Heterogeneous Two-facility Location Games with Minimum Distance Requirement*

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ABSTRACT

We study the mechanism design problem of a social planner for locating two heterogeneous facilities on a line interval $[0, 1]$, where a set of n strategic agents report their locations and a mechanism determines the locations of the two facilities. Unlike prior work on two-facility location games, we consider the requirement of the minimum distance d between the two facilities. As the two facilities are heterogeneous and have additive effects on agents, we model that the cost of an agent is the sum of his distances to both facilities and the social cost is the total cost of all agents. In the two-facility location game to minimize the social cost, we show that the optimal solution can be computed in polynomial time and prove that carefully choosing one optimal solution as output is strategyproof. In the obnoxious two-facility location game for maximizing the social utility, a mechanism outputting the optimal solution is not strategyproof and we propose new deterministic group strategyproof mechanisms with provable approximation ratios. Moreover, we establish a lower bound $\frac{7-d}{6}$ for the approximation ratio achievable by deterministic strategyproof mechanisms. Finally, we study the two-facility location game with triple-preference, where each of the two facilities may be favorable, obnoxious, indifferent for any agent. We further allow each agent to misreport his location and preference towards the two facilities and design a deterministic group strategyproof mechanism with approximation ratio 4.

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1 INTRODUCTION

In this paper, we study the two-facility location games with minimum distance requirement between the two facilities. Here, the minimum distance requirement means the distance between the two facilities should be at least a certain value. Its origin, the two-facility location game, models the scenario where the social planner is going to build two facilities on a line segment with some agents who want to minimize (maximize) their own costs (utilities). The agents are required to report their locations as private information, which will then be mapped to the two facilities' locations by a mechanism, with the purpose of optimizing the social cost (utility).

We find that the minimum distance requirement models well the real life where the social planner builds two heterogeneous facilities to serve agents. There is some natural conflict for them to co-locate in practice. Consider the first scenario (fitting the two-facility location game) that the social planner plans to deploy an Internet café and a primary school in a street, where all agents prefer living close for easy access to both internet surfing and education resource. In the worry that some pupils in the primary school may develop addiction to computer games after class, the two facilities should be some distance away from each other. Since each agent needs services from both heterogeneous facilities, similar to [17], the cost of each agent should be the sum of his Euclidean distances to the two facilities. Consider the second scenario (fitting the obnoxious two-facility location game) for building a refuse landfill and a sewage treatment plant, where all agents prefer living far away from both. The minimum distance requirement should also be considered here since the refuse landfill may contaminate water output by the sewage treatment plant and may need future expansion. The third scenario (fitting the two-facility location game with triple-preference) is that the social planner plans to build a park and a kindergarten with at least some distance in between. Some agents prefer to live close by to play in the park, others want to stay away to avoid the crowd around the park and the rest of agents are indifferent to the park; Some agents prefer to stay close to the kindergarten to pick up kids, others want to stay away to avoid noise of the kindergarten and the rest of agents are indifferent to the kindergarten.

An agent may have a chance to improve his benefit, i.e., decrease his cost or increase his utility by misreporting his location. Therefore, we emphasize on strategyproofness of a mechanism, which guarantees that an agent cannot acquire any benefit by misreporting. We aim to design deterministic strategyproof mechanisms whose performances approximate well the optimal social cost/utility. The evaluation of a mechanism is mainly conducted by the approximation ratio, which is the worst ratio between the social cost/utility of the mechanism output and the optimal social cost/utility considering all possible agent profiles.

We summarize our key novelty and results as follows.

- *Facility location games for two heterogeneous facilities with minimum distance requirement:* To the best of our knowledge, this is the first time that the minimum distance constraint is included in the two-facility location games for strategyproof mechanism design.
- *Mechanism design for the two-facility location game:* In Section 3, each agent prefers to stay close to both two facilities and may misreport his location. Given heterogeneous natures for the two facilities, the cost of an agent is the sum of his distances to both facilities and the social cost is the total cost of all agents. We design a mechanism outputting an optimal solution and prove it is strategyproof.
- *Mechanism design for the obnoxious two-facility location game:* In Section 4, each agent prefers to stay far away from the two facilities. The optimal solution is no longer strategyproof and we design new deterministic group strategyproof mechanisms with provable approximation ratios. We further prove the lower bound for the approximation ratio achievable by strategyproof mechanisms.
- *Mechanism design for the two-facility location game with triple-preference:* In Section 5, we extend to the general case with triple-preference, where each of the two facilities may be favorable, obnoxious or indifferent for any agent. Besides locations, agents can also be allowed to misreport their preferences towards the two facilities. We design a deterministic group strategyproof mechanism with approximation ratio 4 and obtain the lower bound.

1.1 Related work

In the algorithmic view of locating one-facility, [15] first studied strategyproof mechanisms with provable approximation ratios on a line. [14] and [16] provided characterizations of deterministic strategyproof mechanisms on line, tree, and cycle networks. For the obnoxious facility game, the mechanism design to improve the social utility was first studied by [3]. They presented a 3-approximation deterministic group strategyproof mechanism and proved a lower bound of 2. [9] characterized strategyproof mechanisms with exactly two candidates in the general metric and showed that there exists a lower bound 3 for strategyproof mechanism in any metric, matching the upper bound 3 in [3]. [24] extended mechanism design for both games with weighted agents on a line and provided the lower and upper bounds on the optimal social utility. Combining the above two models together, the dual-preference game was studied in [25] and [6], where some agents want to be close to the facility while the others want to be far away from the facility.

For the two-facility location game, [11] improved the lower bounds for the two homogeneous facilities scenario and the scenario when one agent possesses multiple locations. [10] considered the cost of an agent to be the distance between his own location and the nearest facility in a general metric space. [20] proposed a class of percentile mechanisms in the form of generalized median mechanisms. [17] and [18] initiated the study on two heterogeneous facility location games in the graph where the cost of an agent is the sum of his distances to both facilities. [23] proposed the optional preference model for the facility location game with two heterogeneous facilities on a line, where agents are allowed to have optional preference. [1] studied heterogeneous k-facility location games on the line segment where the preferences of the agents over the facilities are the private information and the locations of agents are known to the social planner. [7] proposed a fractional preference model for the facility location game with two facilities that serve the similar purpose on a line where each agent has his location information as well as fractional preference towards the two facilities.

In addition, [21] extended the original model by fully characterizing the deterministic false-name-proof facility location mechanisms for locating a single facility on a line. Then [19] extended the model by characterizing the possible outcomes of false-name-proof mechanisms on a line for locating two facilities on a line as well as on a circle. [22] studied variable populations in the static and dynamic facility location models and proposed a class of online social choice functions for the dynamic model. [4] considered a multi-stage facility reallocation problems on the line, where a facility is being moved between stages based on the locations reported by n agents and characterized the optimal mechanisms both in the offline setting and in the online setting. Other extensions of the facility location game can be found in [2, 5, 8, 12].

Besides, for minimum distance requirement, [13] proposed non-strategic version of the two-facility location problem but our paper is the first in the strategic two-facility location game. Unlike prior work on two-facility location games, we consider the requirement of the minimum distance and agents' strategic behavior of preferences towards the two facilities.

