Phase Transition of the 2-Choices Dynamics on Core-Periphery Networks*

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ABSTRACT

Consider the following process on a network: Each agent initially holds either opinion *blue* or *red*; then, in each round, each agent looks at two random neighbors and, if the two have the same opinion, the agent adopts it. This process is known as the *2-Choices* dynamics and is arguably the most basic non-trivial *opinion dynamics* modeling voting behavior on social networks. Despite its apparent simplicity, 2-Choices has been analytically characterized only on networks with a strong expansion property – under assumptions on the initial configuration that establish it as a fast *majority consensus* protocol.

In this work, we aim at contributing to the understanding of the 2-Choices dynamics by considering its behavior on a class of networks with core-periphery structure, a well-known topological assumption in social networks. In a nutshell, assume that a densely-connected subset of agents, the *core*, holds a different opinion from the rest of the network, the *periphery*. Then, depending on the strength of the cut between the core and the periphery, a phase-transition phenomenon occurs: Either the core's opinion rapidly spreads among the rest of the network, or a *metastability* phase takes place, in which both opinions coexist in the network for super-polynomial time. The interest of our result is twofold. On the one hand, by looking at the 2-Choices dynamics as a simplistic model of competition among opinions in social networks, our theorem sheds light on the *influence* of the core on the rest of the network, as a function of the core's connectivity towards the latter. On the other hand, to the best of our knowledge, we provide the first analytical result which shows a heterogeneous behavior of a simple dynamics as a function of structural parameters of the network. Finally, we validate our theoretical predictions with extensive experiments on real networks.

KEYWORDS

Opinion Dynamics; 2-Choices Dynamics; Core-Periphery Structure; Consensus; Metastability; Phase Transition; Social Networks

ACM Reference Format:
Emilio Cruciani, Emanuele Natale, André Nusser, and Giacomo Scornavacca. 2018. Phase Transition of the 2-Choices Dynamics on Core-Periphery Networks. In *Proc. of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018), Stockholm, Sweden, July 10–15, 2018,* IFAAMAS, 9 pages.

*Work carried out at Max Planck Institute for Informatics, Saarbrücken, Germany.

Proc. of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018), M. Dastani, G. Sukthankar, E. André, S. Koenig (eds.), July 10–15, 2018, Stockholm, Sweden.

1 INTRODUCTION

Opinion dynamics (for short, dynamics) are simplistic mathematical models for the competition of agents' opinions on social networks [38]. In a nutshell, given a network where each agent initially supports an opinion (from a finite set), a dynamics is a simple rule which agents apply to update their opinion based on that of their neighbors. Proving theoretical results on dynamics is a challenging mathematical endeavor which may require the development of new analytical techniques [4, 7].

As dynamics are aimed at modeling the spread of opinions, a central issue is to understand under which conditions the network reaches a *consensus*, i.e. a state where the whole network is taken over by a single opinion [39]. In this respect, most efforts have been directed toward obtaining *topology-independent* results, which disregard the initial opinions' placement on the network [5, 6, 17].

The *trivial* example of dynamics is the so-called *Voter Model*, in which in each round each agent copies the opinion of a random neighbor. This classical model arises in physics and computer science and, despite its apparent simplicity, some properties have been proven only recently [9, 29, 37]. The simplest non-trivial example is then arguably the *2-Choices* dynamics, in which agents choose two random neighbors and switch to their opinion only if they coincide [14] (see Definition 3.2). Still, the analysis of the 2-Choices dynamics has been limited to networks with good expansion properties, and the theoretical guarantees provided so far are essentially independent from the positioning of initial opinions [15].

In this work, we aim at contributing to the general understanding of the evolution of simple opinion dynamics in richer classes of network topologies by studying their behavior theoretically and empirically on core-periphery networks. Core-periphery networks are typical economic and social networks which exhibit a core-periphery structure, a well-known concept in the analysis of social networks in sociology and economics [11, 19], which defines a bipartition of the agents into *core* and *periphery*, such that certain key features are identified.

We consider an axiomatic framework that has been introduced in previous work in computer science [3], which is based on two parameters only, *dominance* and *robustness*. The ranges for these parameters in the theorems we obtain include the values of the parameters in the the experimental part of this work, in which our results are validated on important datasets of real-world networks.

Intuitively, the core is a set of agents that *dominates* the rest of the network. In order to do so, it maintains a large amount of external influence on the periphery, higher than or at least comparable to the internal influence that the periphery has on itself. Similarly, to maintain its *robustness*, namely to hold its position and stick to its opinions, the core must be able to resist the "outside" pressure in the form of external influence. To achieve that, the core must maintain a higher (or at least not significantly lower) influence on itself than the external influence exerted on it by the periphery. Both, high dominance and high robustness, are essential for the core to be able to maintain its dominating status in the network. Moreover, it seems natural for the core size to be as small as possible. In social-network terms this is motivated by the idea that if membership in a social elite entails benefits, then keeping the elite as small as possible increases the share for each of its members.

The above requirements are formalized in the following axioms [3]. Given a network $G = (V, E)$ and two subsets of agents $A, B \subseteq V$, let $c(A, B) = \{(u, v) \mid u \in A, v \in B, (u, v) \in E\}$ be the *set of cut edges* between A and B. The *density* of a set $X \subseteq V$ is defined as $\delta(X) = |c(X, X)|/|X|$. Let c_d and c_r be two positive constants and let $V = \mathcal{C} \dot{\cup} \mathcal{P}$, where $\mathcal{C}$ is the set of agents in the core and $\mathcal{P}$ the set of agents in the periphery. Then, the axioms are as follows:

- *Dominance*: The core's influence dominates the periphery, i.e. $|c(\mathcal{C}, \mathcal{P})| \geqslant c_d \cdot |c(\mathcal{P}, \mathcal{P})|$.
- *Robustness*: The core can withstand outside influence from the periphery, i.e. $|c(\mathcal{C}, \mathcal{C})| \geqslant c_r \cdot |c(\mathcal{P}, \mathcal{C})|$.
- *Compactness*: The core is a minimal set satisfying the above dominance and robustness axioms.[1]
- *Density*: The core is denser than the whole network, i.e. $\delta(\mathcal{C}) > \delta(V)$.

Our analytical and experimental results leverage on the dominance and robustness axioms only (see Definition 3.1), showing how assumptions on the values of c_d and c_r are sufficient to provide a good characterization of the behavior of the dynamics.

We consider the 2-Choices dynamics in core-periphery networks when starting from *natural* initial configurations in which the core and the periphery have different opinions. Our experiments on real-world networks show that the execution of the 2-Choices dynamics tends to fall mainly within two opposite kinds of possible behavior:

Consensus: The opinion of the core spreads in the periphery and takes over the network in a short time.

Metastability: The periphery *resists* and, although the opinion of the core may continuously "infect" agents in the periphery, most of them retain the initial opinion.

By comprehending the underlying principles which govern the aforementioned phenomena, we aim at a twofold contribution:

- We seek for the first results on basic non-trivial opinion dynamics, such as 2-Choices, in order to characterize its behavior (i) on new classes of topologies other than networks with strong expansion and (ii) as a function of the process' initial configuration.
- We look for a *dynamic* explanation for the axioms of core-periphery networks: By investigating the interplay between the core-periphery axioms and the evolution of simple opinion dynamics, we want to get insights on *dynamical properties* which are implicitly responsible for causing social and economic agents to form networks with a core-periphery structure.

