Sequential Allocation Rules are Separable: Refuting a Conjecture on Scoring-Based Allocation of Indivisible Goods

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ABSTRACT

Baumeister et al. [2] introduced scoring allocation correspondences and rules, parameterized by an aggregation function $\star$ (such as $+$ and min) and a scoring vector s. Among the properties they studied is *separability*, a.k.a. *consistency* [15], a central property important in many social decision contexts. Baumeister et al. [2] show that some common scoring allocation rules fail to be separable and conjecture that "(perhaps under mild conditions on s and $\star$), no positional scoring allocation rule is separable." We refute this conjecture by showing that (1) the family of sequential allocation rules—an elicitation-free protocol for allocating indivisible goods based on picking sequences [9]—*is* separable for each coherent collection of picking sequences, and (2) every sequential allocation rule can be expressed as a scoring allocation rule for a suitable choice of scoring vector and social welfare ordering.

KEYWORDS

computational social choice; fair division; separability; sequential allocation rule; scoring allocation rule

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1 INTRODUCTION

In many applications of artificial intelligence, an important task is to allocate indivisible goods to a number of agents. This problem has been studied both from an economic and a computational perspective, and we refer to the book by Moulin [11] and to recent book chapters and surveys [3, 10, 13] for an overview.

Any allocation procedure takes as its input the preferences of the agents over the goods. There are many conceivable ways of expressing such preferences, ranging from attractively simple (but imprecise) approval schemes, where agents merely declare whether or not they like each good, to very precise (but complex) cardinal preferences, where agents assign an exact numerical value to each good. Ordinal preferences, where agents rank the items in a linear order, represent a reasonable and frequently employed compromise between the two extremes.

When it comes to concrete allocation procedures based on ordinal preferences, two approaches stand out in the literature: The first is based on the simple idea that agents take turns picking their favorite items among those currently still available, which was first formalized as the family of *sequential allocation rules* by Kohler and Chandrasekaran [9] and studied later on by many authors, e.g., [1, 2, 4, 7]; the second, inspired by scoring rules from voting theory, was first employed by Brams et al. [5, 6], and then defined and studied in more generality by Baumeister et al. [2] and Nguyen et al. [12] as the family of *scoring allocation correspondences*, parameterized by a scoring vector s (such as Borda, k-approval, or lexicographic scoring) and a social welfare aggregation function $\star$ (such as, typically, utilitarian and egalitarian social welfare, expressed by $+$ and (lexi)min).

In both of these families, the most common examples are naturally defined for any number of agents and goods, raising the question of whether these procedures yield consistent results when passing to subsets of agents. This is a central property, which can be studied in the context of many social decision problems (see Thomson [15] for an extensive survey). Baumeister et al. [2] call this property *separability* in the context of allocation rules. Perhaps a bit surprisingly, they show that many common scoring allocation rules (using, e.g., Borda or lexicographic scoring with utilitarian or egalitarian social welfare) defy separability, and they conjecture that "(perhaps under mild conditions on s and $\star$), no positional scoring allocation rule is separable." In fact, adding to their examples [2], we show that separability fails also for k-approval scoring with utilitarian or egalitarian social welfare (Proposition 3.6).

Our main results, however, refute the above conjecture of Baumeister et al. [2]. Namely, we show:

(1) Sequential allocation rules commonly do satisfy separability. More precisely, we identify a natural condition on picking sequences (satisfied by the most commonly employed sequences), which ensures separability of the corresponding sequential allocation rule (Theorem 3.14).
(2) While this might at first seem to set up a stark dichotomy between the two large classes of allocation rules, quite the opposite is the case: Sequential allocation rules form a central, well-studied subclass of the scoring allocation rules (Theorem 3.10).

2 PRELIMINARIES

In this section we will properly define the concepts that we sketched in the Introduction.

First we define the basic notions of ordinal preference profiles and the allocation rules based on them:

Definition 2.1. Let $n \geq 2$ be a natural number and set $N = \{1, \ldots, n\}$ (called the set of *agents*). Furthermore, let G be a finite

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set (called the set of *goods* or *items*). An *allocation of G to N* is a tuple $\pi = (\pi_1, \dots, \pi_n)$, with $\pi_i \subseteq G$ for $i \in \{1, \dots, n\}$, such that the π_i form a partition of G, i.e., $\pi_1 \cup \dots \cup \pi_n = G$ and $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$. The set of such allocations will be denoted by $\Pi(G, n)$. By a *(singleton-based) preference on G* we mean a strict total order, i.e., a relation $>$ that is *transitive* (if $a > b$ and $b > c$ then $a > c$) and *trichotomous* (exactly one of $a > b$, $a = b$, and $b > a$ holds) on G. The set of preferences on G will be denoted by $\mathscr{P}(G)$.

An n-tuple $(>_1, \dots, >_n) \in \mathscr{P}(G)^n$ of such preferences will be called a *(singleton-based) preference profile of N over G*.

A map $\mathscr{P}(G)^n \to \Pi(G, N)$ assigning to each preference profile of N over G an allocation of G to N is called a *(singleton-based) allocation rule*.

More generally, a map $\mathscr{P}(G)^n \to 2^{\Pi(G,n)} \setminus \{\emptyset\}$, assigning to each preference profile a nonempty subset of allocations, is called a *(singleton-based) allocation correspondence*.

The scoring allocation correspondences that we will study proceed in three steps:

(1) Employing a *scoring vector*, derive from the preferences a utility vector for each possible allocation π, that specifies each agent's individual utility for the allocation π.
(2) Aggregate the individual utilities using an *aggregation function*, yielding a single collective utility (or social welfare) of the outcome π.
(3) Pick the outcome(s) that maximize(s) collective utility.

As an additional step, we can specify a way to break ties in order to make sure that there is always a single winning allocation, if needed, which yields a scoring allocation rule.

Let us describe the first step of constructing a vector of individual utilities for each possible allocation:

Definition 2.2. A *scoring vector* is any vector $s = (s_1, \dots, s_m) \in \mathbb{Q}^m_{\geq 0}$ of rational numbers with $s_1 \geq s_2 \geq \dots \geq s_m \geq 0$ and $s_1 > 0$. Given a preference $>$ over G and a good $g \in G$, define the *rank of g with respect to* $>$ as $\text{rank}(g, >) = |\{g' \in G \mid g' > g\}| + 1$. Given a scoring vector $s = (s_1, \dots, s_m)$ and a total order $>$ over G, we define the utility of a bundle $X \subseteq G$ according to $>$ and s as

$$u_{>,s}(X) = \sum_{g \in X} s_{\text{rank}(g,>)}.$$

We can now define the utility vector of any given allocation $\pi = (\pi_1, \dots, \pi_n)$ with respect to a preference profile $P = (>_1, \dots, >_n)$ over G and a scoring vector s as follows:

$$u_{P,s}(\pi) = (u_{>_1,s}(\pi_1), \dots, u_{>_n,s}(\pi_n)) \in \mathbb{Q}^n_{\geq 0}.$$

It is best to think of the $u_{>_i,s}(\pi_i)$ as approximations to the "real utilities" of the agents. We can imagine that each agent has some private way of rating sets of items, but (for reasons of economy of data or because we might consider it an undue burden on the agents to come up with a rating for all sets) we only ask for a ranking of individual items. Then we use the scoring vector to "reconstruct" some plausible rating on all sets. From this point of view the $u_{>_i,s}(\pi_i)$ are a stand-in or a proxy for the unknown actual utilities.