2 SYSTEM MODEL

Let $N = \{1, 2, \cdots, n\}$ be the set of agents located on a line interval $I = [0, 1]$. We denote $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ as the n agents' location profile, which is private information and needs to be reported by each agent. Without loss of generality, we assume $x_i \le x_{i+1}$ for any $1 \le i \le n-1$.

In the two-facility location game, a mechanism f outputs two facilities' locations (y_1, y_2) based on a given location profile $\mathbf{x}$, i.e., $(y_1, y_2) = f(\mathbf{x}) : I^n \to I^2$. Denote the minimum distance requirement between the two facilities as $d \in [0, 1]$, i.e., $|y_2 - y_1| \ge d$. Since the two facilities are heterogeneous, the cost of agent i is denoted as the sum of his distances to the two facilities, i.e.,

$$c_i(f(\mathbf{x}), x_i) = |y_1 - x_i| + |y_2 - x_i|. \quad (1)$$

Let $\mathbf{x}_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ be the location profile without agent i. Let $\mathbf{x}_S$ be the location profile with all agent $i \in S \subseteq N$ and $\mathbf{x}_{-S}$ be the location profile without any agent $i \in S \subseteq N$. The social cost of a mechanism $f(\mathbf{x})$ on $\mathbf{x}$ is denoted as the sum of costs of n agents, i.e.,

$$SC(f(\mathbf{x}), \mathbf{x}) = \sum_{i=1}^{n} c_i(f(\mathbf{x}), x_i). \quad (2)$$

As agents may misreport their locations to change y_1 and y_2 for their own benefits, strategyproofness of $f(\mathbf{x})$ is important to ensure. Next we formally define the strategyproofness and the group strategyproofness respectively.

Definition 2.1. A mechanism is strategyproof in the two-facility location game if no agent can benefit from misreporting his location. Formally, given agent i, profile $\mathbf{x} = \{x_i, \mathbf{x}_{-i}\} \in I^n$, and any misreported location $x_i' \in I$, it holds that $c_i(f(x_i, \mathbf{x}_{-i}), x_i) \le c_i(f(x_i', \mathbf{x}_{-i}), x_i)$.

Definition 2.2. A mechanism is group strategyproof in the two-facility location game if for any group of agents, at least one of them cannot benefit if they misreport simultaneously. Formally, given a non-empty set $S \subseteq N$, profile $\mathbf{x} = \{\mathbf{x}_S, \mathbf{x}_{-S}\} \in I^n$, and the misreported $\mathbf{x}_S' \in I^{|S|}$, there exists $i \in S$, satisfying $c_i(f(x_S, \mathbf{x}_{-S}), x_i) \le c_i(f(x_S', \mathbf{x}_{-S}), x_i)$.

In the facility location game, we are interested in designing strategyproof mechanisms that also perform well with respect to minimizing the social cost. For a location profile, let $OPT_1(\mathbf{x})$ be the optimal (minimum) social cost. A mechanism f has an approximation ratio γ, if for any possible profile $\mathbf{x} \in I^n, SC(f, \mathbf{x}) \le \gamma OPT_1(\mathbf{x})$.

In the obnoxious two-facility location game, the agents prefer to be far away from the two facilities. We define agent i's utility as $u_i(f(\mathbf{x}), x_i) = |y_1 - x_i| + |y_2 - x_i|$. The objective is to maximize the social utility $SU(f(\mathbf{x}), \mathbf{x}) = \sum_{i=1}^{n} u_i(f(\mathbf{x}), x_i)$.

Definition 2.3. A mechanism is strategyproof in the obnoxious two-facility location game if no agent can benefit from misreporting his location. Formally, given agent i, profile $\mathbf{x} = \{x_i, \mathbf{x}_{-i}\} \in I^n$, and any misreported location $x_i' \in I$, it holds that $u_i(f(x_i, \mathbf{x}_{-i}), x_i) \ge u_i(f(x_i', \mathbf{x}_{-i}), x_i)$.

The definition of the group strategyproofness in the obnoxious two-facility location game can be similarly defined by following Definition 2.2. For a location profile $\mathbf{x}$, let $OPT_2(\mathbf{x})$ be the optimal (maximum) social utility. A mechanism f has an approximation ratio γ, if for any profile $\mathbf{x} \in I^n, OPT_2(\mathbf{x}) \le \gamma SU(f, \mathbf{x})$. Combining the above two models, the two-facility location game with triple-preference will be shown later where we will further allow agents to misreport their preferences towards the two facilities.

3 TWO-FACILITY LOCATION GAMES

In this section, we consider the cost of agent i as $c_i = |y_1 - x_i| + |y_2 - x_i|$. Assume, without loss of generality, that facility 1 is on the left of facility 2, i.e., $0 \le y_1 \le y_2 \le 1$. From (1) and (2), we rewrite the social cost as function $g = SC(f(\mathbf{x}), \mathbf{x})$ of two variables $(y_1, y_2) \in D$, and the social optimal cost $OPT_1(\mathbf{x})$ can be obtained by solving:

$$\begin{aligned} &\min_{y_1,y_2} g(y_1, y_2|\mathbf{x}) = \min_{y_1,y_2} \sum_{i=1}^{n} (|y_1 - x_i| + |y_2 - x_i|), \\ &\text{s.t. } (y_1, y_2) \in D = \{(y_1, y_2)|y_2 - y_1 \ge d, 0 \le y_1, y_2 \le 1\}, \\ &\quad \text{given } 0 \le x_i \le 1, \text{for } i = 1, ..., n \text{ and } 0 \le d \le 1. \end{aligned} \quad (3)$$

The feasible region D of (y_1, y_2) is an isosceles right triangle with three corners $(0, d), (1-d, 1), (0, 1)$ and is closed convex.

PROPOSITION 3.1. *g is a convex function with $(y_1, y_2) \in D$ and can obtain its minimum in D at $y_2 - y_1 = d$.*

PROOF. According to the property of convex functions, agent i's cost function $|y_1 - x_i| + |y_2 - x_i|$ with two variables (y_1, y_2) is convex, since function $|y_1 - x_i|$ with variable y_1 and function $|y_2 - x_i|$ with variable y_2 are both convex. Therefore, social cost $\sum_{i=1}^{n}(|y_1 - x_i| + |y_2 - x_i|)$ is convex in (y_1, y_2). As D is a convex set, g is a continuous convex function in D. To obtain the optimal social cost OPT_1, we consider the linear optimization problem (3). Further, we define $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$, as the boundary of the closed convex set D, where $\partial D_1 = \{(y_1, y_2)|y_2 - y_1 = d, 0 \le y_1 \le 1 - d\}$, $\partial D_2 = \{(y_1, y_2)|y_1 = 0, d \le y_2 \le 1\}$, $\partial D_3 = \{(y_1, y_2)|y_2 = 1, 0 \le y_1 \le 1 - d\}$. Obviously, $D \backslash \partial D$ is the largest open convex subset of D.

Next we prove that the optimal point in $(y_1, y_2) \in D$ can be obtained in ∂D_1. It is known that

$$\arg\min_{y} \sum_{i=1}^{n} |y - x_i| = \begin{cases} [x_{\frac{n}{2}}, x_{(\frac{n}{2}+1)}], & \text{if } n \text{ is even;} \\ x_{\frac{n+1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \quad (4)$$

We have two cases according to the parity of n.

