenarios, C_s and C'_s, of student s starting from the same stage of QRDA. If (1) each school that appears in C'_s also appears in C_s (the order is immaterial), (2) student s applies to all the schools in C_s, and (3) all the actions of $\mathcal{R}(C'_s)$ happen in the same stage, then all the actions in $\mathcal{R}(C'_s)$ also happen in $\mathcal{R}(C_s)$.*

Proof. The first action in $\mathcal{R}(C'_s)$ is "student s applies to school c," where c is the first school that appears in C'_s. Since c also appears in C_s, and s applies to all the schools in C_s, $\mathcal{R}(C_s)$ also includes this action. For an inductive step, assume the first $i-1$ actions in $\mathcal{R}(C'_s)$ also happen in $\mathcal{R}(C_s)$, and consider the i-th action of $\mathcal{R}(C'_s)$. The i-th action in $\mathcal{R}(C'_s)$ must be either (i) "student s' applies to school c'" or (ii) "school c' rejects student s'."

In case (i) with $s' = s$, since school c' must appear in C_s and s applies to all the schools in C_s, $\mathcal{R}(C_s)$ also includes this action. In case (i) with $s' \neq s$, there must be a previous action, "school c'' rejects student s'," in $\mathcal{R}(C'_s)$. From the inductive assumption, this action also happens in $\mathcal{R}(C_s)$. Thus, the action "student s' applies to school c'" also happens in $\mathcal{R}(C_s)$.

In case (ii), let $S'_{c'}$ be the set of students who applied to c' before the i-th action in $\mathcal{R}(C'_s)$, and let $S_{c'}$ be the set of all the students applying to c' until all actions in $\mathcal{R}(C_s)$ are executed. Here, $S'_{c'} \subseteq S_{c'}$ holds since every application before the i-th action in $\mathcal{R}(C'_s)$ also appears in $\mathcal{R}(C_s)$. Since in the i-th action of $\mathcal{R}(C'_s)$, s' is rejected by school c', she is not among c''s most preferred $q^k_{c'}$ students in set $S'_{c'}$. Since the quotas of schools are non-increasing as QRDA continues, in some stage k' ($k' \ge k$), student s' must not be among the most preferred $q^{k'}_{c'}$ students in $S_{c'}$. Thus, the action "school c' rejects student s'" eventually occurs in $\mathcal{R}(C_s)$. □

Now we are ready to prove our main theorem.

Theorem 3.5. *QRDA is strategyproof.*

PROOF. Assume student s is assigned to a better school when she misreports. Without loss of generality, we assume her true preference is $c_1 \succ_s c_2 \succ_s \cdots \succ_s c_m$, and s is assigned to school c_j in stage k when misreporting while assigned to c_i in stage k' under her true preference, where $c_j \succ_s c_i$. If $k' \leq k$, s cannot benefit from misreporting, since (i) the standard DA is strategyproof [4, 27] and (ii) the standard DA satisfies a property called resource monotonicity, i.e., DA's outcome is weakly less preferred by each student if the quotas decrease [6]. Thus, $k < k'$ must hold.

Let C_s be $(c_1, c_2, \ldots, c_{i-1})$, which is based on the true preference of s and truncated before c_i. Then the last action in $\mathcal{R}(C_s)$ must be "school c_{i-1} rejects student s." On the other hand, let C'_s be a sequence of schools to which s applies when s misreports, in which the last school is c_j. For C'_s, the following two cases are possible: (i) c_j is the least preferred school for s within C'_s based on her true preference $\succ_s$ or (ii) C'_s contains at least one school that is less desired than c_j based on $\succ_s$.

In case (i), each school c that appears in C'_s also appears in C_s. Thus, we can apply Lemma 3.4. Let $\dot{X}$ denote the set of contracts obtained by assigning all students except s by DA with respect to q^k_C. Assume that when s enters the market with C_s, she is assigned to school c' (where $c' \neq c_j$) and infeasible matching $\dot{X}^k$ is obtained. Also, when s enters the market with C'_s, she is assigned to c_j and feasible matching $\ddot{X}^k$ is obtained. From these facts, at least one of the following four cases (which are not necessarily mutually exclusive) must be true:

(1) c_j is strictly minimum in $\dot{X}$, i.e., $|\dot{X}_{c_j}| < |\dot{X}_c|$ holds for each $c \neq c_j$.