Before we go through a detailed example, let us define some commonly used scoring vectors:

Definition 2.3. For any $m \geq 1$, the vectors

$$\text{borda} = (m, m-1, \dots, 2, 1) \text{ and } \text{lex} = (2^{m-1}, \dots, 2^1, 2^0)$$

are called the *Borda* scoring vector and the *lexicographic* scoring vector, respectively. For any given scoring vector $s \in \mathbb{Q}^m_{\geq 0}$ and $M > \sum_{i=1}^m s_i$, we set

$$(s, M)\text{-qi} = \left(1 + \frac{s_1}{M}, 1 + \frac{s_2}{M}, \dots, 1 + \frac{s_m}{M}\right),$$

and call this the (s, M)*-quasi-indifference* scoring vector. Often, the particular choice of M is immaterial and (s, M)-qi is then simply denoted by s-qi.

For $m \geq k \geq 1$, the vector $k\text{-app} = (1, \dots, 1, 0, \dots, 0)$ with ones in the first k of m entries is called the k*-approval* scoring vector.

Example 2.4. Consider three preferences over a set of nine goods, $G = \{a, b, c, d, e, f, g, h, i\}$, given by:

$$a >_1 b >_1 c >_1 d >_1 e >_1 f >_1 g >_1 h >_1 i,$$
$$b >_2 a >_2 f >_2 i >_2 g >_2 d >_2 c >_2 h >_2 e,$$
$$d >_3 a >_3 b >_3 f >_3 e >_3 i >_3 h >_3 c >_3 g.$$

Then $P = (>_1, >_2, >_3) \in \mathscr{P}(G)^3$ is a preference profile of three agents over G. For conciseness and legibility, we will from here on write concrete examples in the following short form:

$$P = (abcdefghi,\ bafigdche,\ dabfeihcg).$$

Now consider the bundle $X = \{b, h, i\}$. The ranks of these three items with respect to the first agent's preference are

$$\text{rank}(b, >_1) = 2,\ \text{rank}(h, >_1) = 8,\ \text{and}\ \text{rank}(i, >_1) = 9.$$

For a scoring vector s, we can then compute the utility of X with respect to $>_1$ as

$$u_{>_1,s}(X) = s_2 + s_8 + s_9.$$

Consider these four scoring vectors:

$$\text{borda} = (9, 8, 7, 6, 5, 4, 3, 2, 1),$$
$$\text{lex} = (256, 128, 64, 32, 16, 8, 4, 2, 1),$$
$$\text{borda-qi} = (1.09, 1.08, 1.07, 1.06, 1.05, 1.04, 1.03, 1.02, 1.01),$$
$$\text{3-app} = (1, 1, 1, 0, 0, 0, 0, 0, 0).$$

We then have $u_{>_1,\text{borda}}(X) = 8 + 2 + 1 = 11$, $u_{>_1,\text{lex}}(X) = 128 + 2 + 1 = 131$, $u_{>_1,\text{borda-qi}}(X) = 1.08 + 1.02 + 1.01 = 3.11$, and $u_{>_1,\text{3-app}}(X) = 1 + 0 + 0 = 1$.

Consider allocation $\pi = (\{b, h, i\}, \{a, c, f, \}, \{d, e, g\})$ (or $\pi = (bhi, acf, deg)$, for short) of G to N. Just like the utility of $\pi_1 = X$ for $>_1$, we can calculate that of $\pi_2 = \{a, c, f\}$ for $>_2$ and of $\pi_3 = \{d, e, g\}$ for $>_3$, obtaining the utility vector of π with respect to P and our four scoring vectors:

$$u_{P,\text{borda}}(\pi) = (11, 18, 15), \qquad u_{P,\text{lex}}(\pi) = (131, 196, 273),$$
$$u_{P,\text{borda-qi}}(\pi) = (3.11, 3.18, 3.15), \quad u_{P,\text{3-app}}(\pi) = (1, 2, 1).$$

A good allocation rule should choose a "best" allocation in some sense. Looking at $\pi' = (ce, dfgi, abh)$, we have

$$u_{P,\text{borda}}(\pi') = (12, 22, 18), \qquad u_{P,\text{lex}}(\pi') = (80, 120, 196),$$
$$u_{P,\text{borda-qi}}(\pi') = (2.12, 4.22, 3.18), \quad u_{P,\text{3-app}}(\pi') = (1, 1, 2).$$

Comparing the utility vectors for π and π', note that different choices of scoring vectors can give completely opposite assessments:

Judging by the utilities derived from Borda scoring, all agents would seem to prefer π' to π, whereas lexicographic scoring suggests that all agents should prefer π to π'.

To know what a "best" allocation is, we need a way of comparing utility vectors.

Definition 2.5. Let $N = \{1, \ldots, n\}$ be a set of agents, G a set of m goods, and $s \in \mathbb{Q}^m_{\geq 0}$ a scoring vector. A weak order (i.e., a transitive, reflexive, and complete relation) $\succsim$ on $\mathbb{Q}^n_{\geq 0}$ is called a *social welfare ordering*. Given such an ordering, we define a function $F_{s,\succsim}\colon \mathscr{P}(G)^n \to 2^{\Pi(G,n)} \setminus \{\emptyset\}$ by prescribing that $\pi \in F_{s,\succsim}(P)$ if and only if the utility vector associated to π (with respect to the given preference profile P and scoring vector s) is maximal under the order $\succsim$, i.e.,

$$\pi \in F_{s,\succsim}(P) \Longleftrightarrow \forall \pi' \in \Pi(G,n) : u_{P,s}(\pi) \succsim u_{P,s}(\pi').$$

We sometimes call such a π a *winning allocation*. For short, we may also write

$$F_{s,\succsim}(P) = \underset{\pi \in \Pi(G,n)}{\arg\max}{}^{\succsim} u_{P,s}(\pi).$$

Such an $F_{s,\succsim}$ is called a *scoring allocation correspondence*.