Case 1: n is even. Due to (4), function $g(y_1, y_2|\mathbf{x})$ obtains its local minimum at the point $(y_1, y_2) \in E = [x_{\frac{n}{2}}, x_{\frac{n}{2}+1}] \times [x_{\frac{n}{2}}, x_{\frac{n}{2}+1}]$, given $(y_1, y_2) \in [0, 1] \times [0, 1]$. E is a square area and also a closed convex set. We have two subcases depending on the relationship between E and D.

1) $E \cap D \ne \emptyset$. The necessary condition for $E \cap D \ne \emptyset$ is that $E \cap \partial D_1 \ne \emptyset$, due to the shapes of triangle D and square E in $[0, 1]^2$. See convex sets E and D in Figure 1. The optimal point in $(y_1, y_2) \in D$ can be obtained in $E \cap \partial D_1$ which is a non-empty subset of ∂D_1.

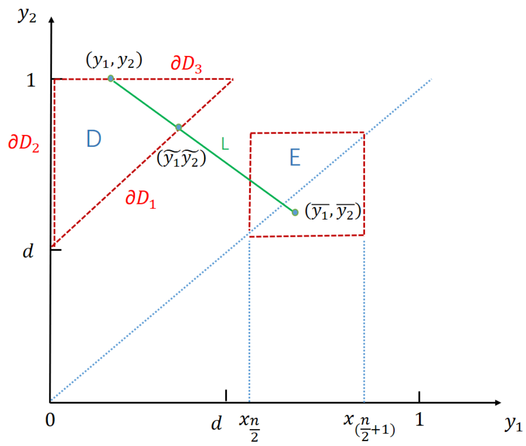

Figure 1: Connections between the line segment L and convex sets E, D.

2) $E \cap D = \emptyset$. Since g is a convex function, by (4), its local minimum can only be obtained at the point $(y_1, y_2) \in E$, given $(y_1, y_2) \in [0, 1] \times [0, 1]$. Since $E \cap D = \emptyset$, there is no local minimum point but one global minimum point (optimal point) in D. The optimal point in D can only be obtained at a point in ∂D. Otherwise, if the optimal point is obtained at a point in open set $D \backslash \partial D$, this point must be the point of local minimum, which contradicts the fact that g can only obtain its local minimum at the point $(y_1, y_2) \in E$ and $E \cap D = \emptyset$.
Further, this optimal point can be obtained in ∂D_1. Otherwise, we assume the optimal point (y_1, y_2) is only obtained in $\partial D_2 \cup \partial D_3$ as shown in Figure 1. Then we can always draw a line segment L connecting the optimal point $(y_1, y_2) \in \partial D_2 \cup \partial D_3$ and any point $(\bar{y_1}, \bar{y_2}) \in E$. This line segment L must intersect with line segment ∂D_1 at point $(\widetilde{y_1}, \widetilde{y_2})$. Note that $(\widetilde{y_1}, \widetilde{y_2})$ is between points (y_1, y_2) and $(\bar{y_1}, \bar{y_2})$ on line segment L. Recall that the value of g at point $(\widetilde{y_1}, \widetilde{y_2})$ is greater than the values of g at points (y_1, y_2) and $(\bar{y_1}, \bar{y_2})$. However, this contradicts the fact that function g is also convex in convex line segment L.

Case 2: n is odd. Due to (4), function $g(y_1, y_2|\mathbf{x})$ can obtain its local minimum at the point $(y_1, y_2) = (x_{\frac{n+1}{2}}, x_{\frac{n+1}{2}})$, given $(y_1, y_2) \in [0, 1] \times [0, 1]$. This point cannot overlap with D, i.e., $(x_{\frac{n+1}{2}}, x_{\frac{n+1}{2}}) \cap D = \emptyset$, since any point in D must satisfy that $y_2 - y_1 > 0$. Thus we can follow the similar proof in subcase 2) of Case 1 and draw the same conclusion that this optimal point in D can be obtained in ∂D_1. □

According to Proposition 3.1, we can consider an equivalent linear optimization problem to replace (3):

$$\min_{y_1, y_2} g(y_1, y_2|\mathbf{x}) \text{ s.t. } (y_1, y_2) \in \partial D_1 = \{(y_1, y_2)|y_2 - y_1 = d, y_1 \in [0, 1 - d]\}. \quad (5)$$

Thus, we let $y_2 = y_1 + d$ and only need to find the solution of facility 1's location to solve (5), which we denote as $y_1^\star(d; \mathbf{x})$:

$$y_1^\star(d; \mathbf{x}) = \arg\min_{y_1 \in [0, 1-d]} \sum_{i=1}^{n} (|y_1 - x_i| + |y_1 + d - x_i|). \quad (6)$$

Before solving (6), it is widely known that if n is even,

$$\arg\min_y \sum_{i=1}^{n} |y - x_i| = [x_{\frac{n}{2}}, x_{(\frac{n}{2}+1)}]. \quad (7)$$

Define location profile $\mathbf{x} - d = \{x_1 - d, x_2 - d, \ldots, x_n - d\}$. From (6) and (7), we have $y_1^\star(d; \mathbf{x}) \in [\tilde{x}_n, \tilde{x}_{n+1}]$, where $\tilde{x}_i$ is denoted as the i-th order statistic of the set $\{\mathbf{x} - d, \mathbf{x}\} \in I^{2n}$. Since $y_1^\star(d; \mathbf{x})$ should be within feasible interval $[0, 1 - d]$, we have the solution of facility 1's location, which is

$$y_1^\star(d; \mathbf{x}) \in [\tilde{x}_n, \tilde{x}_{n+1}] \cap [0, 1 - d] = [\max\{0, \tilde{x}_n\}, \min\{1 - d, \tilde{x}_{n+1}\}]$$

We can also choose a special $y_1^\star(d; \mathbf{x})$ to make it strategyproof as shown in Mechanism 1 below. However, we should note that an arbitrary choice of $y_1^\star(d; \mathbf{x})$ in the range cannot guarantee strategyproofness.

Mechanism 1. $(y_1, y_2) = (y_1^\star(d; \boldsymbol{x}), y_1^\star(d; \boldsymbol{x}) + d)$ *where*

$$y_1^\star(d; \boldsymbol{x}) = \max\{0, \tilde{x}_n\} = \begin{cases} 0, & \text{if } x_n \le d; \\ \tilde{x}_n, & \text{if } x_n > d. \end{cases} \quad (8)$$

Theorem 3.2. *Mechanism 1 is strategyproof.*

Proof. If $x_n \le d$, then $x_1 - d$, $x_2 - d$, $\ldots$, $x_n - d \le 0$. Thus $\tilde{x}_n = x_n - d \le 0$ and $y_1^\star(d; \mathbf{x}) = \max\{0, \tilde{x}_n\} = 0$. Otherwise if $x_n > d$, then $\tilde{x}_n \ge \min\{x_1, x_n - d\} \ge 0$. Thus $y_1^\star(d; \mathbf{x}) = \max\{0, \tilde{x}_n\} = \tilde{x}_n$. Therefore we design $y_1^\star(d; \mathbf{x})$ in (8) and $f(\mathbf{x}) = (y_1^\star(d; \mathbf{x}), y_1^\star(d; \mathbf{x}) + d)$. Suppose that agent i misreports his location from x_i to x_i'. Define $\mathbf{x}' = \{x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\}$, $x_n(\mathbf{x}') = \max\{\mathbf{x}'\}$, $\tilde{x}_n(\mathbf{x}')$ as the i-th order statistic of the set $\{\mathbf{x}' - d, \mathbf{x}'\} \in I^{2n}$ and $y_1^\star(d; \mathbf{x}')$ as the location of facility 1 in (8) after agent i's misreporting. We divide our discussion of strategyproofness into three cases according to x_i.