(2) $|\dot{X}_{c_j}| = q^k_{c_j}$ and a student is rejected when student s applies to school c_j in scenario C'_s. Then student s' ($s' \neq s$) is eventually assigned to c'' ($c'' \neq c_j$), such that c'' is strictly minimum in $\dot{X}$.

(3) c' is strictly maximum in $\dot{X}^k$, i.e., $|\dot{X}^k_{c'}| = |\dot{X}_{c'}| + 1 > |\dot{X}^k_c| = |\dot{X}_c|$ holds for each $c \neq c'$.

(4) $|\dot{X}_{c'}| = q^k_{c'}$ and a student is rejected when s applies to school c' in scenario C_s. Then student s'' ($s'' \neq s$) is eventually assigned to $\tilde{c}$ ($\tilde{c} \neq c'$), such that $\tilde{c}$ is strictly maximum in $\dot{X}^k$, i.e., $|\dot{X}^k_{\tilde{c}}| = |\dot{X}_{\tilde{c}}| + 1 > |\dot{X}^k_c| = |\dot{X}_c|$ holds for each $c \neq \tilde{c}$.

For case (1), the last action in $\mathcal{R}(C'_s)$ must be "student s applies to school c_j," which also appears in $\mathcal{R}(C_s)$. Assume this action occurs in stage $k'' \leq k'$.

Since c_j is strictly minimum in $\dot{X}$, we obtain $|\dot{X}_{c_j}| < \lfloor n/m \rfloor$ for the following reason. Let u denote $|\dot{X}_{c_j}|$. Then for each school $c \neq c_j$, $|\dot{X}_c| \geq u + 1$ holds. Since the total number of students in $\dot{X}$ is $n - 1$, and there are $m - 1$ schools except c_j, we obtain $(u+1)(m-1) + u \leq n - 1$. By transforming this formula, we obtain $u \leq n/m - 1$. Since $n/m - 1 < \lfloor n/m \rfloor$ holds, we obtain $u < \lfloor n/m \rfloor$.

From Lemma 3.2, since the maximum quota of each school is at least $\lfloor n/m \rfloor$, c_j can accept another student. As the mechanism continues, the number of students assigned to the most popular school in each stage never increases. Thus, when c_j accepts another student, the obtained matching is feasible, and the mechanism terminates. Therefore, in stage k'', the mechanism terminates when s applies to c_j. However, this contradicts our assumption that the last action in $\mathcal{R}(C_s)$ is "student s is rejected by school c_{i-1}."

For case (2), we can use a similar argument as case (1) and show that the mechanism terminates with a feasible matching in $\mathcal{R}(C_s)$, which contradicts our assumption.

In the rest of this proof, we assume cases (1) and (2) do not occur. For case (3), let t denote $|\dot{X}_{c'}|$. Since $\dot{X}^k$ is infeasible and $\ddot{X}^k$ is feasible, if the number of students of the most popular school becomes $t + 1$, then the matching becomes infeasible. If the number of students of that school is at most t, then the matching becomes feasible. Assume the last action in $\mathcal{R}(C'_s)$ is "student s' applies to school c_ℓ," such that $|\ddot{X}^k_{c_\ell}| = |\dot{X}_{c_\ell}| + 1$ holds. Since $\ddot{X}^k$ is feasible, $|\ddot{X}^k_{c_\ell}| = |\dot{X}_{c_\ell}| + 1 \leq t$ must hold. According to Lemma 3.4, action "student s' applies to school c_ℓ" also appears in $\mathcal{R}(C_s)$. Assume this action happens in stage k'' ($k'' \leq k'$).

Then from Lemma 3.3, case (a) continues to hold until stage k'' in $\mathcal{R}(C_s)$. Otherwise, case (b) holds and the number of assigned students for each school becomes at most t. Then the matching becomes feasible, and the mechanism terminates. Thus, the number of assigned students of c_ℓ remains $|\dot{X}_{c_\ell}| < t$. At stage k'' in $\mathcal{R}(C_s)$, case (b) must hold. The maximum quota of c_ℓ must be at least t (since before stage k'', there exists a school with $t + 1$ students). Thus, when s' applies to school c_ℓ, an available seat exists in c_ℓ, and s' will be accepted. Furthermore, every school accepts at most t students. Thus, the obtained matching is feasible, and the mechanism terminates. This contradicts the assumption that the last action in $\mathcal{R}(C_s)$ is "school c_{i-1} rejects student s."