Since the set $\Pi(G,n)$ of allocations is finite and $\succsim$ is complete in the above definition, there must be at least one allocation $\pi \in F_{s,\succsim}(P)$, though there might be more than one. The ordering $\succsim$ will typically be induced by some function $W\colon \mathbb{Q}^n_{\geq 0} \to \mathbb{R}$ (called an *aggregation*, *collective utility*, or *social welfare function*), by letting

$$u \succsim_W v \iff W(u) \geq W(v).$$

In this case, we write $F_{s,W}$ instead of $F_{s,\succsim_W}$. Hence,

$$F_{s,W}(P) = \underset{\pi \in \Pi(G,n)}{\arg\max}\ W(u_{P,s}(\pi)).$$

Example 2.6. Two of the most common choices of social welfare functions $W\colon \mathbb{Q}^n_{\geq 0} \to \mathbb{R}$ are the *utilitarian social welfare function* that returns the sum of the agents' individual utilities and the *egalitarian* or *Rawlsian social welfare function*,[1] that returns their minimum:

$$F_{s,+}(P) = \underset{\pi \in \Pi(G,n)}{\arg\max}\ u_{>_1,s}(\pi_1) + \cdots + u_{>_n,s}(\pi_n),$$
$$F_{s,\min}(P) = \underset{\pi \in \Pi(G,n)}{\arg\max}\ \min\{u_{>_1,s}(\pi_1), \ldots, u_{>_n,s}(\pi_n)\}.$$

The min function seems like a natural choice for a social welfare function, but it does have certain drawbacks: For example, when using the ordering induced by min on $\mathbb{Q}^n_{\geq 0}$, the utility vector $(2,5,4,6)$ would not be considered preferable to $(2,3,2,4)$, even though the latter strictly increases the utilities of all agents but one (who happens to be worst-off in both vectors). In more technical terms, min fails to be monotonic, a property closely related to Pareto-optimality. This can be fixed by refining min to the *leximin social welfare ordering*, first introduced by Sen [14]:

Definition 2.7. For $u, v \in \mathbb{Q}^n$, define

$$u \succsim^{\mathrm{lm}} v \iff u^* \geq^{\mathrm{lex}} v^*,$$

where u^* denotes the vector arising from u by sorting the components in ascending order, and $\geq^{\mathrm{lex}}$ denotes the lexicographic ordering on $\mathbb{Q}^n$.

We shall denote the scoring allocation correpondences based on this ordering $\succsim^{\mathrm{lm}}$ simply by $F_{s,\mathrm{leximin}}$.

Note that $v \succsim^{\mathrm{lm}} u$ implies $\min(v) \geq \min(u)$, so we have

$$F_{s,\mathrm{leximin}}(P) \subseteq F_{s,\min}(P)$$

for all scoring vectors $s \in \mathbb{Q}^n_{\geq 0}$ and all $P \in \mathscr{P}(G)^n$.

Example 2.8. The utilitarian social welfare function and leximin social welfare ordering can be considered as the extreme cases of an entire spectrum of social welfare functions. Define the maps $W_p\colon \mathbb{Q}^n_{>0} \to \mathbb{Q}$ for $p \leq 1$ by

$$W_p(u) = \begin{cases} u_1^p + \cdots + u_n^p & \text{if } p > 0, \\ \log(u_1) + \cdots + \log(u_n) & \text{if } p = 0, \\ -u_1^p - \cdots - u_n^p & \text{if } p < 0. \end{cases}$$

This family plays a very prominent role in the study of social welfare functions. Its members can be axiomatically characterized as those increasing, continuous functions on $\mathbb{R}^n_{>0}$ satisfying independence of common scale and the Pigou–Dalton principle; see Section 3.2 in the book by Moulin [11] for details. They are also closely related to the family of power means, which have long been studied in mathematics; see, e.g., Hardy et al. [8].

Note that W_1 is simply the sum function. If we postcompose W_0 with the exponential function, we obtain the more common representation $\exp(W_0(u)) = u_1 \cdot \ldots \cdot u_n$, which is called the *Nash social welfare function*. The leximin ordering can be understood as the limit of the orders $\succsim_{W_p}$ as $p \to -\infty$ in the following sense:

$$u \succsim^{\mathrm{lm}} v \iff \exists M < 0\colon \forall p < M\colon W_p(u) \geq W_p(v).$$

While the W_p for $p > 0$ can be extended from $\mathbb{Q}^n_{>0}$ to $\mathbb{Q}^n_{\geq 0}$ (the term $u_1^p + \cdots + u_n^p$ still makes sense when one or more of the u_i is 0), the given definition of $W_p(u)$ for $p \leq 0$ does not make sense when u contains a zero component. This can pose a problem in the context of scoring allocation rules, as our utility vectors can contain zero entries. To fix this, one may define, in the case $p \leq 0$, that $W_p(u) = -\infty$ for all $u \in \mathbb{Q}^n_{\geq 0}$ such that $u_i = 0$ for some $i \in \{1, \ldots, n\}$. As a result, all the orderings $\succsim_{W_p}$ are defined on $\mathbb{Q}^n_{\geq 0}$. For $p \leq 0$, they rate any vector with a zero component worse than all vectors without zero components.

Example 2.9. Recall from Example 2.4 the preference profile

$$P = (abcdefghi,\ bafigdche,\ dabfeihcg)$$

and the two allocations

$$\pi = (bhi, acf, deg) \text{ and } \pi' = (ce, dfgi, abh).$$

All social welfare orderings we encountered rate $u_{P,\mathrm{borda}}(\pi')$ better than $u_{P,\mathrm{borda}}(\pi)$ and $u_{P,\mathrm{lex}}(\pi)$ better than $u_{P,\mathrm{lex}}(\pi')$. For borda-qi-scoring and utilitarian social welfare, we have

$$\sum_{i=1}^{3} u_{P,\text{borda-qi}}(\pi)_i = 9.44 < 9.52 = \sum_{i=1}^{3} u_{P,\text{borda-qi}}(\pi')_i,$$

but

$$\min(u_{P,\text{borda-qi}}(\pi)) = 3.11 > 2.12 = \min(u_{P,\text{borda-qi}}(\pi')),$$

[1] These terms are also sometimes used for the leximin social welfare ordering, defined below.

and thus also

$$u_{P,\text{borda-qi}}(\pi) \succ^{\text{lm}} u_{P,\text{borda-qi}}(\pi'),$$

using egalitarian social welfare. So, in this case, the two notions disagree about which allocation is preferable. Note that neither π nor π' is optimal with respect to any of our scoring vectors and social welfare functions. In fact,

$$\begin{aligned} F_{\text{borda},+}(P) &= F_{\text{lex},+}(P) = F_{\text{borda-qi},+}(P) \\ &= \{(ace, bfgi, dh), (ac, bfgi, deh)\}. \end{aligned}$$

Note also that $F_{3\text{-app},+}(P)$ consists of 729 distinct allocations, which is typical of scoring vectors with few distinct entries. Finally,

$$F_{\text{borda},\text{leximin}}(P) = F_{\text{borda-qi},\text{leximin}}(P) = \{(ace, bgi, dfh)\},$$

while

$$F_{\text{lex},\text{leximin}}(P) = \{(ac, bgi, defh)\}$$

and

$$F_{\text{lex},\min}(P) = \{(ac, bgi, defh), (ac, bghi, def), (ach, bgi, def)\}.$$

3 SEPARABILITY

In this section, we provide our main results that refute a conjecture of Baumeister et al. [2] on separability of scoring allocation rules. We start by reviewing their example that many common scoring allocation correspondences fail to be separable, then discuss the notion of weak separability, and briefly discuss two examples where separability fails for trivial reasons. Finally, we refute the conjecture of Baumeister et al. [2] by showing, first, that all sequential allocation rules are scoring allocation rules for a suitable choice of scoring vector and social welfare ordering (Theorem 3.10) and, second, that sequential allocation rules indeed are separable under mild conditions on the picking sequence (Theorem 3.14).