Case 1: $y_1^\star(d; \mathbf{x}) \le x_i \le y_1^\star(d; \mathbf{x}) + d$. Agent i has no incentive to misreport his location since he has obtained the minimum cost d.

Case 2: $x_i < y_1^\star(d; \mathbf{x})$. Agent i misreports his location from x_i to x_i'. We divide this case into two parts according to x_i'.

(1) $x_i' < x_i$. We have two choices of $y_1^\star(d; \mathbf{x})$ in (8).
  (a) If $x_n \le d$, then we choose 0 as $y_1^\star(d; \mathbf{x})$ in (8). We have $y_1^\star(d; \mathbf{x}) = 0$, which contradicts the fact that $y_1^\star(d; \mathbf{x}) > x_i \ge 0$. Hence, this choice never exists.
  (b) If $x_n > d$, then we choose $\tilde{x}_n$ as $y_1^\star(d; \mathbf{x})$ in (8). After agent i's misreporting, since $x_i' < x_i < y_1^\star(d; \mathbf{x})$, we still have $x_n(\mathbf{x}') = x_n > d$ and $y_1^\star(d; \mathbf{x}') = \tilde{x}_n(\mathbf{x}')$ in (8). Since $x_i' < x_i < y_1^\star(d; \mathbf{x}) = \tilde{x}_n$, we have $\tilde{x}_n(\mathbf{x}') = \tilde{x}_n$, which means $y_1^\star(d; \mathbf{x}') = \tilde{x}_n$ and new locations of the two facilities do not change.

(2) $x_i' > x_i$. After agent i misreports, we have $x_n(\mathbf{x}') \ge x_n$ and thus $y_1^\star(d; \mathbf{x}') \ge y_1^\star(d; \mathbf{x})$ in (8). Hence, $c_i(f(\mathbf{x}'), x_i) = 2y_1^\star(d; \mathbf{x}') + d - 2x_i \ge c_i(f(\mathbf{x}), x_i) = 2y_1^\star(d; \mathbf{x}) + d - 2x_i$ and agent i increases his cost.

Case 3: $x_i > y_1^\star(d; \mathbf{x}) + d$. Agent i misreports his location from x_i to x_i'. We divide this case into two parts.

(1) $x_i' < x_i$. After agent i's misreporting, we have $x_n(\mathbf{x}') \le x_n$, $\tilde{x}_n(\mathbf{x}') \le \tilde{x}_n$ and thus $y_1^\star(d; \mathbf{x}') \le y_1^\star(d; \mathbf{x})$ in (8).

Hence, $c_i(f(\mathbf{x}'), x_i) = 2x_i - 2y_1^\star(d;\mathbf{x}') - d \geq c_i(f(\mathbf{x}), x_i) = 2x_i - 2y_1^\star(d;\mathbf{x}) - d$ and agent i increases his cost.

(2) $x_i' > x_i$. We have two choices of $y_1^\star(d;\mathbf{x})$ in (8).

(a) If $x_n \leq d$, then we choose 0 as $y_1^\star(d;\mathbf{x})$ in (8). We have $x_i \leq x_n \leq d$, which contradicts the fact that $x_i > y_1^\star(d;\mathbf{x}) + d$. Hence, this choice never exists.

(b) If $x_n > d$, then we choose $\tilde{x}_n$ as $y_1^\star(d;\mathbf{x})$ in (8). After agent i's misreporting, since $x_i' > x_i > y_1^\star(d;\mathbf{x}) + d$, we still have $x_n(\mathbf{x}') \geq x_n > d$ and $y_1^\star(d;\mathbf{x}') = \tilde{x}_n(\mathbf{x}')$ in (8). Since $x_i' - d > x_i - d > y_1^\star(d;\mathbf{x}) = \tilde{x}_n$, we have $\tilde{x}_n(\mathbf{x}') = \tilde{x}_n$, which means $y_1^\star(d;\mathbf{x}') = \tilde{x}_n$ and new locations of the two facilities do not change.

Therefore, Mechanism 1 is strategyproof. □

4 OBNOXIOUS TWO-FACILITY LOCATION GAMES

In this section, we study the obnoxious two-facility location game, where all agents dislike the two facilities. We need to solve the problem: $\max g(y_1, y_2|\mathbf{x})$ s.t. $(y_1, y_2) \in D$.

Proposition 4.1. *Social utility g can reach its maximum if (y_1, y_2) is at one out of three points $(0, d)$, $(1-d, 1)$, $(0, 1)$.*

Proof. Since function g is a convex function according to Proposition 3.1, g has a global maximum in D. Further, g can obtain its global maximum on boundary ∂D. Otherwise, if g obtains its global maximum at point (y_1, y_2) in open set $D \backslash \partial D$, that point must be a local maximum point, which contradicts the fact that g is a convex function. Then finding the maximum point of g on D is equivalent to finding the maximum point of g on $\partial D = \partial D_1 \cup \partial D_2 \cup \partial D_3$. ∂D_i's $(i = 1, 2, 3)$ are line segments and thus convex sets. Similarly, the convex function g in each line segment ∂D_i reaches its maximum at the boundary of ∂D_i, i.e., two endpoints of ∂D_i. Overall, the maximum point (y_1, y_2) of g over the three line segments can only be among the three corner points of ∂D : $(0, d), (1-d, 1), (0, 1)$. □

It is easy to obtain OPT_2 by using Proposition 4.1. However, a mechanism outputting OPT_2 the optimal solution is not strategyproof given $d < 1$. Take an example when $d = 0$, the obnoxious two-facility location game degenerates to the obnoxious one-facility location game, where the optimal location is not strategyproof according to [3]. Next, we propose strategyproof mechanisms.

Mechanism 2. *Given a location profile $\boldsymbol{x}$, return $f(\boldsymbol{x}) = (y_1, y_2) = (0, 1)$.*

Theorem 4.2. *Mechanism 2 is group strategyproof with approximation ratio $\gamma = 2 - d$.*

Proof. Mechanism 2 is group strategyproof since (y_1, y_2) is fixed at $(0, 1)$.

The social utility of Mechanism 2 is $SU((0, 1), \mathbf{x}) = n$. For any agent i's utility, we have

$$d \leq |y_1 - y_2| \leq u_i((y_1, y_2), x_i) = |y_1 - x_i| + |y_2 - x_i| \leq |y_1 + y_2 - 2x_i| \leq 2 - d,$$

due to $|y_2 - y_1| \geq d$. Thus, for the optimal utility, we have

$$OPT_2(\mathbf{x}) = \max_{(y_1, y_2) \in D} \sum_{i=1}^{n} (|y_1 - x_i| + |y_2 - x_i|) \leq (2 - d)n.$$

Therefore, $\gamma = \frac{OPT_2(\mathbf{x})}{SU((0,1), \mathbf{x})} \leq 2 - d$, which is within $[1, 2]$. □

Mechanism 2 does not take agents' locations into account. By counting agents' numbers in different location intervals, we propose Mechanism 3 which selects (y_1, y_2) among all the three candidate optimal points $(0, d), (1-d, 1), (0, 1)$.