For case (4), we can use a similar argument as case (3) and show that the mechanism terminates with a feasible matching in $\mathcal{R}(C_s)$, which contradicts our assumption.

Furthermore, for case (ii), we can create a new scenario C''_s by removing all the schools that are less desired than c_j based on $\succ_s$ from C'_s. Then if s is assigned to c_j in $\mathcal{R}(C''_s)$, we obtain the same contradiction as case (i) by comparing $\mathcal{R}(C''_s)$ and $\mathcal{R}(C_s)$. Thus, action "school c_j rejects student s" must appear in $\mathcal{R}(C''_s)$. Then by Lemma 3.4, this action also appears in $\mathcal{R}(C'_s)$, but this is also a contradiction. □

To examine the time complexity of QRDA, we assume an alternative execution of DA in each stage used in the proof of Theorem 3.5: for stage k, instead of running DA from scratch, we start from the matching obtained in stage $k - 1$, and continue the execution when a student is rejected.

THEOREM 3.6. *The time complexity of QRDA is $O(n \times m)$, assuming school-feasibility can be checked in a constant time.*

PROOF. QRDA repeatedly applies DA (Mechanism 1). Since a student is rejected by each school at most once, each step in Mechanism 1 is executed at most $n \times m$ times in total. Thus, the time complexity of QRDA is $O(n \times m)$. □

3.3 Comparison with Baseline Mechanism

To the best of our knowledge, there exists no mechanism that is fair, strategyproof, and can handle ratio constraints. One way to handle ratio constraints is to use an indirect approach, i.e., to transform ratio constraints into other types of constraints by sacrificing flexibility to some extent. In this subsection, we present an indirect approach in which ratio constraints are transformed into standard maximum quotas, i.e., artificial maximum quotas are defined such that the obtained matching by the standard DA is guaranteed to satisfy the ratio constraints. Such a mechanism is called Artificial Cap Deferred Acceptance mechanism (ACDA). ACDA is used in Japanese medical resident matching programs [20] to handle regional maximum quotas as well as a baseline mechanism in many works related to distributional constraints [9, 13, 14].

Without loss of generality, assume $q_{c_1} \leq q_{c_2} \leq \ldots \leq q_{c_m}$ holds. The following lemma holds:

Lemma 3.7. *The matching obtained by the standard DA satisfies the ratio constraints defined by α if q_C satisfies the following condition:*

$$\alpha \leq \frac{n - \sum_{i=2}^{m} q_{c_i}}{q_{c_m}}. \tag{2}$$

Proof. Assume matching $\ddot{X}$ is obtained in the following method. We first assign q_{c_m} students to c_m, $q_{c_{m-1}}$ students to c_{m-1}, and so on. Finally, $n - \sum_{i=2}^{m} q_{c_i}$ students are assigned to c_1 (or no student is assigned to c_1 if $n - \sum_{i=2}^{m} q_{c_i}$ is negative). Then for any matching $\dot{X}$ that is school-feasible in a standard market with quotas q_C, $r(\dot{X}) \geq r(\ddot{X})$ holds. Also, from Eq. (2), $r(\ddot{X}) \geq \alpha$ holds. Thus, $r(\dot{X}) \geq \alpha$ holds. □

If we knew beforehand which schools are popular/unpopular, we might be able to find q_C that satisfies Eq. (2) to maximize the student welfare. Otherwise, one simple and reasonable way for finding appropriate q_C is using σ (which is also used in QRDA). ACDA based on σ is defined as follows:

Mechanism 3 (ACDA (based on σ)).