[1]The core is a minimal set and not necessarily the minimum.

Original Contribution

In order to understand what network key properties are responsible for the aforementioned dichotomy between a long *metastable* and a fast *consensus* behavior, we theoretically investigate a class of networks belonging to the core-periphery model.

To further simplify the theoretical analysis, in Theorem 3.4 we initially consider the setting in which *agents in the core are stubborn*, i.e. they don't change their initial opinion.[2] We later show, in Corollary 3.7, how to substitute this assumption with milder hypotheses on the core's structure.

The common difficulty in analyzing opinion dynamics is the lack of general tools which allow to rigorously handle their intrinsic nonlinearity and stochastic dependencies [6, 14, 17, 38, 39]. Hence, the difficulty usually resides in identifying some crucial key quantities for which ad-hoc analytical bounds on the expected evolution are derived. Our approach is yet another instance of such efforts: In Section 3 we provide a careful bound on the expected change of the number of agents supporting a given opinion. Together with the use of Chernoff bounds, we obtain a concentration of probability around the expected evolution. Rather surprisingly, our analysis on the concentrated probabilistic behavior turns out to identify a *phase transition* phenomenon (Corollaries 3.5 and 3.7):

> There exists a universal constant $c^\star = \frac{\sqrt{2}-1}{2}$ such that, on any core-periphery network of n agents, if the dominance parameter c_d is greater than $c^\star$, an *almost-consensus* is reached within $O(\log n)$ rounds, with high probability;[3] otherwise, if c_d is less than $c^\star$, a *metastable* phase in which the periphery retains its opinion takes place, namely, although the opinion of the core may continuously "infect" agents in the periphery, most of them will retain the initial opinion for $n^{\omega(1)}$ rounds, with high probability.

We validate our theoretical predictions by extensive experiments on real-world networks chosen from a variety of domains including social networks, communication networks, road networks, and the web. We thoroughly discuss our results in Section 4, however, we briefly want to highlight the key results of our experiments in the following. The experiments showed some weaknesses of the core extraction heuristic used in [3]. To avoid those drawbacks, we designed a new core extraction heuristic which repeatedly calculates densest-subgraph approximations. Our experimental results on real-world networks show a strong correlation with the theoretical predictions made by our model. They further suggests an empirical threshold larger than $c^\star$ for which the aforementioned correlation is even stronger. We discuss which aspects of the current theoretical model may be responsible for such discrepancy, and thus identify possibilities for a model which is even more accurate.

[2]We remark that the evolution of the 2-Choices dynamics, together with the latter assumption on the stubbornness of the core, can be regarded as a SIS-like epidemic model [23, 30]. In such a model, the network is the subgraph induced by the periphery and the infection probability is given by the 2-Choices dynamics, which also determines a certain probability of *spontaneous infection* (that in the original process corresponds to the fact that agents in the periphery interact with agents in the core). This interpretation of our results may be of independent interest.

[3]We further emphasize that our analysis is not only *mean-field*. In addition to describing how the process evolves in expectation, we show that the process does not deviate significantly from how it behaves in expectation *with high probability* (w.h.p.), i.e. with probability at least $1 - O(n^{-c})$ for some constant $c > 0$.

We remark that our investigation represents an original contribution with respect to the line of research on *consensus* discussed in Section 2, as it shows a drastic change in behavior for the 2-Choices dynamics on an arguably *typical* broad family of social networks which is not directly characterized by expansion properties. In particular, the convergence to the core's opinion in our theoretical and experimental results is a highly nontrivial fact when compared to previous analytical works (see Section 4 for more details).

Overall, our theoretical and experimental results highlight new potential relations between the typical core-periphery structure in social and economic networks and the behavior of simple opinion dynamics – both, in terms of getting insights into the driving forces that may determine certain structures to appear frequently in real-world networks, as well as in terms of the possibility to provide analytical predictions on the outcome of simplistic models of interaction in networks of agents.

2 RELATED WORK

Simple models of interaction between pairs of nodes in a network are studied since the first half of the 20th century in statistical mechanics where mathematical models of ferromagnetism led to the study of Ising and Potts models [36]. A different perspective later came from diverse sciences such as economics and sociology where averaging-based opinion dynamics such as the DeGroot model were investigated [20, 24, 27, 28, 32].

More recently, computer scientists have started to contribute to the investigating of opinion dynamics for mainly two reasons. First, with the advent of the Internet, huge amounts of of data from social networks are now available. As the law of large numbers often allows to assume crude simplifications on the agents' behavior in such networks [19], investigating opinion dynamics allows for a more fine-grained understanding of the evolution of such systems. Second and somehow complementary to the previous motivation, technological systems of computationally simple agents, such as mobile sensor networks, often require the design of computationally primitive protocols. These protocols end up being surprisingly similar, if not identical, to many opinion dynamics which emerge from a simplistic mathematical modeling of agents' behavior in fields such as sociology, biology, and economy [4, 29, 39].

A substantial line of work has recently been devoted to investigating the use of simple opinion dynamics for solving the plurality consensus problem in distributed computing. The goal in this problem setting is for each node to be aware of the most frequent initially supported opinion after a certain time [4–6, 8, 13, 16, 18, 21, 26]. The seminal work by Hassin and Peleg [29] introduced for the first time the study of a synchronous-time version of the Voter Model in statistical mechanics. In this model, in each discrete-time round, each node looks at a random neighbor and copies her opinion. The Voter Model is considered the *trivial* opinion dynamics, in the sense that it is arguably the simplest conceivable rule by which nodes may meaningfully update their opinion as a function of their neighbors' opinion. Many properties of this process are understood by known mathematical techniques such as an elegant duality with the coalescing random walk process [1]. In particular, it is known that the Voter Model is not a *fast* dynamics as for the time it takes before consensus on one opinion is reached in the network. For that reason, the 2-Choices dynamics has been considered [14]. In such dynamics, in each round, each node looks at the opinion of *two* random neighbors and, if these two are the same, adopts it. This process can arguably be considered the *simplest* non-trivial type of opinion dynamics.

The authors of [14] consider any initial configuration in which each node is supporting one out of two possible opinions. They proved that in such a configuration, under the assumption that the initial bias (i.e. advantage of an opinion) is greater than a function of the network's *expansion* (measured in terms of the second eigenvalue of the network) [31], the whole network will support the initially most frequent opinion with high probability after a polylogarithmic number of rounds. The results of [14] on the 2-Choices dynamics were later refined with milder assumptions on the initial bias with respect to the network's expansion [15] and generalized to more opinions [17]. Until now, the 2-Choices dynamics constitutes one of the few processes whose behavior has been characterized on non-complete topologies – however, assuming good expansion properties. Their techniques should be easily adaptable in order to handle similar dynamics, such as the 3-Majority dynamics [5, 6, 25].

On a different note, for the deterministic Majority dynamics where in each round each node updates her opinion with the most frequent one among her neighbors: Substantial effort has been devoted to investigating how small the cardinality of a *monopoly* can be, i.e. a set of nodes supporting a given opinion O such that, when running the Majority dynamics, eventually the network reaches consensus on O [40, 41]. The previous line of investigation has focused on determining the existence of network classes on which the monopoly has a size which is upper bounded by a small function. This question has been settled by [10], who proved the existence of a family of networks with constant-size monopoly. We emphasize that this line of investigation is peculiar as it deals with existential questions related to specific network classes, as opposed to the typical research questions that we discussed so far, which ask for general characterizations of the behavior of the considered process.