Mechanism 3. *Denote $l_1 = \frac{1}{2}(1-d)$ and $l_2 = \frac{1}{2}(1+d)$. Given a location profile $\boldsymbol{x}$, if more than $\frac{n}{2}$ agents are located in $[0, l_1]$, $f(\boldsymbol{x}) = (y_1, y_2) = (1-d, 1)$, if more than $\frac{n}{2}$ agents are located in $[l_2, 1]$, $f(\boldsymbol{x}) = (y_1, y_2) = (0, d)$, and otherwise, $f(\boldsymbol{x}) = (y_1, y_2) = (0, 1)$.*

Theorem 4.3. *Mechanism 3 is group strategyproof with approximation ratio $\gamma = \max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\}$.*

Proof. We first prove group strategyproofness. We have, for any $x_i \in [0, l_1]$,

$$u_i((1-d, 1), x_i) \geq u_i((0, 1), x_i) \geq u_i((0, d), x_i); \quad (9)$$

for any $x_i \in [l_2, 1]$, $u_i((0, d), x_i) \geq u_i((0, 1), x_i) \geq u_i((1-d, 1), x_i)$; for any $x_i \in (l_1, l_2)$, $u_i((0, 1), x_i) \geq u_i((1-d, 1), x_i)$, $u_i((0, d), x_i)$. Let $S \subseteq N$ be an agent coalition. We must prove that the agents in S cannot all gain by misreporting. We denote n_1, n_2, n_3 as the numbers of agents in $[0, l_1]$, $[l_2, 1]$, (l_1, l_2) without misreporting, respectively. n_1', n_2', n_3' are the numbers of agents in $[0, l_1]$, $[l_2, 1]$, (l_1, l_2) with misreporting, respectively. The new location profile is $\mathbf{x}'$ to mislead the two facilities' locations to (y_1', y_2'). We have three cases.

Case 1: $n_1 > \frac{n}{2}$, thus $(y_1, y_2) = (1-d, 1)$.

(1) If $n_1' > \frac{n}{2}$, then $(y_1', y_2') = (1-d, 1)$ and $u_i(f(\mathbf{x}), x_i) = u_i(f(\mathbf{x}'), x_i)$ for any agent $i \in N$.

(2) If $n_2' > \frac{n}{2}$, then $(y_1', y_2') = (0, d)$. Since $n_2 + n_3 \leq \frac{n}{2}$, at least one agent i in $[0, l_1]$ misreports his location to $x_i' \in [l_2, 1]$. Thus $u_i(f(\mathbf{x}'), x_i) \leq u_i(f(\mathbf{x}), x_i)$ due to (9).

(3) Otherwise, $(y_1', y_2') = (0, 1)$. Since $n_1' \leq \frac{n}{2}$, at least one agent i in $[0, l_1]$ misreports his location to $x_i' \in [l_2, 1] \cup (l_1, l_2)$. Due to (9), $u_i(f(\mathbf{x}'), x_i) \leq u_i(f(\mathbf{x}), x_i)$

Case 2: $n_2 > \frac{n}{2}$, thus $(y_1, y_2) = (0, d)$. Strategyproofness analysis of Case 2 is the same as Case 1.

Case 3: Otherwise, $(y_1, y_2) = (0, 1)$. We have three subcases and can similarly follow the proof of Case 1 to draw the same conclusion that f is group strategyproof.

Next, we analyze the ratio γ. We have three cases.

Case 1: $n_1 > \frac{n}{2}$ and then $(y_1, y_2) = (1-d, 1)$. We have

$$\begin{aligned} SU((1-d, 1), \mathbf{x}) \geq & \sum_{x_i \in [0, l_1]} (2 - d - 2x_i) + n_3 d + n_2 d \\ \geq & n_1 + n_2 d + n_3 d, \end{aligned} \quad (10)$$

$$\begin{aligned} SU((0, d), \mathbf{x}) \leq & \sum_{x_i \in [0, l_1]} (x_i + |d - x_i|) + n_3 + n_2(2 - d) \\ \leq & n_1 \max\{d, 1 - 2d\} + n_2(2 - d) + n_3, \end{aligned} \quad (11)$$

$$SU((0, 1), \mathbf{x}) = n. \quad (12)$$

Due to (10), (11), (12) and $n_1 > \frac{n}{2}$, we have

$$\frac{SU((0,d),\mathbf{x})}{SU((1-d,1),\mathbf{x})} \le \frac{n_1 \max\{d, 1-2d\} + (n-n_1)(2-d)}{n_1 + (n-n_1)d}$$
$$\le \frac{\frac{n}{2}\max\{d,1-2d\} + \frac{n}{2}(2-d)}{\frac{n}{2} + (n-\frac{n}{2})d} = \max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\},$$
$$\frac{SU((0,1),\mathbf{x})}{SU((1-d,1),\mathbf{x})} \le \frac{n}{n_1 + (n-n_1)d} \le \frac{n}{\frac{n}{2} + \frac{n}{2}d} = \frac{2}{1+d},$$

and thus according to Proposition 4.1,

$$\gamma = \frac{\max\{SU((0,d),\mathbf{x}), SU((0,1),\mathbf{x})\}}{SU((1-d,1),\mathbf{x})} = \max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\}.$$

Case 2: $n_2 > \frac{n}{2}$. The analysis is similar to Case 1.

Case 3: otherwise, $(y_1, y_2) = (0,1)$. By (11),

$$SU((0,d),\mathbf{x}) \le n_1 + n_2(2-d) + n_3 = n_2(1-d) + n$$
$$\le \frac{n}{2}(1-d) + n \quad (13)$$

and we have $SU((1-d,1),\mathbf{x}) \le \frac{n}{2}(1-d) + n$. Thus

$$\gamma = \max\{\frac{SU((0,d),\mathbf{x})}{SU((0,1),\mathbf{x})}, \frac{SU((1-d,1),\mathbf{x})}{SU((0,1),\mathbf{x})}\}$$
$$\le \frac{\frac{n}{2}(1-d) + n}{n} = \frac{3-d}{2}.$$

In conclusion, $\gamma \le \max\{\max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\}, \frac{3-d}{2}\} = \max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\}$, which is within $[1,3]$. □

By combining Mechanism 2 and Mechanism 3, we have the following mechanism which is also group strategyproof and can obtain a smaller approximation ratio for $d \in [0,1]$.

Mechanism 4. *Given a location profile* $\boldsymbol{x}$*, if* $d \le 2-\sqrt{3}$*, use Mechanism 2 to return* (y_1, y_2)*; if* $d > 2-\sqrt{3}$*, use Mechanism 3 to return* (y_1, y_2)*.*

Mechanism 4 has an approximation ratio

$$\gamma = \min\{2-d, \max\{\frac{3-3d}{1+d}, \frac{2}{1+d}\}\} \in [1,2].$$

The next theorem establishes the lower bound for any deterministic strategyproof mechanism.