Initialization:
For all $c \in C$, $q_c^1 \leftarrow q_{max}$, $k \leftarrow 1$.
Stage k ($k \geq 1$):
Step 1 *If q_C^k satisfies Eq. (2), then run the standard DA in market $(S, C, X, \succ_S, \succ_C, q_C^k)$ and return the obtained matching.*

Step 2 *Otherwise, for school $c' = \sigma(k)$, $q_{c'}^{k+1} \leftarrow q_{c'}^k - 1$, and for $c \neq c'$, $q_c^{k+1} \leftarrow q_c^k$. Go to **Stage** $k+1$.*

Theorem 3.8. *ACDA (based on σ) is strategyproof and returns a feasible and fair matching.*

Proof. ACDA terminates when Eq. (2) holds. Assume ACDA continues to reduce the maximum quotas since Eq. (2) does not hold. Eventually, there will be stage k such that the following conditions hold: $\sum_{c \in C} q_c^k = n$ and for all $c \in C$, $\lfloor n/m \rfloor \leq q_c \leq \lceil n/m \rceil$. In this case, $n - \sum_{i=2}^{m} q_{c_i} = \lfloor n/m \rfloor$, and $q_{c_m} = \lceil n/m \rceil$. Thus, Eq. (2) holds. Then ACDA must terminate at stage k' (where $k' \leq k$) and the obtained matching satisfies the ratio constraints. Also, the result is identical to the matching obtained by the standard DA for the market $(S, C, X, \succ_S, \succ_C, q_C^{k'})$. Since DA is fair [12], ACDA is also guaranteed to be fair. Furthermore, since stage k where ACDA terminates is determined independently from $\succ_S$ and the standard DA is strategyproof, ACDA is also strategyproof. □

Theorem 3.9. *All students weakly prefer the matching obtained by QRDA over that of ACDA (based on σ).*

Proof. If ACDA terminates at stage k, the matching obtained by the standard DA for the market $(S, C, X, \succ_S, \succ_C, q_C^k)$ satisfies the ratio constraints. Since ACDA and QRDA use the same quota reduction sequence σ, QRDA also terminates if it reaches stage k. Thus, QRDA must terminate at stage k', where $k' \leq k$. Since we have $q_c^{k'} \geq q_c^k$ for any $c \in C$ and DA satisfies resource monotonicity, as described in the first paragraph of the proof of Theorem 3.5, each student weakly prefers the matching obtained by QRDA over that of ACDA. □

Since QRDA always obtains a (weakly) better matching for students than ACDA, it is natural to assume that QRDA will be less *wasteful* than ACDA, i.e., more students claim empty seats in ACDA compared to QRDA. However, we cannot guarantee this property, i.e., Theorem 3.11 holds. For its proof, we use the following example:

Example 3.10.
$S = \{s_1, s_2, s_3, s_4, s_5\}$, $C = \{c_1, c_2, c_3, c_4\}$, $\alpha = 1/2$,
$\succ_{s_1}$: $c_1 \succ_{s_1} c_4 \succ_{s_1} c_2 \succ_{s_1} c_3$,
$\succ_{s_2}$: $c_2 \succ_{s_2} c_3 \succ_{s_2} c_1 \succ_{s_2} c_4$,
$\succ_{s_3}$: $c_1 \succ_{s_3} c_2 \succ_{s_3} c_3 \succ_{s_3} c_4$,
$\succ_{s_4}$: $c_4 \succ_{s_4} c_1 \succ_{s_4} c_3 \succ_{s_4} c_2$,
$\succ_{s_5}$: $c_1 \succ_{s_5} c_2 \succ_{s_5} c_3 \succ_{s_5} c_4$,

$\succ_{c_1}$: $s_2 \succ_{c_1} s_4 \succ_{c_1} s_1 \succ_{c_1} s_3 \succ_{c_1} s_5$,
$\succ_{c_2}$: $s_4 \succ_{c_2} s_3 \succ_{c_2} s_5 \succ_{c_2} s_1 \succ_{c_2} s_2$,
$\succ_{c_3}$: $s_1 \succ_{c_3} s_5 \succ_{c_3} s_3 \succ_{c_3} s_2 \succ_{c_3} s_4$, and
$\succ_{c_4}$: $s_5 \succ_{c_4} s_2 \succ_{c_4} s_3 \succ_{c_4} s_4 \succ_{c_4} s_1$.