Recently, a more systematic and general study of opinion dynamics has been carried out in [7, 38, 39]. It characterizes the evolution and other mathematical aspects of dynamics – such as the Majority dynamics, the Voter Model, the DeGroot model, and others – on different network classes, such as Erdős-Rényi random networks and expander networks. We follow [39] in adopting the term "opinion dynamics" to refer to the class of processes discussed above.

Finally, a similar perspective to ours has been adopted in [34], where the authors show a phase-transition in a mean-field analysis of the behavior of deterministic majority in an asynchronous-update model, in which only few nodes update their opinion at each round, on "coupled fully connected networks" composed of overlapping complete graphs.

Core-Periphery model

It has long been observed in sociology that many economic and social networks exhibit a *core-periphery structure* [11, 42–44], namely, it is generally possible to group nodes into two classes, a *core* and a *periphery*, such that the former exhibits a dense internal topology while the latter is sparse and loosely connected, with specific properties relating the two. Such architectural principle has been linked, for example, to the easiness with which individuals solve routing problems in networks subject to the small-world phenomenon [19].

The qualitative notion of core-periphery structure was translated into quantitative relations in the axiomatic approach of [3], which later also applied the algorithmic properties that follow from the core-periphery structure to the design of efficient distributed networks [2]. In some sense, our theoretical and empirical results may be regarded as a *functional* justification for the presence of a core-periphery structure in networks, as the latter turns out to play a decisive role in determining a certain kind of evolution for basic opinion dynamics such as the 2-Choices.

3 THEORETICAL ANALYSIS

We give a formal analysis of the 2-Choices dynamics on a specific network topology, i.e. on Core-Periphery networks. We use colors instead of opinions to facilitate intuitive understanding of the analysis. Specifically, we consider the setting in which the agents belonging to the core C initially support the color *blue* while the remaining agents, from the periphery $\mathcal{P}$, support the color *red*. We show that, depending on the value of some parameters describing the core-periphery structure of the network, either the opinion of the core rapidly spreads among almost the whole network (*Almost-consensus*: almost all the agents support the *blue* color after a few rounds) or most of the periphery resists for a long time (*Metastability*: most agents in $\mathcal{P}$ remain *red*).

First, we formally describe our characterization of Core-Periphery networks and of the 2-Choices dynamics. Then, we prove two technical results which will be exploited in order to provide a rigorous analysis of the 2-Choices dynamics on Core-Periphery networks.

Definition 3.1. For every n and $\varepsilon, c_r, c_d \in \mathbb{R}^+$, with $1/2 \leqslant \varepsilon \leqslant 1$, we define an $(n, \varepsilon, c_r, c_d)$-Core-Periphery network $G = (C \dot\cup \mathcal{P}, E)$ as a network with $|C| = n^\varepsilon$ and $|\mathcal{P}| = n$ and such that: (i) for each agent $u \in C$, it holds that $|N(u) \cap C| = c_r \cdot |N(u) \cap \mathcal{P}|$; (ii) for each agent $v \in \mathcal{P}$, it holds that $|N(v) \cap C| = c_d \cdot |N(v) \cap \mathcal{P}|$; $N(v)$ is the set of neighbors of agent v. We call C the *core* and $\mathcal{P}$ the *periphery*.

The definition we just provided matches the requirements of the core-periphery structure as axiomatized by Avin et al. [3]. However, observe that it is more restrictive: the values c_r and c_d define properties that hold for each agent of the network and not only globally, i.e. for the partition induced by the core.

The 2-Choices dynamics can be formally described as follows.

Definition 3.2 (2-Choices dynamics). Given a network $G = (V, E)$ with an initial binary coloring of the agents $c : V \mapsto \{red, blue\}$, the 2-Choices dynamics proceeds in synchronous rounds: in each round, each agent u chooses two neighbors v, w uniformly at random with replacement; if $c(v) = c(w)$, then u updates its own color to $c(v)$, otherwise u keeps its color.

In order to analyze the 2-Choices dynamics on Core-Periphery networks, we present a more general technical result that will be exploited later. In fact, both in the analysis of the *almost-consensus* and of the *metastability*, we can focus on the worst-case scenario for the core and for the periphery: Each time an agent in one of the two sets picks a neighbor in the other set, that neighbor has the initial color of the set it belongs to. Alternatively, such a scenario can be seen as the following variant of the 2-Choices dynamics.

Definition 3.3 (Biased-2-Choices(p, σ) dynamics). Let $p \in [0, 1]$ be a constant and let $\sigma \in \{red, blue\}$ be a color. We define the Biased-2-Choices(p, σ) dynamics as a variation of the 2-Choices dynamics: Every time an agent picks a neighbor, with probability p that neighbor supports color σ regardless of its actual color.

The technical result we present considers a network of agents running the Biased-2-Choices(p, σ) dynamics, all having the same initial color different from σ. Informally, it shows that there exists a value $p^\star$ such that if the agents are running the Biased-2-Choices(p, σ) dynamics with $p > p^\star$, then the color σ rapidly spreads among almost the whole network, while if $p < p^\star$, then most of the network does not support the color σ for a superpolynomial number of rounds.

For a set of agents A, let the *volume* of A be defined as $vol(A) = \sum_{v \in A} d_v$, where d_v is the degree of v.

THEOREM 3.4. *Let $G = (V, E)$ be a network of n agents such that each agent v has a color σ_v and $d_v = \omega(\log n)$ neighbors. Let $p \in [0, 1]$ be a constant. Then, starting from a configuration where all agents initially support the* red *color and letting the agents run Biased-2-Choices(p, blue), it holds that:*

- Almost-consensus: *If $p > 3 - 2\sqrt{2}$, then the agents reach a configuration such that the* volume *of agents supporting the* blue *color is $(1 - o(1))vol(V)$ within $O(\log n)$ rounds, w.h.p.*
- Metastability: *If $p < 3 - 2\sqrt{2}$, then the* volume *of the* blue *agents never exceeds $\frac{1-3p}{4(1-p)} vol(V)$ for any* poly(n) *number of rounds, w.h.p.*

PROOF. Let $B^{(t)}$ be the set of *blue* agents and $R^{(t)} = V \setminus B^{(t)}$ be the set of *red* agents at time t. For any agent v, let $N_R(v) = N(v) \cap R^{(t)}$ be the set of *red* neighbors and $N_B(v) = N(v) \cap B^{(t)}$ the set of *blue* neighbors of v. Furthermore, let $r_v^{(t)}$ be the number of *red* neighbors of v at time t, i.e. $r_v^{(t)} = |N_R(v)|$. Let $\phi_v^{(t)} = r_v^{(t)}/d_v$ be the fraction of *red* agents in the neighborhood of v; let $\phi_{\min}^{(t)} = \min_{v \in V} \phi_v^{(t)}$ and $\phi_{\max}^{(t)} = \max_{v \in V} \phi_v^{(t)}$ be, respectively, the minimum and maximum fractions of *red* neighbors among all agents in V. Let $\mathbf{c}^{(t)} \in \{red, blue\}^n$ be the configuration of the colors of the agents at time t. In the following, for the sake of readability, whenever we omit the time index, we refer to the value at time t, e.g. ϕ_v stands for $\phi_v^{(t)}$. Similarly, we concisely denote with $\mathbf{P}_R(v) = \mathbf{P}(v \in R^{(t+1)} \mid \mathbf{c}^{(t)})$ the probability that agent v will be supporting the *red* color in the next round of the Biased-2-Choices(p, *blue*), i.e.