Theorem 4.4. *Given* $d \in [0,1]$*, for any* $n \ge 2$ *agents, any deterministic strategyproof mechanism* f *has an approximation ratio* γ *of at least* $\frac{7-d}{6}$*.*

Proof. Assume $N = \{1,2,3\}$. Let f be a deterministic mechanism. Consider the profile $\mathbf{x} = \{x_1, x_2, x_3\} = \{\frac{1+3d}{4}, \frac{1+d}{2}, \frac{3+d}{4}\}$ and $f(\mathbf{x}) = (y_1, y_2)$. Note that given $d \in [0,1]$, $d \le x_1 \le x_2 \le x_3 \le 1$. For the social utility, we have $SU((0,d),\mathbf{x}) = 3$,

$$SU((1-d,1),\mathbf{x}) = \frac{6-6d+|3-7d|+|2-6d|+|1-5d|}{4} \le 3,$$

and $SU((0,1),\mathbf{x}) = 3$. Hence, the optimal solution of $\mathbf{x}$ is

$$OPT_2(\mathbf{x}) = \max\{SU((0,d),\mathbf{x}), SU((1-d,1),\mathbf{x}), SU((0,1),\mathbf{x})\} = 3. \quad (14)$$

Denote $\mathbf{x}'$ as the location profile after one of the three agents misreports. Let $(y_1', y_2') = f(\mathbf{x}')$ and (y_1', y_2') satisfies $y_2' - y_1' \ge d$ and $0 \le y_1', y_2' \le 1$. We have the following cases.

Case 1: $y_1 > x_1$ and $y_2 \le x_3$. We have $d \le y_2 - y_1 \le x_3 - x_1 = \frac{1-d}{2}$, which implies $d \le \frac{1}{3}$. Since $x_1 < y_1 \le y_2 \le x_3$, by similar analysis and conclusion of proposition 4.1, the social utility can reach its maximum when (y_1, y_2) is one out of (x_1, x_1+d), (x_1, x_3), (x_3-d, x_3), which is

$$SU((y_1,y_2),\mathbf{x}) \le \max\{SU((x_1,x_1+d),\mathbf{x}), SU((x_1,x_3),\mathbf{x}), SU((x_3-d,x_3),\mathbf{x})\}$$

$$= \max\{\frac{5-5d+|5d-1|}{4}, \frac{3(1-d)}{2}, \frac{5-5d+|5d-1|}{4}\}$$
$$\le 1.5. \quad (15)$$

Accordingly, by (14) and (15), $\gamma \ge \frac{OPT_2(\mathbf{x})}{SU((y_1,y_2),\mathbf{x})} \ge 2$.

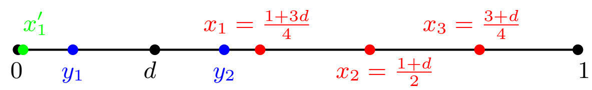

Figure 2: Case 2 for the proof of Theorem 4.4.

Case 2: $y_1 \le x_1$ and $y_2 \le x_1$, as shown in Figure 2. In this case, $u_1((y_1,y_2),x_1) \le 2x_1 - d = \frac{1+d}{2}$. Consider $x_1' = 0$ and $\mathbf{x}' = \{x_1', x_2, x_3\}$. Note that $SU((0,d),\mathbf{x}') = \frac{5+d}{2} \le 3$,

$$SU((1-d,1),\mathbf{x}') = \frac{11-7d+|2-6d|+|1-5d|}{4} \begin{cases} > 3, & \text{if } d \in [0, \frac{1}{9}); \\ \le 3, & \text{if } d \in [\frac{1}{9}, 1]. \end{cases}$$

and $SU((0,1),\mathbf{x}') = 3$. Thus the optimal solution of $\mathbf{x}'$ is

$$OPT_2(\mathbf{x}') = \max\{SU((1-d,1),\mathbf{x}'), SU((0,1),\mathbf{x}')\} \ge 3. \quad (16)$$

As f is strategyproof and agent 1 cannot gain by misreporting from x_1 to x_1', the utility of agent 1 must satisfy

$$u_1((y_1',y_2'),x_1) = |y_1' - x_1| + |y_2' - x_1| \le u_1((y_1,y_2),x_1) \le \frac{1+d}{2}$$
$$\Leftrightarrow \begin{cases} 2x_1 - \frac{1+d}{2} \le y_1' + y_2' \le 2x_1 + \frac{1+d}{2}, \\ |y_1' - y_2'| \le \frac{1+d}{2}, \end{cases}$$
$$\Leftrightarrow \begin{cases} y_1' + y_2' \le 1 + 2d, \\ y_2' - y_1' \le \frac{1+d}{2}. \end{cases} \quad (17)$$

By (17), the domain of (y_1', y_2') is a convex polygon with corner points:

(1) $(0,d)$, $(0, \frac{1+d}{2})$, $(\frac{1+d}{2}, \frac{1+3d}{2})$, $(\frac{1+3d}{4}, \frac{3+5d}{4})$ if $d \in [0, \frac{1}{5}]$;
(2) $(0,d)$, $(0, \frac{1+d}{2})$, $(\frac{1+d}{2}, \frac{1+3d}{2})$, $(\frac{1-d}{2}, 1)$, $(2d, 1)$ if $d \in (\frac{1}{5}, \frac{1}{3}]$;
(3) $(0,d)$, $(0, \frac{1+d}{2})$, $(\frac{1-d}{2}, 1)$, $(1-d, 1)$ if $d \in (\frac{1}{3}, 1]$.

Hence, for the profile $\mathbf{x}'$, by similar analysis and conclusion of Proposition 4.1, the social utility of $\mathbf{x}'$ under f can obtain its maximum if (y_1, y_2) is at one out of all corner points. By some calculations, the social utility of $\mathbf{x}'$ under f is

$$SU((y_1', y_2'), \mathbf{x}') \leq \begin{cases} SU((0,d), \mathbf{x}'), & \text{if } d \in [0, \frac{1}{5}]; \\ \max\{SU((0,d), \mathbf{x}'), SU((\frac{1-d}{2}, 1), \mathbf{x}'), \\ SU((2d, 1), \mathbf{x}')\}, & \text{if } d \in (\frac{1}{5}, \frac{1}{3}]; \\ \max\{SU((0,d), \mathbf{x}'), SU((\frac{1-d}{2}, 1), \mathbf{x}'), \\ SU((1-d, 1), \mathbf{x}')\}, & \text{if } d \in (\frac{1}{3}, 1], \end{cases}$$

$$= \begin{cases} \frac{5+d}{2}, & \text{if } d \in [0, \frac{1}{5}]; \\ \max\{\frac{5+d}{2}, \frac{5+d}{2}, 3-2d\}, & \text{if } d \in (\frac{1}{5}, \frac{1}{3}]; \\ \max\{\frac{5+d}{2}, \frac{5+d}{2}, 2+d\}, & \text{if } d \in (\frac{1}{3}, 1], \end{cases} = \frac{5+d}{2}. \tag{18}$$

Accordingly, by (16) and (18), $\gamma \geq \frac{OPT_2(\mathbf{x}')}{SU((y_1', y_2'), \mathbf{x}')} \geq \frac{5}{5+d}$.

Case 3: $y_1 \leq x_1$ and $x_1 < y_2 \leq x_3$. In this case,

$$\begin{aligned} SU((y_1, y_2), \mathbf{x}) &= (x_1 + x_2 + x_3 - y_1) + (x_3 - x_1 + |y_2 - x_2|) \\ &= x_2 + 2x_3 - y_1 + |y_2 - x_2| \\ &\leq x_2 + 2x_3 - 0 + (x_3 - x_2) = \frac{3(3+d)}{4}. \end{aligned} \tag{19}$$

Accordingly, by (14) and (19), $\gamma \geq \frac{OPT_2(\mathbf{x})}{SU((y_1, y_2), \mathbf{x})} = \frac{4}{3+d}$.