Theorem 3.11. *A case exists where the number of students who claim empty seats in QRDA is larger than that of ACDA (based on σ).*

Proof. Assume the situation in Example 3.10. QRDA sets $q_{max} = 2$, which satisfies Eq. (1). In stage 1, s_1 and s_3 are assigned to c_1, s_2 and s_5 are assigned to c_2, and s_4 is assigned to c_4. Since no student is assigned to c_3, this matching is not school-feasible. Thus, q_{c_1} is reduced to 1. Then in Stage 2, the obtained feasible matching is as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_1\} & \{s_3, s_5\} & \{s_2\} & \{s_4\} \end{pmatrix}.$$

Here student s_3 claims an empty seat in school c_1, and student s_5 claims an empty seat in school c_1.

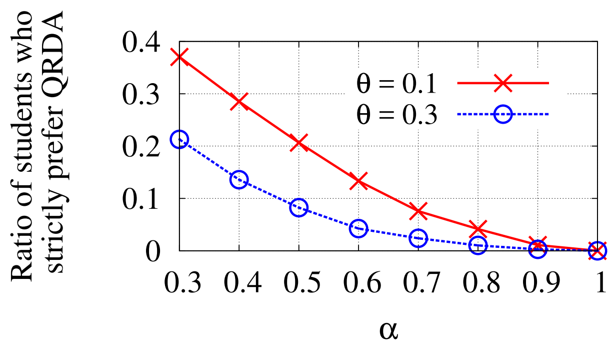
On the other hand, in ACDA, the maximum quotas of c_1, c_2, c_3 are set to 1, and the maximum quota of c_4 is set to 2. The obtained matching is as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \{s_2\} & \{s_3\} & \{s_5\} & \{s_1, s_4\} \end{pmatrix}.$$

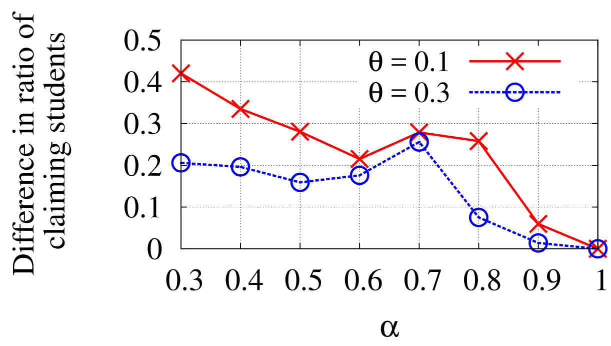
Here only student s_1 claims an empty seat (of school c_1). Other students, for example, s_3, can no longer claim an empty seat, since by moving her from c_2, the obtained matching is not school-feasible. □

4 EXPERIMENTAL EVALUATION

First, in terms of student welfare, Theorem 3.9 guarantees that students weakly prefer QRDA over ACDA. We performed a computer simulation to examine the quantitative difference. Then, in terms

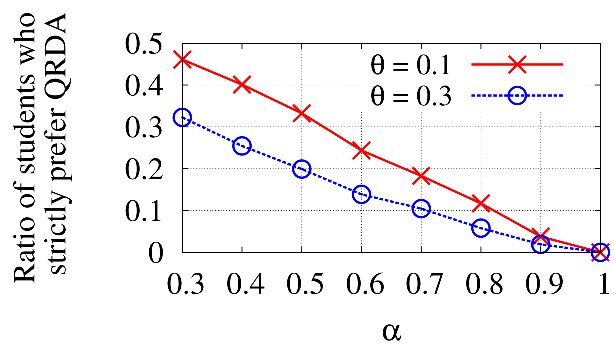


(a) Ratio of students who strictly prefer QRDA

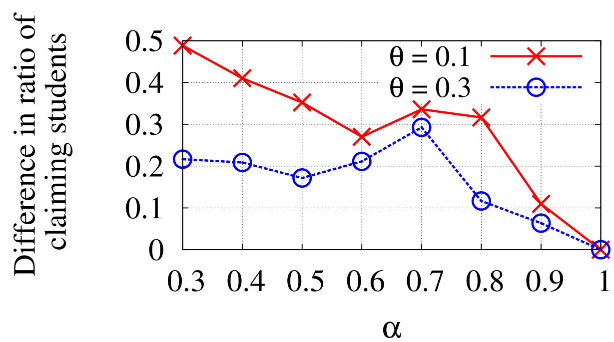


(b) Difference in ratio of claiming students

Figure 1: Comparison between QRDA and ACDA



(a) Ratio of students who strictly prefer QRDA



(b) Difference in ratio of claiming students

Figure 2: Comparison between QRDA and ESDA

of nonwastefulness, Theorem 3.11 shows that we cannot guarantee that QRDA is always better than ACDA. However, we expect that a situation like Example 3.10 is rather extreme and would not happen very often; on average, QRDA will surpass ACDA. We also confirmed this conjecture by computer simulation.