$$\mathbf{P}_R(v) = \begin{cases} 1 - \big(p + (1-p)(1-\phi_v)\big)^2 & \text{if } v \in R, \\ (1-p)^2 \phi_v^2 & \text{if } v \in B. \end{cases}$$

Furthermore, notice that:

$$\min_{w \in R} \mathbf{P}_R(w) = 1 - \big(p + (1-p)(1-\phi_{\min})\big)^2,$$
$$\min_{w \in B} \mathbf{P}_R(w) = (1-p)^2 \phi_{\min}^2.$$

Given a configuration $\mathbf{c}^{(t)}$, we can give a lower bound for the expected fraction of *red* neighbors of any agent v as follows:

$$\mathbf{E}\left[\phi_v^{(t+1)} \,\middle|\, \mathbf{c}^{(t)}\right] = \frac{1}{d_v}\left(\sum_{w \in N_R(v)} \mathbf{P}_R(w) + \sum_{w \in N_B(v)} \mathbf{P}_R(w)\right)$$

$$
\begin{aligned}
&\geqslant \frac{1}{d_v}\left(|N_R(v)| \min_{w\in R} \mathbf{P}_R(w) + |N_B(v)| \min_{w\in B} \mathbf{P}_R(w)\right)\\
&= \frac{r_v}{d_v} \min_{w\in R} \mathbf{P}_R(w) + \left(1-\frac{r_v}{d_v}\right)\min_{w\in B}\mathbf{P}_R(w)\\
&= \frac{r_v}{d_v}\left(1-\left(p+(1-p)(1-\phi_{\min})\right)^2\right) + \left(1-\frac{r_v}{d_v}\right)(1-p)^2\phi_{\min}^2\\
&= \frac{r_v}{d_v}\left(1-\left(p+(1-p)(1-\phi_{\min})\right)^2 - (1-p)^2\phi_{\min}^2\right) + (1-p)^2\phi_{\min}^2\\
&\geqslant \phi_{\min}\left(1-\left(p+(1-p)(1-\phi_{\min})\right)^2 - (1-p)^2\phi_{\min}^2\right) + (1-p)^2\phi_{\min}^2\\
&= \phi_{\min}\left(1-\left(p+(1-p)(1-\phi_{\min})\right)^2 + (1-p)^2(1-\phi_{\min})\phi_{\min}\right)\\
&= \phi_{\min}\left(1-2(1-p)^2\phi_{\min}^2 + (1-p)(3-p)\phi_{\min} - 1\right).
\end{aligned}
$$

Note that we could cancel out 1 and -1, however, leaving them facilitates the analysis. In the steps above, we can lower bound r_v/d_v because its coefficient, i.e. $(1-(p+(1-p)(1-\phi_{\min}))^2-(1-p)^2\phi_{\min}^2)$, is non-negative for any $p, \phi_{\min} \in [0,1]$.

Conversely, we can upper bound the expectation using $\phi_{\max}$, i.e.

$$\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] \leqslant \phi_{\max}\left(1-2(1-p)^2\phi_{\max}^2 + (1-p)(3-p)\phi_{\max} - 1\right).$$

Notice that the lower and the upper bound for the expectation have the same form. In fact, defining the functions

$$f(\phi) = 2(1-p)^2\phi^2 - (1-p)(3-p)\phi + 1,$$
$$g(\phi) = \phi(1-f(\phi)),$$

the lower and the upper bound for the expectation can respectively be written as $g(\phi_{\min})$ and $g(\phi_{\max})$. Thus, analyzing $f(\phi)$, we can see for which values of p the function $g(\phi)$ is increasing or decreasing.

Before proving the *almost-consensus* and the *metastability* configurations that can be reached by the agents running the Biased-2-Choices(p, *blue*), we study $f(\phi)$ in order to characterize the bounds for the expectation. The roots of $f(\phi)$ are in $\frac{3-p\pm\sqrt{p^2-6p+1}}{4(1-p)}$ and the derivative of $f(\phi)$ is $f'(\phi) = 4(1-p)^2\phi - (1-p)(3-p)$. It follows that $f(\phi)$ has a minimum point in $\bar{\phi} = \frac{3-p}{4(1-p)}$. Moreover, the sign of $f(\bar{\phi})$ exclusively depends on p. In fact

$$f(\bar{\phi}) > 0 \qquad \text{if } p > 3-2\sqrt{2}, \tag{1}$$
$$f(\bar{\phi}) < 0 \qquad \text{if } p < 3-2\sqrt{2}, \tag{2}$$

since in (1) the discriminant of $f(\phi)$ is negative, while in (2) it is positive.

Almost-consensus. Let $p > 3-2\sqrt{2}$. Let $f(\bar{\phi}) = \varepsilon$ be the local minimum of f. Notice that ε is positive because of (1) and it is a constant since it only depends on p and $\bar{\phi}$, which are both constants.

Due to the convexity of $f(\phi)$, it holds that $f(\phi) \geqslant \varepsilon$. Thus, for any $v \in V$, we have that

$$\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] \leqslant g(\phi_{\max}) = \phi_{\max}(1-f(\phi_{\max})) \leqslant \phi_{\max}(1-\varepsilon),$$

as $\varepsilon > 0$. Thus, we can apply the multiplicative form of the Chernoff bounds [22, Theorem 1.1] and get that

$$
\begin{aligned}
&\mathbf{P}\left(\phi_v^{(t+1)} > (1-\varepsilon^2)\phi_{\max} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&= \mathbf{P}\left(\phi_v^{(t+1)} > (1+\varepsilon)(1-\varepsilon)\phi_{\max} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\leqslant \mathbf{P}\left(\phi_v^{(t+1)} > (1+\varepsilon)\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] \;\middle|\; \mathbf{c}^{(t)}\right)\\
&= \mathbf{P}\left(r_v^{(t+1)} > (1+\varepsilon)d_v\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\leqslant e^{-(1+\varepsilon)^2 d_v \phi_{\max}(1-\varepsilon)/3} \leqslant e^{-2\log n} = n^{-2},
\end{aligned}
$$

where in the last inequality we can assume that the configuration $\mathbf{c}^{(t)}$ is such that $\phi_{\max} \geqslant \frac{6\log n}{(1+\varepsilon)^2(1-\varepsilon)d_v} = o(1)$, since $d_v = \omega(\log n)$ by definition. Thus, using the union bound over all the agents, we get that $\phi_{\max}^{(t+1)} \leqslant (1-\varepsilon^2)\phi_{\max}$ w.h.p. Formally:

$$
\begin{aligned}
&\mathbf{P}\left(\exists v \in V : \phi_v^{(t+1)} > (1-\varepsilon^2)\phi_{\max} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\quad \leqslant \sum_{v\in V} \mathbf{P}\left(\phi_v^{(t+1)} > (1-\varepsilon^2)\phi_{\max} \;\middle|\; \mathbf{c}^{(t)}\right) \leqslant \sum_{v\in V} n^{-2} = n^{-1}.
\end{aligned}
$$

Such a multiplicative decrease rate of the expected maximum fraction of *red* neighbors implies that $\phi_{\max}$ is in the order of $o(1)$ within $O(\log n)$ rounds of the Biased-2-Choices(p, *blue*). Again, applying the union bound, we get that this happens w.h.p.