3.1 Reviewing an Example of Baumeister et al.

We have introduced allocation correspondences as being defined for some fixed number of agents and goods. As we saw in our examples, however, many correspondences can be defined in a uniform way for any number of agents and goods. We introduce these formally as *correspondence schemes* or *rule schemes*, though we will often de-emphasize the distinction between a correspondence and a correspondence scheme when it is not particularly relevant.

Definition 3.1. A family of allocation rules $F^{(n,G)}\colon \mathscr{P}(G)^n \to \Pi(G,n)$ for all $n \geq 1$ and all nonempty finite sets G is called an *allocation rule scheme*. Similarly, a family of allocation correspondences $F^{(n,G)} : \mathscr{P}(G)^n \to 2^{\Pi(G,n)} \setminus \{\emptyset\}$ for all $n \geq 1$ and all nonempty finite sets G is called an *allocation correspondence scheme*.

In order to define a *scoring allocation correspondence scheme*, we need a family of scoring vectors $s^{(m)} \in \mathbb{Q}^m_{\geq 0}$ for each $m \geq 1$ and a family of social welfare orderings/aggregation functions on $\mathbb{Q}^n_{\geq 0}$ for each n.

Definition 3.2. An *extended scoring vector* is a family

$$s = (s^{(m)})_{m\geq 1}$$

of scoring vectors $s^{(m)} \in \mathbb{Q}^m_{\geq 0}$ for each $m \geq 1$.

The scoring vectors in Definition 2.3 can indeed be viewed as *extended* scoring vectors, which we will often continue to denote simply by borda, lex, k-app, and s-qi. Similarly, our main examples of social welfare functions/orderings (+, min, leximin, and the social welfare functions W_p defined in Example 2.8) all actually form families, as they can all be defined on $\mathbb{Q}^n_{\geq 0}$ for all $n \geq 1$. Given an extended scoring vector $s = (s^{(m)})$ and a family of social welfare orderings $\succsim = (\succsim^{(n)})$ (or aggregation functions $W = (W^{(n)})$), $F^{(n,G)}_{s,\succsim} = F_{s^{|G|},\succsim^{(n)}}$ defines an allocation correspondence scheme.

Based on the notion of *consistency* that can be studied in many social decision problems (see, e.g., [15]), Baumeister et al. [2] define *separability* as follows.

Definition 3.3. Let $N = \{1,\ldots,n\}$ be a set of agents and G a finite set of goods. For subsets $N' \subseteq N$ and $G' \subseteq G$ and a preference profile $P = (>_1,\ldots,>_n) \in \mathscr{P}(G)^n$, we denote by $P|_{N',G'}$ the restriction of P to N' and G', i.e., the tuple with components $>_i \cap (G' \times G')$ for $i \in N'$. Similarly, for an allocation $\pi \in \Pi(G,n)$, $\pi|_{N'}$ denotes the restriction of π to N', i.e., the tuple containing only the components π_i for $i \in N'$.

(1) An allocation rule scheme F satisfies *separability* if for any preference profile P with $F(P) = \pi$ and each partition $N = N^1 \dot\cup N^2$ (the symbol $\dot\cup$ denoting disjoint union), we have $F(P|_{N^1,G^1}) = \pi|_{N^1}$ and $F(P|_{N^2,G^2}) = \pi|_{N^2}$, where $G^j = \bigcup_{i\in N^j} \pi_i$ for $j \in \{1,2\}$.
(2) An allocation correspondence scheme F satisfies *separability* if for every preference profile P, every $\pi \in F(P)$, and each partition $N = N^1 \dot\cup N^2$, we have $\pi|_{N^1} \in F(P|_{N^1,G^1})$ and $\pi|_{N^2} \in F(P|_{N^2,G^2})$ with G^j as before.

Separability answers the following question: Imagine we use an allocation rule F to distribute a set of goods among agents. Now we split the agents into two groups. Each group holds a subset of the items, allocated in some way among them. Would we have gotten the same allocations on the subsets if we had asked F to distribute each subset of items to each subset of agents in the first place?

Crucially, separability is a property that concerns the coherence between the allocation rules for different numbers of agents and items in an allocation rule scheme. Maybe somewhat surprisingly, Baumeister et al. [2] find that none of the scoring allocation correspondences for the standard choices of scoring vectors and social welfare orderings satisfy separability. For completeness, we review their appealing argument.

Example 3.4. For completeness, we review Example 3 by Baumeister et al. [2]. Consider the preference profile

$$P = (adcfhgbei,\ beahgcdfi,\ cfabidegh).$$

For any choice of strictly decreasing scoring vector s, $F_{s,+}$ simply assigns each good to an agent who ranks it highest (possibly yielding more than one winning allocation, if a good is ranked equally by several agents). Hence, $F_{s,+}(P)$ consists of only the allocation $\pi = (ad, begh, cfi)$. Restricting P to the subset $N^1 = \{1,2\}$ of agents and the goods $G^1 = \{a,b,d,e,g,h\}$ they received under π, we obtain the preference profile $P' = P|_{N^1,G^1} = (adhgbe,\ beahgd)$. But $F_{s',+}(P') = (adgh, be) \neq \pi|_{N^1,G^1}$, for any strictly decreasing scoring vector s'.

For $\star \in \{\min, \text{leximin}\}$ and any strictly decreasing scoring vector s, we have that $F_{s,\star}(P)$ consists of the unique allocation $\pi' = (adh, beg, cfi)$, because any allocation ρ in order to satisfy $\min(u_{P,s}(\rho)) \geq s_1 + s_2 + s_5 = \min(u_{P,s}(\pi'))$ must give each agent three goods, two of which must be her favorite and second-favorite and the third at least her fifth-favorite, or better. It is now easy to check that the only way to satisfy this is by having $\rho = \pi'$. Restricting to the subset $N^1 = \{1, 2\}$ of agents and their goods $G^1 = \{a, b, d, e, g, h\}$ once again yields the preference profile P' from above. Now, $F_{s',\text{leximin}}(P')$, for any strictly decreasing scoring vector s', consists of only the allocation $(adg, beh) \neq \pi'|_{N^1,G^1}$.

3.2 Weak Separability

In fact, Baumeister et al. [2] prove something slightly stronger than just the failure of separability for allocation correspondences as defined above. In the definition, it was demanded that *every* $\pi \in F(P)$ have the relevant coherence property for restrictions. Even if separability in this sense fails, it is still possible, that for *some* $\pi \in F(P)$ it holds that $\pi|_{N^1} \in F(P|_{N^1,G^1})$ and $\pi|_{N^2} \in F(P|_{N^2,G^2})$. In this case, there would still be some hope that there exists a clever tie-breaking procedure, somehow always picking one of the "good" $\pi \in F(P)$, and so yielding a separable allocation rule. But in Example 3.4, the sets of winning allocations in all cases consist of only a single element, so *none* of the $\pi \in F(P)$ have the required property (and so there is no hope that separability can be salvaged by clever tie-breaking). That is, the example shows that the treated scoring allocation correspondences fail even the following weaker condition:

Definition 3.5. Let $N = \{1, \dots, n\}$ be a set of players and G a finite set of goods. An allocation correspondence scheme F satisfies *weak separability* if for every preference profile P, there is some $\pi \in F(P)$ such that for every partition $N = N^1 \dot\cup N^2$ we have

$$\pi|_{N^1} \in F(P|_{N^1,G^1}) \text{ and } \pi|_{N^2} \in F(P|_{N^2,G^2})$$

with $G^j = \bigcup_{i \in N^j} \pi_i$ for $j \in \{1, 2\}$.