Furthermore, we examine another indirect approach in which ratio constraints are transformed into individual minimum/maximum constraints [9]. Here, each school c has its minimum quota p_c as well as its maximum quota q_c. At least p_c students are assigned to school c. Fragiadakis et al. [9] present a strategyproof and fair mechanism called Extended Seat Deferred Acceptance mechanism (ESDA). Let us examine how to transform ratio constraints into individual minimum/maximum quotas. Assume all schools have the same minimum/maximum quotas $\hat{p}$ and $\hat{q}$. Then, it is clear that if $\hat{p}/\hat{q} \geq \alpha$ holds, the ratio constraints are satisfied. Then, the next question is how to determine $\hat{q}$ and $\hat{p}$ appropriately. We use the following method: choose $\hat{q}$ and $\hat{p} = \lceil \alpha \cdot \hat{q} \rceil$ as the maximum values that satisfy $n \geq \hat{q} + (m-1)\hat{p}$. In other words, we choose $\hat{q}$ to the largest value such that the ratio constraints are satisfied and there exist enough students to satisfy minimum quotas $\hat{p}$. By choosing a large maximum quota, we can allocate more students to popular schools.[5]

[5] We tried several alternative methods for choosing $\hat{q}$ and $\hat{p}$ but the obtained results were similar to the current method.

We considered a market with $n = 800$ students and $m = 20$ schools and generated student preferences with the Mallows model [3, 25, 31]. We drew strict preference $\succ_s$ of student s whose probability is expressed as:

$$\Pr(\succ_s) = \frac{\exp(-\theta \cdot d(\succ_s, \succ_{\hat{s}}))}{\sum_{\succ'_s} \exp(-\theta \cdot d(\succ'_s, \succ_{\hat{s}}))}.$$

Here $\theta \in \mathbb{R}$ denotes a spread parameter, $\succ_{\hat{s}}$ is a central preference (uniformly randomly chosen from all possible preferences in our experiment), and $d(\succ_s, \succ_{\hat{s}})$ represents the Kendall tau distance between $\succ_s$ and $\succ_{\hat{s}}$. The distance is measured by the number of ordered pairs in $\succ_s$ that are inconsistent with those in $\succ_{\hat{s}}$. When $\theta = 0$, it becomes identical to the uniform distribution and converges to $\Pr(\succ_{\hat{s}})$ as θ increases. The priority ranking of each school c is drawn uniformly at random. We created 100 problem instances for each parameter setting.

In Fig. 1 (a), we show the ratio of students who strictly prefer QRDA over ACDA depending on α. Due to Theorem 3.9, no student strictly prefers ACDA. Thus, the remainder are indifferent. We set θ to 0.1 and 0.3 (Fig. 1 (a)). When $\alpha = 0.3$ and $\theta = 0.1$, approximately 38% of the students strictly prefer QRDA's outcome. When $\alpha = 0.7$ and $\theta = 0.1$, approximately 8% of the students strictly prefer QRDA's outcome. We expect that policymakers will prefer QRDA over ACDA since it is never worse than ACDA and a non-negligible amount of students strictly prefer QRDA. As α becomes smaller, i.e., the set of school-feasible matchings expands, more students strictly prefer QRDA; there is more room for improvement in QRDA compared to ACDA. When $\theta = 0.3$, students' preferences are more similar and the competition among them becomes more severe. In such a case, the improvement obtained by QRDA is smaller than where student preferences are more diverse (i.e., $\theta = 0.1$).