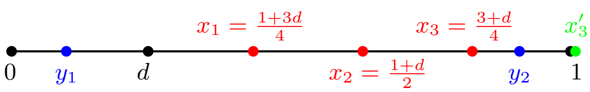

Figure 3: Case 4 for the proof of Theorem 4.4.

Case 4: $y_2 > x_3$, as shown in Figure 3. In this case, $u_3((y_1, y_2), x_3) = y_2 - y_1 \leq 1$. Consider $x_3' = 1$ and $\mathbf{x}' = \{x_1, x_2, x_3'\}$. Note that $SU((0,d), \mathbf{x}') = 2x_1 - d + 2x_2 - d + 2x_3' - d = \frac{7-d}{2} \geq 3$, $SU((1-d, 1), \mathbf{x}') = \frac{5-d+|3-7d|+|2-6d|}{4} \leq 3$ and $SU((0,1), \mathbf{x}') = 3$. Thus the optimal solution of $\mathbf{x}'$ is

$$OPT_2(\mathbf{x}') = SU((0,d), \mathbf{x}') = \frac{7-d}{2}. \tag{20}$$

As f is strategyproof and agent 3 cannot gain by misreporting from x_3 to x_3', the utility of agent 3 must satisfy

$$u_3((y_1', y_2'), x_3) = |y_1' - x_3| + |y_2' - x_3| \leq u_3((y_1, y_2), x_3) \leq 1$$

$$\Leftrightarrow 2x_3 - 1 \leq y_1' + y_2' \leq 2x_3 + 1 \Leftrightarrow \frac{1+d}{2} \leq y_1' + y_2'. \tag{21}$$

By (21), the feasible region of (y_1', y_2') is a convex quadrangle with corner points $(0, \frac{1+d}{2})$, $(0, 1)$, $(1-d, 1)$ and $(\frac{1-d}{4}, \frac{1+3d}{4})$. Hence, for the profile $\mathbf{x}'$, by similar analysis and conclusion of Proposition 4.1, the social utility of $\mathbf{x}'$ under f can obtain its maximum if (y_1, y_2) is at one out of all four corner points, which is

$$\begin{aligned} SU((y_1', y_2'), \mathbf{x}') &\leq \max\{SU((0, \frac{1+d}{2}), \mathbf{x}'), SU((0,1), \mathbf{x}'), \\ &\quad SU((1-d, 1), \mathbf{x}'), SU((\frac{1-d}{4}, \frac{1+3d}{4}), \mathbf{x}')\} \\ &= \max\{\frac{5+d}{2}, 3, \frac{5-d+|3-7d|+|2-6d|}{4}, 2+d\} = 3. \end{aligned} \tag{22}$$

Accordingly, by (20) and (22), $\gamma \geq \frac{OPT_2(\mathbf{x}')}{SU((y_1', y_2'), \mathbf{x}')} = \frac{7-d}{6}$.

In conclusion, f has an approximation ratio γ of at least

$$\min\{2, \frac{6}{5+d}, \frac{4}{3+d}, \frac{7-d}{6}\} = \frac{7-d}{6} \in [1, \frac{7}{6}].$$

□

5 TWO-FACILITY LOCATION GAMES WITH TRIPLE-PREFERENCE

In this section, we design the deterministic strategyproof mechanism for the two-facility location game with triple-preference. Each agent has his own preference towards one out of the two heterogeneous facilities and we denote preference of agent i to facility j as p_i^j which is -1, 0 or 1. An agent i with $p_i^j = 1$ prefers to be close to facility j, an agent i with $p_i^j = -1$ prefers to be far away from facility j and an agent i with $p_i^j = 0$ is indifferent to facility j where $j = 1, 2$. We denote $p_i = \{p_i^1, p_i^2\} \in \{-1, 0, 1\}^2$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$ represents the profile of all n agents' preferences. We allow any agent to misreport both his location and his preferences. The social planner needs to gather information of both agents' locations $\mathbf{x}$ and preferences $\mathbf{p}$ to determine the two facilities' locations (y_1, y_2). Given the two facilities's locations $(y_1, y_2) = f(\mathbf{x}, \mathbf{p})$, we define agent i's utility towards facility j as

$$u_i^j = \begin{cases} |y_j - x_i|, & \text{if } p_i^j = -1; \\ 1, & \text{if } p_i^j = 0; \\ 1 - |y_j - x_i|, & \text{if } p_i^j = 1. \end{cases} \tag{23}$$

Note that in the case that $p_i^j = -1$, to make the approximation ratio positive and meaningful, we require non-negative utilities and purposely add 1 to the utility; in the case of $p_i^j = 0$, we define agent i's utility as 1. Those methods are widely used (e.g., [1, 25]). Denote agent i's utility as

$$u_i = u_i^1 + u_i^2. \tag{24}$$

The social utility of a mechanism f is defined as:

$$SU(f(\mathbf{x}, \mathbf{p}), (\mathbf{x}, \mathbf{p})) = \sum_{i=1}^{n} u_i(f(\mathbf{x}, \mathbf{p}), x_i, p_i). \tag{25}$$

$OPT_3(\mathbf{x}, \mathbf{p})$ is the optimal social utility. Next, we formally define the strategyproofness in the two-facility location game with triple-preference.

Definition 5.1. A mechanism f is strategyproof in the two-facility location game with triple-preference if no agent can benefit from misreporting his location or preferences. Formally, given agent i, location profile $\mathbf{x} = \{x_i, \mathbf{x}_{-i}\} \in I^n$, preference profile $\mathbf{p} = \{p_i^1, p_i^2, \mathbf{p}_{-i}\} \in \{-1, 0, 1\}^{2n}$, any misreported location $x_i' \in I$ and any misreported preferences $\{p_i'^1, p_i'^2\} \in \{-1, 0, 1\}^2$, it holds that

$$\begin{aligned} &u_i(f((x_i, \mathbf{x}_{-i}), (p_i^1, p_i^2, \mathbf{p}_{-i})), x_i, p_i) \\ \geq &u_i(f((x_i', \mathbf{x}_{-i}), (p_i'^1, p_i'^2, \mathbf{p}_{-i}), x_i, p_i). \end{aligned}$$

The group strategyproofness in the two-facility location game with triple-preference can be similarly defined as in Definition 2.2. A mechanism f has an approximation ratio γ, if for any profile $\mathbf{x} \in I^n$ and $\mathbf{p} \in \{-1, 0, 1\}^{2n}$, $OPT_3(\mathbf{x}, \mathbf{p}) \leq \gamma SU(f, \mathbf{x}, \mathbf{p})$.