While Example 3.4 demonstrates that, for a strictly decreasing scoring vector s, $F_{s,+}$, $F_{s,\min}$, and $F_{s,\text{leximin}}$ fail even weak separability, another example given by Baumeister et al. [2] is not of this kind. In their Example 4, they consider the (extended) scoring vector plurality $= 1\text{-app} = (1, 0, \dots, 0) \in \mathbb{Q}^m_{\geq 0}$ and the preference profile $P = (abc,\ abc,\ cba)$. Then they observe that $\pi = (a, \emptyset, bc) \in F_{\text{plurality},\star}(P)$ for $\star \in \{+, \min, \text{leximin}\}$, but restricting to $N^1 = \{2, 3\}$ and their goods $G^1 = \{b, c\}$ yields the preference profile $P' = (bc, cb)$, with $\pi|_{N^1} = (\emptyset, bc) \notin \{(b, c)\} = F_{\text{plurality},\star}(P')$.

This example is somewhat unsatisfying as it relies on a "bad" choice of winning allocation π. The set of winning allocations $F_{\text{plurality},\star}(P)$ in this case also contains the allocation $\pi' = (a, b, c)$. As can be easily checked, π' *does* satisfy the separability condition: For any partition $N = N^1 \dot\cup N^2$ and G^1, G^2 as before, we have $\pi'|_{N^1} \in F_{\text{plurality},\star}(P'|_{N^1,G^1})$ and $\pi'|_{N^2} \in F_{\text{plurality},\star}(P'|_{N^2,G^2})$.

However, by suitably modifying this example, we show that, for $\star \in \{+, \text{leximin}\}$, $F_{\text{plurality},\star}$ fails even weak separability. In fact, we will show something more general.

Proposition 3.6. *Let $k \geq 1$. For $\star \in \{+, \text{leximin}\}$, $F_{k\text{-app},\star}$ is not even weakly separable.*

Proof. Consider the preference profile P for $2k + 2$ agents over $2k + 1$ goods

$$G = \{a_1, \dots, a_k, b_1, \dots, b_k, c\}$$

given as follows: Agents $1, \dots, k + 1$ share the preference

$$>_1, \dots, >_{k+1}:\ a_1\ a_2\ \cdots\ a_k\ c\ b_1\ b_2\ \cdots\ b_k,$$

while agents $k + 2, \dots, 2k + 2$ share the preference

$$>_{k+2}, \dots, >_{2k+2}:\ b_1\ b_2\ \cdots\ b_k\ c\ a_1\ a_2\ \cdots\ a_k.$$

Let us denote $A = \{1, \dots, k + 1\}$ and $B = \{k + 2, \dots, 2k + 2\}$.

Now, $F_{k\text{-app},+}(P)$ consists of all allocations that give the goods $a_1, \dots, a_k$ to agents from A and the goods $b_1, \dots, b_k$ to agents from B (and c to any agent).

Note that under any allocation, one of the agents from A and one of the agents from B will have utility 0 with respect to k-app. By giving the items $a_1, \dots, a_k$ to *distinct* agents from A and the items $b_1, \dots, b_k$ to *distinct* agents from B, we can however make sure that all but two agents have positive utilities, so $F_{k\text{-app},\text{leximin}}(P)$ contains exactly these allocations.

In particular, $F_{k\text{-app},\text{leximin}}(P) \subseteq F_{k\text{-app},+}(P)$. Let $\pi \in F_{k\text{-app},+}(P)$. As noted, at least one of the agents from A will not receive any of the goods $a_1, \dots, a_k$ under π, and we can assume without loss of generality that $k + 1$ is such an agent (hence $\pi_{k+1} = \emptyset$ or $\pi_{k+1} = \{c\}$). Similarly, assume without loss of generality that agent $2k + 2$ receives none of the items $b_1, \dots, b_k$.

Let i be the agent with $c \in \pi_i$. Distinguish two cases:

Case 1: $i \in A$. Then consider the subset of agents $N^1 = A \cup \{2k + 2\}$. Under π, these receive the goods $G^1 = \{a_1, \dots, a_k, c\}$, with agent $2k + 2$ receiving nothing. The induced preference profile $P' = P|_{N^1,G^1}$ has $k + 1$ preferences

$$>_1|_{G^1}, \dots, >_{k+1}|_{G^1}:\ a_1\ a_2\ \cdots\ a_k\ c$$

and one preference

$$>_{2k+2}|_{G^1}:\ c\ a_1\ a_2\ \cdots\ a_k.$$

The restricted allocation $\pi|_{N^1}$ gives all items from G^1 to the first $k + 1$ agents. Such an allocation cannot be in $F_{k\text{-app},\star}(P')$, since the allocation $\rho = (a_1, a_2, \dots, a_k, \emptyset, c)$ is superior to $\pi|_{N^1}$, both in the leximin ordering and as measured by utilitarian social welfare.

Case 2: $i \in B$. Consider the subset $N^1 = \{k + 1\} \cup B$ of agents. Under π these receive the goods $G^1 = \{b_1, \dots, b_k, c\}$. The induced preference profile $P' = P|_{N^1,G^1}$ has one preference

$$>_{k+1}|_{G^1}:\ c\ b_1\ b_2\ \cdots\ b_k$$

and $k + 1$ preferences

$$>_{k+2}|_{G^1}, \dots, >_{2k+2}|_{G^1}:\ b_1\ b_2\ \cdots\ b_k\ c.$$

The restricted allocation $\pi|_{N^1}$ gives all items from G^1 to the latter $k + 1$ agents. Such an allocation cannot be in $F_{k\text{-app},\star}(P')$, since the allocation $\rho = (c, b_1, b_2, \dots, b_k, \emptyset)$ is superior to $\pi|_{N^1}$, both in the leximin ordering and as measured by utilitarian social welfare. ❑

Note that the counterexamples in the proof, maybe atypically, involve instances with more agents than goods, but they could easily be modified by introducing additional goods (undesirable to everyone).

3.3 Two Trivial Examples of Separability

Motivated by their examples, which show failure of separability in many common cases, Baumeister et al. [2] conjecture that *"(perhaps under mild conditions on s and ⋆), no positional scoring allocation rule is separable"*. It is clear that some conditions will indeed have to be put on s and $\star$, for it is easy enough to find silly counterexamples to the conjecture when s and $\star$ are unrestricted.

Example 3.7. (1) Consider an arbitrary extended scoring vector s with all components non-zero (e.g., s could be the Borda, lexicographic, or Borda-quasi-indifference scoring vector) and max as an aggregation function. The resulting scoring allocation correspondence $F_{s,\max}$ will simply assign all goods to a single player (i.e., $\pi \in F_{s,\max}(P)$ if and only if π satisfies $\pi_i = G$ for some i). This is obviously a separable allocation correspondence.