Metastability. Let $p < 3-2\sqrt{2}$. Define $f(\bar{\phi}) = -\varepsilon$ to be the local minimum of f. Recall that ε is positive because of (2) and it is a constant since it only depends on the constants p and $\bar{\phi}$.

Then, using the fact that $g(\phi)$ is monotonically nondecreasing,[4] for every $\phi \geqslant \bar{\phi}$ we have that $g(\phi) \geqslant g(\bar{\phi}) = \bar{\phi}(1-f(\bar{\phi})) = \bar{\phi}(1+\varepsilon)$.

We can now use a multiplicative form of the Chernoff bounds in order to show that if the fraction of *red* neighbors of a agent v is at least $\bar{\phi}$, then the probability that the number of *red* neighbors of v in the next round is lower than $\bar{\phi}$ is negligible. Formally, let $\mathbf{c}^{(t)}$ be an arbitrary configuration such that $\phi_{\min} \geqslant \bar{\phi}$. First, note that due to $g(\phi) \geqslant g(\bar{\phi})$, we have that $\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] \geqslant g(\phi_{\min}) \geqslant g(\bar{\phi}) = \bar{\phi}(1+\varepsilon)$. Then, it follows that

$$
\begin{aligned}
&\mathbf{P}\left(\phi_v^{(t+1)} < \bar{\phi} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&= \mathbf{P}\left(\phi_v^{(t+1)} < \bar{\phi}(1+\varepsilon)\frac{1}{1+\varepsilon} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&= \mathbf{P}\left(\phi_v^{(t+1)} < \bar{\phi}(1+\varepsilon)\left(1-\frac{\varepsilon}{1+\varepsilon}\right) \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\leqslant \mathbf{P}\left(\phi_v^{(t+1)} < \mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right]\left(1-\frac{\varepsilon}{1+\varepsilon}\right) \;\middle|\; \mathbf{c}^{(t)}\right)\\
&= \mathbf{P}\left(r_v^{(t+1)} < d_v\mathbf{E}\left[\phi_v^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right]\left(1-\frac{\varepsilon}{1+\varepsilon}\right) \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\leqslant \exp\left(-d_v\frac{1}{3}\bar{\phi}(1+\varepsilon)\left(1-\frac{\varepsilon}{1+\varepsilon}\right)^2\right)\\
&= e^{-\Omega(d_v)} = e^{-\omega(\log n)} = n^{-\omega(1)}.
\end{aligned}
$$

Applying the union bound over all the agents, we get

$$
\begin{aligned}
&\mathbf{P}\left(\substack{\exists t\in \mathrm{poly}(n)\\ \exists v\in V} : \phi_v^{(t+1)} < \bar{\phi} \;\middle|\; \mathbf{c}^{(t)}\right)\\
&\quad \leqslant \sum_{\substack{t\in\mathrm{poly}(n)\\ v\in V}} \mathbf{P}\left(\phi_v^{(t+1)} < \bar{\phi} \;\middle|\; \mathbf{c}^{(t)}\right) = \sum_{\substack{t\in\mathrm{poly}(n)\\ v\in V}} n^{-\omega(1)} = n^{-\omega(1)}.
\end{aligned}
$$

Thus, with high probability we have that $\phi_{min} \geqslant \bar{\phi}$ for every polynomial number of rounds. Before we can use this to finish the proof, note that $\sum_{v\in B} d_v = \sum_{v\in V}(d_v - r_v)$, by simply counting

[4] Due to space constraints, we omit this standard verification.

the number of *blue* endpoints of an edge in two different ways. Using that $\phi_v \geqslant \phi_{\min} \geqslant \bar{\phi}$ for each v, we have

$$vol\left(B^{(t)}\right) = \sum_{v \in B} d_v = \sum_{v \in V} (d_v - r_v) = \sum_{v \in V} d_v \left(1 - \frac{r_v}{d_v}\right)$$
$$\leqslant (1 - \bar{\phi}) \sum_{v \in V} d_v = \frac{1-3p}{4(1-p)} vol(V).$$

This means, the volume of the *blue* agents never exceeds a fraction of $\frac{1-3p}{4(1-p)}$ of the total volume of the graph w.h.p. ■

Theorem 3.4 has several interesting implications on the behavior of the 2-Choices dynamics on Core-Periphery networks which we describe in the remainder of this section.

In the following, we always assume an initial configuration in which all agents in the core C are *blue* and all agents in the periphery $\mathcal{P}$ are *red*. Now, let $G = (V, E)$ be an $(n, \varepsilon, c_r, c_d)$-Core-Periphery network. Furthermore, let $q_C = |N(u) \cap \mathcal{P}|/d_u$ be the probability that an agent $u \in C$ picks a neighbor in the periphery, and let $q_{\mathcal{P}} = |N(v) \cap C|/d_v$ be the probability that an agent $v \in \mathcal{P}$ picks a neighbor in the core. The relations below follow from Definition 3.1:

$$c_r = \frac{|N(u) \cap C|}{|N(u) \cap \mathcal{P}|} = \frac{1 - q_C}{q_C} \qquad \forall u \in C, \tag{3}$$

$$c_d = \frac{|N(v) \cap C|}{|N(v) \cap \mathcal{P}|} = \frac{q_{\mathcal{P}}}{1 - q_{\mathcal{P}}} \qquad \forall v \in \mathcal{P}. \tag{4}$$

Let $c^\star = \frac{\sqrt{2}-1}{2}$ be the constant which later defines a threshold between *metastability* and *almost-consensus* behavior. We get

$$c_r = \frac{1}{c^\star + \delta_r} \implies q_C = 3 - 2\sqrt{2} + \delta'_r \tag{5}$$

$$c_d = c^\star + \delta_d \implies q_{\mathcal{P}} = 3 - 2\sqrt{2} + \delta'_d \tag{6}$$

for δ_r and δ'_r (δ_d and δ'_d) which are either both positive or both negative. We can now prove a *metastability* phenomenon of the 2-Choices dynamics on Core-Periphery networks.

Corollary 3.5. *Let $c^\star = \frac{\sqrt{2}-1}{2}$ be a universal constant. Let $G = (V, E)$ be an $(n, \varepsilon, c_r, c_d)$-Core-Periphery network such that each agent in the network has $\omega(\log n)$ neighbors. Then, with high probability, we have that for each round t and for any* poly(n) *number of rounds of the 2-Choices dynamics the following two statements hold:*

- *if $c_r > 1/c^\star$ by a constant, then $vol(B^{(t)}) \geqslant \frac{3}{4} vol(C)$*
- *if $c_d < c^\star$ by a constant, then $vol(R^{(t)}) \geqslant \frac{3}{4} vol(\mathcal{P})$*

where $B^{(t)}$ are the blue *agents and $R^{(t)}$ the* red *agents at time t.*

Proof sketch. Consider the following worst case scenario: Every time an agent in the core (periphery) chooses a random neighbor belonging to the periphery (core), then that neighbor is *red* (*blue*). In this scenario, the 2-Choices dynamics can be thought of as two independent Biased-2-Choices(p, σ) in which for the core $p = q_C$ and $\sigma = red$, and for the periphery $p = q_{\mathcal{P}}$ and $\sigma = blue$. From $c_r > 1/c^\star$ and $c_d < c^\star$ and Equations (5) and (6), it follows that q_C and $q_{\mathcal{P}}$ are less than $3 - 2\sqrt{2}$. By applying the metastability result of Theorem 3.4, we get that the volume of the adversary never exceeds $\frac{1-3p}{4(1-p)}$ of the network's volume. Since both q_C and $q_{\mathcal{P}}$ are smaller than $3 - 2\sqrt{2}$, we have that $\frac{1-3p}{4(1-p)} \leqslant \frac{1}{4}$ (as the inequality is tight for $p = 0$ and its value is decreasing). Thus, the volumes of *red* (*blue*) agents in the core (periphery) are at most a fraction of $\frac{1}{4}$. Therefore, the volumes of *blue* and *red* agents in the whole network are at least $\frac{3}{4}$ of the volumes of C and $\mathcal{P}$, respectively. ■

For the almost-consensus result, we require a high robustness of the core such that it remains monochromatic for a logarithmic number of rounds. The following lemma is needed to link the robustness with this property.