Define eighteen sets of agents $Q_1 \cup \cdots \cup Q_{18} = N$ as shown in Table 1, depending on their locations and preferences. Then define the following sets based on $Q_1 \ldots Q_{18}$:

$$R_1 = \{i|p_i^1 = 1\} = Q_1 \cup Q_2 \cup Q_5 \cup Q_6 \cup Q_{11} \cup Q_{16},$$
$$R_2 = \{i|p_i^2 = 1\} = Q_1 \cup Q_3 \cup Q_5 \cup Q_7 \cup Q_9 \cup Q_{14},$$
$$R_3 = \{i|p_i^1 = -1\} = Q_3 \cup Q_4 \cup Q_7 \cup Q_8 \cup Q_{12} \cup Q_{17},$$
$$R_4 = \{i|p_i^2 = -1\} = Q_2 \cup Q_4 \cup Q_6 \cup Q_8 \cup Q_{10} \cup Q_{15}. \quad (26)$$

Then define the social utility function $SU(f(\mathbf{x},\mathbf{p}),(\mathbf{x},\mathbf{p}))$

Set of agents		$\{p_i^1, p_i^2\} =$			
		{1,1}	{1,-1}	{-1,1}	{-1,-1}
$x_i \in$	$[0, \frac{1}{2}]$	Q_1	Q_2	Q_3	Q_4
	$(\frac{1}{2}, 1]$	Q_5	Q_6	Q_7	Q_8

Set of agents		$\{p_i^1, p_i^2\} =$				
		{0,1}	{0,-1}	{1,0}	{-1,0}	{0,0}
$x_i \in$	$[0, \frac{1}{2}]$	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}
	$(\frac{1}{2}, 1]$	Q_{14}	Q_{15}	Q_{16}	Q_{17}	Q_{18}

Table 1: Sets of agents $Q_1 - Q_{18}$.

based on (23), (24), (25), Table 1 and (26):

$$SU((y_1, y_2), (\mathbf{x}, \mathbf{p})) = \sum_{i \in R_1} (1 - |y_1 - x_i|) + \sum_{i \in R_2} (1 - |y_2 - x_i|)$$
$$+ \sum_{i \in R_3} |y_1 - x_i| + \sum_{i \in R_4} |y_2 - x_i| + |Q_{13}| + |Q_{18}| + \sum_{i=9}^{18} |Q_i|. \quad (27)$$

To obtain OPT_3 is to solve

$$\max SU((y_1, y_2), (\mathbf{x}, \mathbf{p})),$$
$$\text{s.t. } (y_1, y_2) \in G = \{(y_1, y_2) | |y_2 - y_1| \geq d, 0 \leq y_1, y_2 \leq 1\}.$$

Note that different from the feasible region D in problem (3), the feasible region G includes two isosceles right triangles, as the two facilities are different to any agent. A mechanism outputting OPT_3 is not strategyproof since this triple-preference game's special case is the obnoxious two-facility location game. We need to design a new deterministic strategyproof mechanism. Define $N = R_5 \cup R_6 \cup R_7$, where

$$R_5 = Q_2 \cup Q_7 \cup Q_{10} \cup Q_{11} \cup Q_{14} \cup Q_{17},$$
$$R_6 = Q_3 \cup Q_6 \cup Q_9 \cup Q_{12} \cup Q_{15} \cup Q_{16},$$
$$R_7 = Q_1 \cup Q_4 \cup Q_5 \cup Q_8 \cup Q_{13} \cup Q_{18}. \quad (28)$$

Mechanism 5. *If* $|R_5| \geq |R_6|$, $f(\boldsymbol{x}, \boldsymbol{p}) = (y_1, y_2) = (0, 1)$, *otherwise,* $f(\boldsymbol{x}, \boldsymbol{p}) = (y_1, y_2) = (1, 0)$.

Theorem 5.2. *Mechanism 5 is group strategyproof with approximation ratio* 4.

Proof. By (23), (24) and Table 1, we have for any agent $i \in R_5$, $u_i((0,1), x_i, p_i) \geq u_i((1,0), x_i, p_i)$; for any agent $i \in R_6$, $u_i((1,0), x_i, p_i) \geq u_i((0,1), x_i, p_i)$; and for any agent $i \in R_7$, $u_i((0,1), x_i, p_i) = u_i((1,0), x_i, p_i)$. Note that, in Mechanism 5, there are only two options of the two facilities' locations, which are $(0,1)$ and $(1,0)$. By following the similar proof of group strategyproofness in Theorem 4.3, obviously, we can also prove Mechanism 5 is group strategyproof.

Next, we prove approximation ratio γ. Without loss of generality, assume that $|R_5| \geq |R_6|$, and therefore $(y_1, y_2) = (0, 1)$. For the optimal utility, because $|R_5| \geq |R_6|$, we have

$$\begin{aligned} OPT_3(\mathbf{x}, \mathbf{p}) &= \max_{(y_1,y_2) \in G} SU((y_1, y_2), (\mathbf{x}, \mathbf{p})) \\ &= \sum_{i \in N} \max_{(y_1,y_2) \in G} (u_i^1 + u_i^2) \leq 2N \\ &= 2(|R_5| + |R_6| + |R_7|) \leq 4|R_5| + 2|R_7|. \quad (29) \end{aligned}$$

By (27) and (28), we have the social utility of Mechanism 5:

$$\begin{aligned} SU((0,1), (\mathbf{x}, \mathbf{p})) &= \sum_{i \in R_1 \cup R_4} (1 - x_i) + \sum_{i \in R_2 \cup R_3} x_i \\ &+ |Q_{13}| + |Q_{18}| + \sum_{i=9}^{18} |Q_i| \\ \geq & (|Q_1| + |Q_2| + |Q_{11}| + |Q_2| + |Q_4| + |Q_{10}|) \times (1 - 0.5) \\ &+ (|Q_5| + |Q_7| + |Q_{14}| + |Q_7| + |Q_8| + |Q_{17}|) \times 0.5 \\ &+ |Q_{13}| + |Q_{18}| + \sum_{i=9}^{18} |Q_i| \\ = & |R_5| + 0.5|R_7| + 0.5(|Q_{10}| + |Q_{11}| + |Q_{14}| + |Q_{17}|) \\ &+ (|Q_9| + |Q_{12}| + |Q_{15}| + |Q_{16}|) + 1.5(|Q_{13} + Q_{18}|) \\ \geq & |R_5| + 0.5|R_7|. \quad (30) \end{aligned}$$

By (29) and (30), we have the approximation ratio,

$$\gamma \leq \frac{OPT_3(\mathbf{x}, \mathbf{p})}{SU((0,1), (\mathbf{x}, \mathbf{p}))} \leq \frac{4|R_5| + 2|R_7|}{|R_5| + 0.5|R_7|} = 4.$$

□

The lower bound result of Theorem 4.4 for the obnoxious two-facility location game can carry over to the two-facility location game with triple-preference as we have remarked that the obnoxious facility location game is a special case of the two-facility location game with triple-preference.

6 CONCLUSIONS

We considered the mechanism design problem of a social planner for locating two facilities with minimum distance requirement on a line interval, where a set of n strategic agents report their locations. In the two-facility location game, we found the optimal solution and proved carefully choosing one optimal solution as output is strategyproof. In the obnoxious two-facility location game, the optimal solution is not strategyproof. We proposed new deterministic group strategyproof mechanisms with provable approximation ratios and obtained the lower bound $\frac{7-d}{6}$. In the two-facility location game with triple-preference, we designed a deterministic group strategyproof mechanism with approximation ratio 4.

In the future, we will study the randomized mechanism design in the obnoxious two-facility location game and in the two-facility location game with triple-preference. Besides, we will consider minimizing the maximum cost of all agents and maximizing the minimum utility of all agents. Moveover, we will extend our model to include more than two facilities or consider the facility location games in more general metric spaces such as circles and trees.

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