(2) Consider the extended scoring vector $\mathbf{1} = (1, 1, \dots, 1)$ and aggregation function min (or leximin). The resulting scoring allocation correspondence $F_{\mathbf{1},\min}$ (or $F_{\mathbf{1},\text{leximin}}$) will simply always return all the even-shares allocations under which all agents will receive the same number of goods up to one good. This allocation correspondence is also clearly separable (the even-shares condition, stating that the sizes of the sets π_i differ by at most 1 for different i, still holds when we restrict to any subset of players).

These two scoring allocation correspondences may not be particularly desirable, but they still give some hints regarding the conjecture: The first example uses an unusual aggregation function but works with many reasonable scoring vectors, whereas the second example employs an unusual scoring vector but eminently reasonable aggregation functions. This shows that we will have to put conditions on both s *and* $\star$ to ensure failure of separability for $F_{s,\star}$. Furthermore, both examples do enjoy certain desirable properties; e.g., they are both anonymous and the leximin- and max-aggregation functions are monotonic.

Still, one major criticism that can be levelled at the examples above is that $F_{s,\max}(P)$ and $F_{\mathbf{1},\min}(P)$ hardly depend on P at all! That is, the resulting allocations can be determined without even looking at the preference profile.

3.4 Refuting the Conjecture

While we might consider the rules above pathological examples of separable scoring allocation rules, we will now show (in Theorems 3.10 and 3.14) that there do exist separable scoring allocation rules that are unquestionably sensible and useful. These are (under mild conditions) the well-studied *sequential allocation rules*, introduced by Kohler and Chandrasekaran [9].

3.4.1 All Sequential Allocation Rules are Scoring Allocation Rules.

Definition 3.8. Let $N = \{1, \dots, n\}$ with $n \geq 1$ be a set of agents and $G = \{g_1, \dots, g_m\}$ a set of m goods. Furthermore, let $\sigma = (\sigma_1, \dots, \sigma_m) \in N^m$. Let $P = (>_1, \dots, >_n)$ be a preference profile of N over G. We now inductively define *partial allocations* $\pi^0, \dots, \pi^m$ (i.e., allocations of some subset of G to N). π^0 is the empty allocation. Assume π^{j-1} has been defined. In π^j, one additional good will be allocated, namely agent σ_j will get to pick one good that is not yet allocated. More precisely, set $\pi_i^j = \pi_i^{j-1}$ for all $i \neq \sigma_j$. Set $\pi_{\sigma_j}^j = \pi_{\sigma_j}^{j-1} \cup \{g^j\}$, where g^j is the good ranked highest in the preference $>_{\sigma_j}$ which is not contained in $\bigcup_{i=1}^n \pi_i^{j-1}$. Finally, set $F_\sigma^{\text{seq}}(P) = \pi^m$.

We call $F_\sigma^{\text{seq}} : \mathscr{P}(G)^n \to \Pi(G, n)$ the *sequential allocation rule* associated to the picking sequence σ.

The two most common choices for picking sequences are the sequence

$$(1, 2, \dots, n,\ 1, 2, \dots, n,\ \dots)$$

and the sequence

$$(1, 2, \dots, n,\ n, n-1, \dots, 1,\ 1, 2, \dots, n, \dots).$$

Example 3.9. Consider three agents with the preference profile $P = (\mathit{beahgcdfi}, \mathit{adcfhgbei}, \mathit{cfabidegh})$ and the picking sequence $\sigma = (1, 2, 3, 1, 2, 3, 1, 2, 3)$. In the first round of picking, agent 1 picks b, then agent 2 picks a, and agent 3 picks c. Then it is agent 1's turn again and she picks e, then agent 2 picks d, and agent 3 picks f. Finally, agent 1 picks h (since a is already gone), agent 2 picks g (since c, f, and h are gone), and agent 3 picks the last remaining item i. So we get the allocation $F_\sigma^{\text{seq}} = (\mathit{beh}, \mathit{adg}, \mathit{cfi})$. Note that this allocation is not in $F_{s,\star}(P)$ for any strictly decreasing scoring vector s and any $\star \in \{+, \min, \text{leximin}\}$ (cf. Example 3.4).

Theorem 3.10. *All sequential allocation rules are scoring allocation rules for a suitable choice of scoring vector and social welfare ordering.*

Proof. Let G be a set of m goods and $n \geq 1$. We will use the scoring vector $s = \text{lex} = (2^{m-1}, \dots, 2^1, 2^0)$. Note that, with this scoring vector, for any preference profile P and allocation π, we can uniquely determine the ranks of the goods each player received (according to her preference ranking) *from the utility vector* $u_{P,s}(\pi) = (u_1, \dots, u_n) \in \mathbb{N}_0^n$ *alone.* That is because

$$u_i = u_{>_i,s}(\pi_i) = \sum_{g \in \pi_i} s_{\text{rank}(g,>_i)} = \sum_{g \in \pi_i} 2^{m-\text{rank}(g,>_i)}$$

is a sum of distinct powers of 2 and every number can be written in only one way as such a sum (its binary expansion). So if, e.g., $m = 5$ and u is such that

$$u_3 = 11 = 2^3 + 2^1 + 2^0 = 2^{5-2} + 2^{5-4} + 2^{5-5},$$

then we know that agent 3 received the goods ranked second, fourth, and fifth in her preference ranking.

Hence, given a utility vector $u = u_{P,s}(\pi) \in \mathbb{N}_0^n$, we can define as $r_{i,k}(u)$ the rank of agent i's kth-favorite good among the ones she received in π_i, or $r_{i,k}(u) = m + 1$ if agent i received fewer than k items. Like the notation indicates, we need to know neither P nor π, but only u, to determine $r_{i,k}(u)$.[2] In the example above, where $u_3 = 11$, we would thus have $r_{3,1}(u) = 2$, $r_{3,2}(u) = 4$ and $r_{3,3}(u) = 5$.

Now let $\sigma \in \{1, \dots, n\}^m$ be a picking sequence. Recall that σ_j denotes the player who gets to pick an item in round j of the

[2] This is crucial: We will use these numbers to define a social welfare ordering. By definition, the social welfare ordering is defined on $\mathbb{Q}_{\geq 0}^n$; it does not "see" the preference profile, only the resulting scores. To explicitly determine $r_{i,k}(u)$ from u alone: Write u_i in its binary expansion $u_i = \sum_{j=0}^l \varepsilon_j 2^j$, where $\varepsilon_j \in \{0, 1\}$ for $j \in \{0, \dots, l\}$. Now determine the index of the kth 1 in this binary expansion, i.e., let j_k be such that $\varepsilon_{j_k} = 1$ and $|\{j \geq j_k \mid \varepsilon_j = 1\}| = k - 1$, and set $r_{i,k}(u) = m - j_k$. If there aren't at least k 1s in the binary expansion of u_i then we set $r_{i,k}(u) = m + 1$.

sequential allocation rule. We will need, for $j \in \{1, \ldots, m\}$, the number $p_j = |\{j' \leq j \mid \sigma_{j'} = \sigma_j\}|$, i.e., p_j tells us how many picks agent σ_j can make up to round j; e.g., if the picking sequence is $\sigma = (1, 2, 3, 3, 2, 1, 1, 2, 3, 3)$, we have $p_7 = 3$ because agent 1 is on her 3rd pick in round 7 (she got to pick before in rounds 1 and 6). Now set

$$r_\sigma(u) = (m - r_{\sigma_1,p_1}(u), \ldots, m - r_{\sigma_m,p_m}(u)).$$

Finally, define a total preorder $\succsim$ on $\mathbb{Z}^n$ by

$$u \succsim u' \iff r_\sigma(u) \geq^{\text{lex}} r_\sigma(u').$$

We claim that the positional scoring allocation rule $F_{\text{lex},\succsim}$ is equal to F^{seq}_σ. The following example illustrates our setup for the proof of this assertion.