To show that QRDA is less *wasteful* than ACDA, we measured the ratio of students who claim empty seats in both mechanisms and show the difference, i.e., by plotting $(|S_{\mathrm{ACDA}}| - |S_{\mathrm{QRDA}}|)/n$, where S_{ACDA} (resp. S_{QRDA}) is the set of students who claimed empty seats in ACDA (resp. QRDA). If this value is positive, it means more students claimed empty seats in ACDA compared to QRDA. We illustrate the results in Fig. 1 (b). For all the instances that we generated, the number of claiming students in QRDA is weakly smaller than that of ACDA. Similar to Fig. 1 (a), as α becomes smaller, i.e., the set of school-feasible matchings expands, the difference becomes larger; there is more possibility to improve the matching using QRDA compared to ACDA, but the trend is less definite compared to Fig. 1 (a).

In Fig. 2 (a), we show the ratio of students who strictly prefer QRDA over ESDA depending on α. We cannot theoretically guarantee that students always weakly prefer QRDA over ESDA. However, for all the instances that we generated, all students weakly prefer QRDA over ESDA. The trend is similar to Fig. 1 (a). Actually, in terms of students welfare, ESDA is worse than ACDA. Indeed, to satisfy minimum quotas, many students are assigned to less preferred schools.

In Fig. 2 (b), we show the difference in the ratio of claiming students between QRDA and ESDA in a similar fashion as Fig. 1 (b). For all the instances that we generated, the number of claiming students in QRDA is weakly smaller than that of ESDA. The trend is similar to Fig. 1 (b).

5 EXTENDED MODEL WITH MINIMUM QUOTAS

The model presented in Section 2 would be appropriate when all schools are about the same size. We can extend the model to the case where schools are of different sizes. Let us assume each school c has its minimum quota p_c, i.e., c must be allocated at least p_c students. Each school has its own minimum quotas, i.e., p_c can vary according to the size of c. Then, we enforce ratio constraints for the number of students assigned beyond p_c.

Formally, the extended model is defined by a tuple $(S, C, X, \succ_S, \succ_C, p_C, \alpha)$. Here, $p_C = (p_c)_{c \in C}$ is a profile of minimum quotas. We assume $\sum_{c \in C} p_c + m \leq n$, i.e., each school can be assigned at least one student beyond its minimum quota. Furthermore, we assume α must be at most $\lfloor (n - \sum_{c \in C} p_c)/m \rfloor / \lceil (n - \sum_{c \in C} p_c)/m \rceil$ to guarantee the existence of a feasible matching. For $\dot{X} \subseteq X$, let us (re-)define $r(\dot{X})$ as follows:

$$r(\dot{X}) = \frac{\min_{c \in C}(|\dot{X}_c| - p_c)}{\max_{c \in C}(|\dot{X}_c| - p_c)}.$$

$\dot{X}$ is school-feasible if $r(\dot{X}) \geq \alpha$ and $|\dot{X}_c| > p_c$ for all $c \in C$.

Let $\hat{n}$ denote $n - \sum_{c \in C} p_c$, i.e., the number of students beyond the sum of minimum quotas. Then, let $\hat{q}_{max}$ be the largest value satisfying the following formula:

$$\alpha \cdot \hat{q}_{max} \leq \left\lfloor \frac{\hat{n} - \hat{q}_{max}}{m - 1} \right\rfloor.$$

We can apply the same mechanism as Mechanism 2, where q_c^1 is initialized to $\hat{q}_{max} + p_c$. In a similar way to Theorems 3.1 and 3.5, we can show that this mechanism returns a feasible and fair matching, and it is strategyproof.

6 CONCLUSIONS

This paper introduced ratio constraints, which explicitly specify the required balance among schools in two-sided matching. Since they do not belong to a known well-behaved class of constraints (i.e., an M-convex set), we cannot use a general mechanism based on DA. We developed a fair and strategyproof mechanism called QRDA and showed that in terms of student welfare, it theoretically outperforms ACDA. Furthermore, we experimentally showed that QRDA is better than ACDA and ESDA in terms of student welfare and nonwastefulness.

Future works will: (i) clarify the class of constraints (broader than an M-convex set) that can be handled by QRDA (or its variant); and (ii) generalize our model such that schools are divided into different types (e.g., small/large schools) and different α values are imposed for different combinations of types (e.g., within small/large schools, α should be 0.9, while between small and large schools, α should be 0.3).

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