Lemma 3.6. *Let ε and δ be two positive constants. Let $G = (V, E)$ be a graph of n^ε agents, and let $0 \leqslant p \leqslant n^{-(\varepsilon+\delta)/2}$. Starting from a configuration such that each agent initially supports the* blue *color, within $O(\log(n))$ rounds of the Biased-2-Choices(p, red) no agent becomes* red*, w.h.p.*

Proof. The probability that an agent v changes its color to *red* at time t, given that all the other agents are still *blue*, is

$$\mathbf{P}\left(v \in R^{(t+1)} \,\middle|\, V = B^{(t)}\right) = p^2 \leqslant \left(n^{-(\varepsilon+\delta)/2}\right)^2 = n^{-(\varepsilon+\delta)}.$$

Applying the union bound over all the agents and over $\tau = O(\log n)$ rounds, we get

$$\mathbf{P}\left(\substack{\exists t \leqslant \tau \\ \exists v \in V} : v \in R^{(t+1)} \,\middle|\, V = B^{(t)}\right) \leqslant \frac{n^\varepsilon \cdot \tau}{n^{\varepsilon+\delta}} = O(n^{-\delta}).$$

Thus, all agents in the graph remain *blue* for any logarithmic number of rounds, w.h.p. ■

If $c_r \geqslant n^{(\varepsilon+\delta)/2}$, by Equation (3) it follows that

$$q_C = \frac{1}{c_r + 1} < \frac{1}{c_r} \leqslant n^{-(\varepsilon+\delta)/2}.$$

Finally, we are ready to prove the *almost-consensus* behavior of the 2-Choices dynamics on Core-Periphery networks.

Corollary 3.7. *Let $c^\star = \frac{\sqrt{2}-1}{2}$ and let ε and δ be two positive constants. Let $G = (V, E)$ be an $(n, \varepsilon, c_r, c_d)$-Core-Periphery network such that each agent in the network has $\omega(\log n)$ neighbors. If $c_r > n^{(\varepsilon+\delta)/2}$ and $c_d > c^\star$ by a constant, then the agents reach a configuration such that the* volume *of* blue *agents is $(1 - o(1))vol(V)$ within $O(\log n)$ rounds of the 2-Choices dynamics, w.h.p.*

Proof sketch. Since $c_r > n^{(\varepsilon+\delta)/2}$, we can apply Lemma 3.6 and thus the agents in the core never change color for $O(\log n)$ rounds, w.h.p. Therefore, for any $O(\log n)$ number of rounds, the process is equivalent to a Biased-2-Choices($q_{\mathcal{P}}$, *blue*) run by the periphery. Since $c_d > c^\star$ and thus $q_{\mathcal{P}} > 3 - 2\sqrt{2}$ by Equation (6), we can apply Theorem 3.4 and get almost-consensus on the *blue* color in a logarithmic number of rounds, w.h.p. ■

4 EXPERIMENTS

In Section 3 we formally studied the 2-Choices dynamics on Core-Periphery networks, observing a phase transition phenomenon that appears on a *dominance* threshold $c^\star = \frac{\sqrt{2}-1}{2}$. Here, we report the results of the empirical data obtained by simulating the 2-Choices dynamics on real-world networks. Furthermore, we discuss our results and compare them with our theoretical analysis. The source code of the experiments is freely available.[5]

[5]https://github.com/chaot4/opinion_dynamics_impl/releases/tag/AAMAS2018

We simulated the 2-Choices dynamics on 70 real-world networks, 25 of them taken from KONECT [33] and 45 of them taken from SNAP [35]. Detailed information regarding the networks and the results of the experiments are reported in Table 1. The networks chosen for the experiments are drawn from a variety of domains including social networks, communication networks, road networks, and web graphs; moreover, they range in size from thousands of nodes and thousands of edges up to roughly one million of nodes and tens of millions of edges. Before simulating the 2-Choices dynamics, we pre-process the networks in order to match the theoretical setting. In particular, for all the networks, we remove the orientation of the edges and all loops, and we work on the largest (w.r.t. the number of nodes) connected component.

The first issue we faced simulating the 2-Choices dynamics was the extraction of the set of agents representing the core. In fact, there is no exact definition of what a *good* core is with respect to *dominance* and *robustness* values. We started by using a simple heuristic to extract the core, namely the k-*rich-club* method [45]: This method establishes the core $\mathcal{C}$ as the set of k agents with highest degree and the periphery $\mathcal{P}$ as the remaining agents. Avin et al. [3] empirically show that when k is at the *symmetry point*, i.e. k is chosen such that $vol(\mathcal{C}) \approx vol(\mathcal{P})$, the core found by this method is sublinear in size with respect to the number of agents of the network. We remark that if $vol(\mathcal{C}) = vol(\mathcal{P})$ then, from the definitions of *robustness* and *dominance*, it follows that $c_r = 1/c_d$.

We initially used the k-*rich-club* method to extract the core but noticed that this simple heuristic produces a core with very low *robustness* values, contrary to what common sense would suggest to be a *good core*. In particular, low robustness values imply that the *dominance* values never go below our theoretical threshold $c^\star$ (see columns $\bar{c}_r$ and $\bar{c}_d$ in Table 1), which hinders the comparison between theoretical and experimental results. Indeed, in our theoretical analysis we assume that the core never changes color, i.e. that the *robustness* is high; however, in the experiments the core was very unstable when using the k-*rich-club* method. The main issue of this method is that it does not take into account the topological structure of the network: For example, if we consider a regular graph with a well defined core-periphery structure (which satisfies Definition 3.1), the k-*rich-club* method would identify the core to be a random subset of nodes.

Therefore, we introduce a novel heuristic for extracting the core which takes the network topology into account by prioritizing the *robustness* of the core over its *dominance*. The procedure, which we refer to as *densest-core* method, is described in Algorithm 1. Informally, it iteratively extracts the densest subgraph from the network and adds it to the core unless the core's volume becomes too large. In order to compute this constrained densest subgraph, it uses a variation of the 2-approximation algorithm [12], which chooses every time the densest subgraph that will not make the core's volume larger than the periphery's volume.

We apply the *densest-core* method to the networks and, as expected, we obtain higher *robustness* and lower *dominance* values compared to the k-*rich-club* method. The data reported in Table 1 shows that the *robustness* of the core extracted by our method is higher in all the considered datasets but one. Indeed, we finally obtain *dominance* values below the theoretical threshold $c^\star$.