Example 3.11. Consider three players who want to divide up eight goods $G = \{a, b, c, d, e, f, g, h\}$. Their preference rankings are as follows:

$$P = (gafebcdh,\ afbechgd,\ aehgdcbf).$$

We use the picking sequence $\sigma = (1, 2, 3, 1, 2, 3, 1, 2)$. The resulting allocation for the sequential allocation rule is

$$F^{\text{seq}}_\sigma(P) = (gfc, abd, eh).$$

To see how our scoring allocation rule above works, consider some allocation, say $\pi = (gde, abfh, c)$. The utilities according to the scoring vector $s = \text{lex} = (2^7, \ldots, 2^1, 2^0)$ are

$$u = u_{P,s}(\pi) = (274, 356, 4).$$

In binary, $u = (100100102, 111001002, 000001002)$. By taking the resulting utility vector and considering the binary expansions of each component, we can recover (without needing to know π or P) the information that agent 1 received her 1st-, 4th-, and 7th-favorite goods, agent 2 received her 1st-, 2nd-, 3rd-, and 6th-favorite goods, and agent 3 merely received her 6th-favorite good in π. Hence, we have

$$\begin{array}{lll} r_{1,1}(u) = 1 & r_{2,1}(u) = 1 & r_{3,1}(u) = 6 \\ r_{1,2}(u) = 4 & r_{2,2}(u) = 2 & \\ r_{1,3}(u) = 7 & r_{2,3}(u) = 3 & \\ & r_{2,4}(u) = 6 & \end{array}$$

For all other i, j, we have $r_{i,j}(u) = 9$, by definition. Now our definition of r_σ yields

$$\begin{aligned} r_\sigma(u_{P,s}(\pi)) &= (8 - r_{1,1}(u), 8 - r_{2,1}(u), 8 - r_{3,1}(u), \\ &\qquad 8 - r_{1,2}(u), 8 - r_{2,2}(u), 8 - r_{3,2}(u), \\ &\qquad 8 - r_{1,3}(u), 8 - r_{2,3}(u)) \\ &= (8 - 1, 8 - 1, 8 - 6, \\ &\qquad 8 - 4, 8 - 2, 8 - 9, \\ &\qquad 8 - 7, 8 - 3) \\ &= (7, 7, 2, 4, 6, -1, 1, 5). \end{aligned}$$

Meanwhile, for the winning allocation under the sequential rule, we can compute

$$r_\sigma(u_{P,s}(F^{\text{seq}}_\sigma(P))) = (7, 7, 6, 5, 5, 5, 2, 0),$$

so indeed, $r_\sigma(u_{P,s}(F^{\text{seq}}_\sigma(P))) \geq^{\text{lex}} r_\sigma(u_{P,s}(\pi))$, i.e., $u_{P,s}(F^{\text{seq}}_\sigma(P)) \succsim u_{P,s}(\pi)$.

By definition, $F_{\text{lex},\succsim}(P)$ contains the allocations π for which the vector $r_\sigma(u_{P,s}(\pi))$ is maximal with respect to the lexicographic order. Which allocations have this property?

As we use the lexicographic order on the vectors $r_\sigma(u_{P,s}(\pi))$, for an allocation π to be optimal, the first component of this vector must be maximal among all possible choices for π.

The first component, by definition, is $m - r_{\sigma_1,p_1}(u_{P,s}(\pi))$, which is maximal if $r_{\sigma_1,p_1}(u_{P,s}(\pi))$ is minimal. This is the rank of agent σ_1's favorite item among the ones she receives in π (note, p_1 is always 1). For this to be minimal, agent σ_1 has to be assigned her favorite item in π.

Let π^1 be the partial allocation that only assigns this one good. Note that this is precisely the partial allocation π^1 that appears in the definition of the sequential allocation rule F^{seq}_σ (recall Definition 3.8).

For an induction, assume that the allocations π for which the first i components of $r_\sigma(u_{P,s}(\pi))$ are optimal (according to the lexicographic ordering) are exactly those that extend the allocation π^i from Definition 3.8. To find the allocations for which the first $i+1$ components of $r_\sigma(u_{P,s}(\pi))$ are optimal, we need to determine all allocations for which the $(i + 1)$st component is optimal among all allocations that extend π^i. The $(i+1)$st component of $r_\sigma(u_{P,s}(\pi))$ is $m - r_{\sigma_i,p_i}(u_{P,s}(\pi))$, which is maximal if $r_{\sigma_i,p_i}(u_{P,s}(\pi))$, the rank of agent σ_i's p_ith favorite good among π_{σ_i}, is minimal. $p_i - 1$ is exactly the number of goods that have already been allocated to agent σ_i in π^i. So minimizing this number means assigning agent σ_i her favorite good among all those that have not been assigned yet in π^i. This yields precisely the partial allocation π^{i+1} from Definition 3.8.

Hence, by induction, the allocations for which $r_\sigma(u_{P,s}(\pi))$ is maximal in the lexicographic order are those that extend $\pi^m = F^{\text{seq}}_\sigma(P)$. Since π^m already allocates all goods, there is only one optimal allocation with respect to the order $\succsim$, namely $F^{\text{seq}}_\sigma(P)$. ❑

3.4.2 Sequential Allocation Rules are Separable. Finally, we need to show that sequential allocation rules, using suitable picking sequences, are indeed separable. The "suitability" here regards the fact that the picking sequences for varying numbers of agents and goods have to be chosen in a coherent way. It is easy to define what "coherent" should mean when varying the number of goods, i.e., the length of the picking sequence. Here, we simply demand that there be some *infinite* picking sequence, e.g., $(1, 2, 3, 1, 2, 3, \ldots)$, and the picking sequence used for m goods simply consists of its first m terms. Defining when a collection $(\sigma^n)_{n\geq 1}$ of such infinite sequences for each number n of agents is coherent, is a little more involved. Essentially, removing t of the agents from the sequence σ^n should give σ^{n-t}, up to relabeling. This is formalized as follows:

Definition 3.12. A collection of infinite picking sequences is a family $\sigma = (\sigma^n)_{n\geq 1}$, where $\sigma^n = (\sigma^n_i) \in \{1, \ldots, n\}^{\mathbb{N}}$ for each $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ and $N^1 \subseteq \{1, \ldots, n\}$ with $|N^1| = r$, let $\sigma^n|_{N^1}$ denote the sequence arising from σ^n by deleting all terms σ^n_i that are not in N^1. Furthermore, set $\sigma^r|^{N^1} = (f(\sigma^r_i)) \in (N^1)^{\mathbb{N}}$, where $f\colon \{1, \ldots, r\} \to N^1$ is the unique order-preserving bijection between these sets.

The collection σ is called *coherent* if $\sigma^n|_{N^1} = \sigma^r|^{N^1}$ for all $n \in \mathbb{N}$ and all subsets $N^1 \subseteq \{1, \ldots, n\}$ with $|N^1| = r$.

The following example makes the intuition behind this definition clear.

Example 3.13. For each $n \in \mathbb{N}$, we can define the infinite picking sequence

$$\sigma^n = (1, 2, \ldots, n,\ n, n-1, \ldots, 1,\ 1, 2, \ldots, n,\ n, n-1, \ldots, 1,\ \ldots).$$

So, for example,

$$\sigma^5 = (1, 2, 3, 4, 5,\ 5, 4, 3, 2, 1,\ 1, 2, 3, 4, 5,\ 5, 4, 3, 2, 1,\ \ldots).$$

Now let $M = \{2, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$, so $m = |M| = 3$. Then we get $\sigma^5|_M$ by simply deleting all 1s and 3s from σ^5:

$$\sigma^5|_M = (2, 4, 5,\ 5, 4, 2,\ 2, 4, 5,\ 5, 4, 2, \ldots).$$

The unique increasing bijection $f\colon \{1, 2, 3\} \to M$ sends $f(1) = 2$, $f(2) = 4$, and $f(3) = 5$. Now

$$\sigma^3 = (1, 2, 3,\ 3, 2, 1,\ 1, 2, 3,\ 3, 2, 1, \ldots).$$

Applying f to all the terms yields

$$\sigma^3|^M = (2, 4, 5,\ 5, 4, 2,\ 2, 4, 5,\ 5, 4, 2, \ldots).$$

So, in this case we see that, indeed, $\sigma^5|_M = \sigma^3|^M$.

It is now easy to convince oneself that for the picking sequences σ^n, this holds for arbitrary $n \geq 1$ and $M \subseteq \{1, \ldots, n\}$. Hence, they form a coherent collection of infinite picking sequences.

Similarly, the collection $\tau = (\tau^n)$ with

$$\tau^n = (1, 2, \ldots, n,\ 1, 2, \ldots, n,\ 1, 2, \ldots, n,\ \ldots)$$

is also coherent.

Given a collection $\sigma = (\sigma^n)$ of infinite picking sequences, we can define the allocation rule scheme, consisting of, for each $n \in \mathbb{N}$ and finite set G with $|G| = m$, the sequence allocation rule $F^{\text{seq}}_{\sigma^n}$, using the first m terms of the infinite sequence σ^n as a picking sequence. As before, we simply denote all of these functions by F^{seq}_{σ}.

Theorem 3.14. *If $\sigma = (\sigma^n)_{n\geq 1}$ is a coherent collection of picking sequences, then F^{seq}_{σ} is a separable allocation rule scheme.*

Proof. Let $N = \{1, \ldots, n\}$ be a set of players and G a set of m goods. Let P be a preference profile of N over G. Let

$$\pi = F^{\text{seq}}_{\sigma^n}(P)$$

be the allocation resulting from the sequential allocation rule. Let $N^1 \subseteq N$ be a subset of the agents with $|N^1| = k$ and let $G^1 = \bigcup_{i\in N^1} \pi_i$ be the set of goods they received. Let

$$\pi' = F^{\text{seq}}_{\sigma^k}(P|_{N^1,G^1}).$$

By coherence of the family σ, the picking sequence σ^k arises from σ^n by deleting all terms not in N^1 and relabeling. This means that the agents from N^1 pick in exactly the same order in the procedure defining $F^{\text{seq}}_{\sigma^k}$ as they did in the procedure for $F^{\text{seq}}_{\sigma^n}$. The only difference is that in $F^{\text{seq}}_{\sigma^k}$ they only get to pick from the set G^1. But the items that are not in G^1 are exactly the ones these agents did not manage to pick in the original procedure $F^{\text{seq}}_{\sigma^n}$ anyway (because they got picked by some agent from $N \setminus N^1$ before an agent from N^1 had the chance).

Hence, each agent from N^1 picks exactly the same items (in the same order) in the procedure $F^{\text{seq}}_{\sigma^k}$ as in $F^{\text{seq}}_{\sigma^n}$, meaning π' is exactly $\pi|_{N^1}$. ❑

The following example illustrates the proof of Theorem 3.14.

Example 3.15. In Example 3.9, we considered the preference profile

$$P = (\textit{beahgcdfi},\ \textit{adcfhgbei},\ \textit{cfabidegh})$$

With the standard picking sequence $\sigma^3 = (1, 2, 3, 1, 2, 3, 1, 2, 3)$ we obtained

$$F^{\text{seq}}_{\sigma^3} = (\textit{beh}, \textit{adg}, \textit{cfi}).$$

Now consider the subset $N^1 = \{2, 3\}$ of agents and the subset $G^1 = \{a, c, d, f, g, i\}$ of goods they received. The restricted preference profile is

$$P|_{N^1,G^1} = (\textit{adcfgi},\ \textit{cfaidg}).$$

Restricting the picking sequence and relabeling gives the standard picking sequence $\sigma^2 = (1, 2, 1, 2, 1, 2)$.

If we now carry out the procedure to compute $F^{\text{seq}}_{\sigma^2}(P|_{N^1,G^1})$, we find indeed that the agents pick exactly the same items in the same order as before (agent 1 picks a, agent 2 picks c, agent 1 picks d, agent 2 picks f, agent 1 picks g, agent 2 picks i), so

$$F^{\text{seq}}_{\sigma^2}(P|_{N^1,G^1}) = (\textit{adg}, \textit{cfi}) = \pi|_{N^1}.$$

4 CONCLUSIONS AND FUTURE WORK

We exhibited scoring allocation rules $F_{\text{lex},\succsim}$, based on lexicographic scoring vectors and specially constructed social welfare orderings, which are equal to sequential allocation rules F^{seq}_{σ} and we showed these to be separable under certain reasonable conditions on the picking sequence σ, thus disproving a conjecture from Baumeister et al. [2]. In view of this, the conjecture that separability might fail for "all" aggregation functions was overly optimistic (or pessimistic, if you will).[3]

Still, the social welfare ordering used in the proof of Theorem 3.10 was tailor-made to imitate sequential allocation rules. So while the conjecture fails in the stated generality, it is still worthwhile considering the question for the well-studied social welfare functions from Example 2.8. Let us then suggest a more realistic, and presumably quite challenging, question:

> *Does separability fail for all scoring allocation rules of the form F_{s,W_p}, where s is a strictly decreasing extended scoring vector and W_p is one of the social welfare functions introduced in Example 2.8?*

For any particular choice of p and s, it is usually easy to find counterexamples for separability experimentally. But it seems much harder to prove general statements covering a whole range of scoring vectors or a whole range of values for p.

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[3]Though the social welfare ordering used in the proof of Theorem 3.10 was described in an algorithmic manner, it is in fact induced by an aggregation function, which could be written down in an explicit, closed form (though such an expression would hardly be enlightening).

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