Table 1: Experimental data. *Source* reports the source of the dataset, i.e. SNAP (S) or KONECT (K). The values c_r and c_d are the *robustness* and *dominance* obtained using the *densest-core* method; the values $\bar{c}_r$ and $\bar{c}_d$ are the *robustness* and *dominance* obtained using the k-*rich-club* method. $\mathcal{C}$ and $\mathcal{P}$ are the fraction of experiments in which the core's and the periphery's color respectively spread to reach an *almost-consensus*, while $\mathcal{M}$ is the fraction of experiments in which there is *metastability*, all having the core extracted with the *densest-core* method. K stands for thousand, M for million.

Dataset (Source)	$\|V\|$	$\|E\|$	c_r ($\bar{c}_r$)	c_d ($\bar{c}_d$)	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{M}$
Chicago (K)	0.8K	1.6K	6.55 (0.10)	0.15 (9.72)	0.00	0.00	1.00
email-Eu-core (S)	0.9K	32.1K	0.75 (0.53)	1.32 (1.88)	0.92	0.08	0.00
Euroroad (K)	1.0K	2.6K	5.53 (0.62)	0.18 (1.61)	0.00	0.00	1.00
Blogs (K)	1.2K	33.4K	0.62 (0.38)	1.57 (2.60)	0.00	0.00	1.00
Traffic Control (K)	1.2K	4.8K	1.25 (0.51)	0.78 (1.96)	0.00	0.00	1.00
Protein (K)	1.4K	3.8K	0.90 (0.33)	1.10 (2.95)	1.00	0.00	0.00
US Airport (K)	1.5K	34.4K	0.54 (0.48)	1.82 (2.10)	0.00	0.00	1.00
Stelzl (K)	1.6K	6.2K	1.03 (0.36)	0.96 (2.73)	1.00	0.00	0.00
Bible (K)	1.7K	18.1K	0.74 (0.54)	1.33 (1.84)	0.98	0.02	0.00
Hamster full (K)	2.0K	32.1K	0.96 (0.66)	1.02 (1.51)	1.00	0.00	0.00
Opsahl OF (K)	2.9K	31.2K	0.76 (0.55)	1.30 (1.81)	1.00	0.00	0.00
OpenFlights (K)	3.3K	38.4K	0.73 (0.50)	1.35 (1.98)	0.80	0.00	0.20
bitcoin-alpha (S)	3.7K	28.2K	0.53 (0.39)	1.87 (2.52)	1.00	0.00	0.00
ego-Facebook (S)	4.0K	176.4K	4.83 (1.53)	0.20 (0.65)	0.00	0.00	1.00
ca-GrQc (S)	4.1K	26.8K	3.33 (1.29)	0.29 (0.77)	0.00	0.00	1.00
US power grid (K)	4.9K	13.1K	3.17 (0.53)	0.31 (1.86)	0.00	0.00	1.00
bitcoin-otc (S)	5.8K	42.9K	0.52 (0.38)	1.88 (2.59)	1.00	0.00	0.00
p2p-Gnutella08 (S)	6.2K	41.5K	1.20 (0.53)	0.82 (1.86)	0.00	1.00	0.00
Route Views (K)	6.4K	25.1K	0.30 (0.16)	3.26 (6.13)	0.96	0.04	0.00
wiki-Vote (S)	7.0K	201.4K	0.60 (0.44)	1.64 (2.24)	1.00	0.00	0.00
p2p-Gnutella09 (S)	8.1K	52.0K	1.08 (0.53)	0.91 (1.86)	0.00	1.00	0.00
ca-HepPh (S)	8.6K	49.6K	1.40 (0.69)	0.71 (1.44)	1.00	0.00	0.00
p2p-Gnutella06 (S)	8.7K	63.0K	0.91 (0.53)	1.09 (1.87)	1.00	0.00	0.00
p2p-Gnutella05 (S)	8.8K	63.6K	0.93 (0.54)	1.06 (1.83)	0.86	0.14	0.00
PGP (K)	10.6K	48.6K	2.54 (1.18)	0.39 (0.84)	1.00	0.00	0.00
p2p-Gnutella04 (S)	10.8K	79.9K	0.91 (0.52)	1.08 (1.90)	1.00	0.00	0.00
ca-HepTh (S)	11.2K	235.2K	3.49 (2.39)	0.28 (0.41)	0.00	0.00	1.00
ca-AstroPh (S)	17.9K	393.9K	1.54 (0.84)	0.64 (1.18)	1.00	0.00	0.00
ca-CondMat (S)	21.3K	182.5K	1.70 (0.68)	0.58 (1.46)	1.00	0.00	0.00
p2p-Gnutella25 (S)	22.6K	109.3K	0.72 (0.41)	1.37 (2.43)	1.00	0.00	0.00
E.A.T. (K)	23.1K	594.1K	0.60 (0.48)	1.64 (2.07)	0.96	0.04	0.00
Cora citation (K)	23.1K	178.3K	1.37 (0.54)	0.72 (1.83)	1.00	0.00	0.00
CAIDA (K)	26.4K	106.7K	0.31 (0.16)	3.13 (6.03)	1.00	0.00	0.00
p2p-Gnutella24 (S)	26.4K	130.7K	0.71 (0.42)	1.39 (2.34)	1.00	0.00	0.00
cit-HepTh (S)	27.4K	704.0K	1.33 (0.74)	0.74 (1.34)	1.00	0.00	0.00
Digg (K)	29.6K	169.5K	0.59 (0.49)	1.67 (2.01)	1.00	0.00	0.00
Linux (K)	30.8K	426.4K	0.47 (0.24)	2.10 (4.14)	0.90	0.10	0.00
email-Enron (S)	33.6K	361.6K	0.71 (0.54)	1.39 (1.84)	1.00	0.00	0.00
cit-HepPh (S)	34.4K	841.5K	1.34 (0.61)	0.74 (1.61)	1.00	0.00	0.00
Internet topology (K)	34.7K	215.4K	0.61 (0.32)	1.62 (3.08)	0.88	0.00	0.12
p2p-Gnutella30 (S)	36.6K	176.6K	0.82 (0.44)	1.21 (2.23)	1.00	0.00	0.00
loc-Brightkite (S)	56.7K	425.8K	0.99 (0.71)	1.00 (1.40)	1.00	0.00	0.00
p2p-Gnutella31 (S)	62.5K	295.7K	0.78 (0.44)	1.27 (2.26)	1.00	0.00	0.00
soc-Epinions1 (S)	75.8K	811.4K	0.72 (0.60)	1.37 (1.65)	1.00	0.00	0.00
Slashdot081106 (S)	77.3K	937.1K	0.51 (0.46)	1.93 (2.13)	0.98	0.02	0.00
soc-Slashdot0811 (S)	77.3K	938.3K	0.51 (0.46)	1.93 (2.13)	1.00	0.00	0.00
ego-Twitter (S)	81.3K	2.6M	1.12 (0.57)	0.89 (1.75)	0.00	0.00	1.00
Slashdot090216 (S)	81.8K	995.3K	0.53 (0.48)	1.87 (2.08)	1.00	0.00	0.00
Slashdot090221 (S)	82.1K	1.0M	0.53 (0.48)	1.87 (2.08)	1.00	0.00	0.00
soc-Slashdot0922 (S)	82.1K	1.0M	0.53 (0.47)	1.87 (2.08)	1.00	0.00	0.00
Prosper loans (K)	89.1K	6.6M	0.82 (0.47)	1.21 (2.10)	0.00	0.00	1.00
Livemocha (K)	104.1K	4.3M	0.47 (0.38)	2.10 (2.56)	0.94	0.06	0.00
Flickr (K)	105.7K	4.6M	2.27 (1.07)	0.43 (0.92)	0.00	0.00	1.00
ego-Gplus (S)	107.6K	24.4M	0.95 (0.54)	1.04 (1.82)	0.00	0.00	1.00
epinions (S)	119.1K	1.4M	0.64 (0.52)	1.53 (1.89)	1.00	0.00	0.00
Github (K)	120.8K	879.7K	0.88 (0.70)	1.12 (1.41)	1.00	0.00	0.00
Bookcrossing (K)	185.9K	867.2K	0.52 (0.34)	1.90 (2.87)	1.00	0.00	0.00
loc-Gowalla (S)	196.5K	1.9M	1.14 (0.80)	0.87 (1.24)	0.02	0.00	0.98
email-EuAll (S)	224.8K	679.8K	0.16 (0.06)	6.19 (14.4)	0.00	0.00	1.00
web-Stanford (S)	255.2K	3.8M	2.52 (0.37)	0.39 (2.68)	0.00	0.00	1.00
amazon0302 (S)	262.1K	1.7M	2.61 (0.44)	0.38 (2.23)	1.00	0.00	0.00
com-DBLP (S)	317.0K	2.0M	1.43 (0.70)	0.69 (1.42)	1.00	0.00	0.00
web-NotreDame (S)	325.7K	2.1M	2.63 (0.60)	0.37 (1.65)	0.00	0.00	1.00
com-amazon (S)	334.8K	1.8M	1.72 (0.32)	0.57 (3.03)	0.98	0.00	0.02
amazon0312 (S)	400.7K	4.6M	2.18 (0.41)	0.45 (2.41)	0.00	0.00	1.00
amazon0601 (S)	403.3K	4.8M	2.08 (0.40)	0.47 (2.44)	0.00	0.00	1.00
amazon0505 (S)	410.2K	4.8M	2.01 (0.41)	0.49 (2.40)	0.00	0.00	1.00
web-BerkStan (S)	654.7K	13.1M	1.60 (0.43)	0.62 (2.30)	0.00	0.00	1.00
web-Google (S)	855.8K	8.5M	1.77 (0.42)	0.56 (2.33)	0.00	0.00	1.00
roadNet-PA (S)	1.0M	3.0M	7.35 (1.01)	0.13 (0.98)	0.00	0.00	1.00

Algorithm 1 Densest-Core Extraction.

```
procedure DensestCore(G)
    C* ← ∅
    do
        C ← ∅; D ← G
        while D ≠ ∅ do
            v ← LowestDegreeNode(D)
            D ← D \ {v}
            if Density(D) > Density(C) and
               FractionOfVolume(C* ∪ D) ⩽ 1/2 then
                C ← D
            end if
        end while
        C* ← C* ∪ C
        G ← G \ C
    while C ≠ ∅
    return C*
```

We proceed as follows: We initialize all the agents in C with *blue* and all the agents in $\mathcal{P}$ with *red*. Then, we simulate the 2-Choices dynamics on each network, keeping track of the volumes of *blue* and *red* agents in each iteration. We consider a simulation *metastable* if within $|V|$ iterations – waiting for a superpolynomial number of rounds would be infeasible – neither the *red* nor the *blue* agents reach a volume greater than 95% of the network's volume. Otherwise, we declare *almost-consensus* on the majority's color. The experiments were repeated 50 times for each network.

As can be observed in Figure 1, there exists an *empirical threshold* $\sigma = 1/2$ which is different from the theoretical one. In fact, in 86% of the datasets with a *dominance* below the threshold, the 2-Choices dynamics ends up in a phase of *metastability*, while in 81% of the datasets with a *dominance* above the threshold the 2-Choices dynamics converges to an *almost-consensus*. The empirical threshold is greater than the theoretical threshold because of several factors: i) in the experiments the core actually changes color to a small extent (while in the theoretical part we ignored such small *perturbations*), and it consequently lowers the probability for an agent in the periphery to pick the core's color; ii) the real-world network we used in the experiments do not have the regularity assumptions of the networks that we consider in the analysis; iii) in the experiments we declare metastability only after $|V|$ iterations and this increases the likelihood of metastable runs. The gap between the theoretical and the empirical threshold should be closed in future work by providing a more fine-grained theoretical analysis which does not assume the adversary's color to be monochromatic and considers more general networks.

We want to highlight that the protocol's convergence to the core's color (as shown in Table 1) is remarkable in light of the fact that the *densest-core* method ensures that the sum of the agents' degrees of the core and of the periphery are equal. More precisely, notice that equal volumes of core and periphery, starting from an initial configuration where two sets support different colors, is sufficient in the Voter Model to say that the two initial colors have the same probability to be the one eventually supported by all agents [29], regardless of the topological structure. Previous works on the 2-Choices dynamics [14] provided convergence results

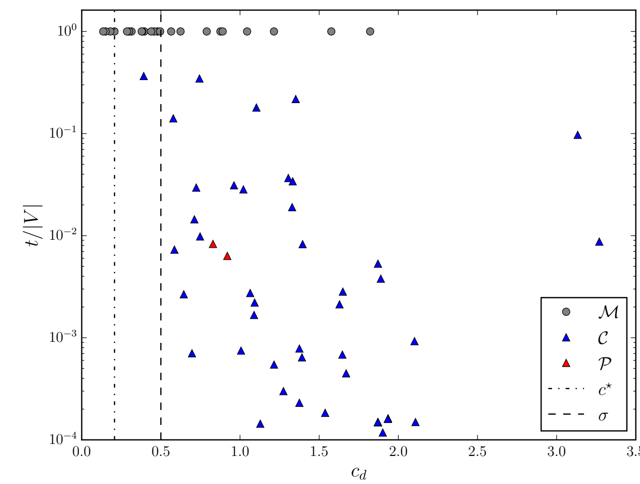

Figure 1: Metastability and almost-consensus of the experiments compared to the theoretical and empirical thresholds $c^\star$ and σ. The 86% of the runs are metastable when $c_d < \sigma$; in 81% of them there is an almost-consensus if $c_d > \sigma$. The value t is the arithmetic mean of the number of rounds until almost-consensus/metastability was declared.

which are parametrized only in the difference of the volumes of the two sets, suggesting a similar behavior. Our experimental results highlight the insufficiency of the initial volume distribution as an accurate predictive parameter, showing that the topological structure of the core plays a decisive role.

5 CONCLUSIONS

We analyzed the 2-Choices dynamics on a class of networks with core-periphery structure, where the *core*, a small group of densely interconnected agents, initially holds a different opinion from the rest of the network, the *periphery*. We formally proved that a *phase-transition* phenomenon occurs: Depending on the dominance parameter c_d characterizing the connectivity of the network, either the core's opinion spreads among the agents of the periphery and the network reaches an *(almost-)consensus*, or there is a *metastability* phase in which none of the opinions prevails over the other.

We validated our theoretical results on several real-world networks. Introducing an efficient and effective method to extract the core, we showed that the same parameter c_d is sufficient to predict the convergence/metastability of the 2-Choices dynamics most of the time. Surprisingly, even if the volumes of core and periphery are equal, the core's opinion wins in most of the cases. These behaviors suggest that in many real-world networks there actually is a core whose initial opinion has a great advantage of spreading in simple opinion dynamics such as the 2-Choices. We think that these results are a relevant step towards understanding which dynamical properties are implicitly responsible for causing social and economic agents to form networks with a core-periphery structure.

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