From Hotelling to Load Balancing: Approximation and the Principle of Minimum Differentiation

Extended Abstract

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ABSTRACT

Competing firms tend to select similar locations for their stores. This phenomenon, called the principle of minimum differentiation, was captured by Hotelling with a landmark model of spatial competition but is still the object of an ongoing scientific debate. Although consistently observed in practice, many more realistic variants of Hotelling's model fail to support minimum differentiation or do not have pure equilibria at all. In particular, it was recently proven for a generalized model which incorporates negative network externalities and which contains Hotelling's model and classical selfish load balancing as special cases, that the unique equilibria do not adhere to minimum differentiation. Furthermore, it was shown that for a significant parameter range pure equilibria do not exist.

We derive a sharp contrast to these previous results by investigating Hotelling's model with negative network externalities from an entirely new angle: approximate pure subgame perfect equilibria. This approach allows us to prove analytically and via agent-based simulations that approximate equilibria having good approximation guarantees and that adhere to minimum differentiation exist for the full parameter range of the model. Moreover, we show that the obtained approximate equilibria have high social welfare.

KEYWORDS

Location Analysis, Facility Location Games, Approximate Pure Subgame Perfect Equilibria, Agent-based Simulation

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1 INTRODUCTION

The choice of a profitable facility location is one of the core strategic decisions for firms competing in a spatial market. Finding the right location is a classical object of research and has kindled the rich and interdisciplinary research area called Location Analysis [6, 14, 32]. In this paper we investigate one of the landmark models of spatial competition and strategic product differentiation where facilities offering the same service for the same price compete in a linear spatial marked. Originally introduced by Hotelling [24] and later extended by Downs [10] to model political competition, the model is usually referred to as the *Hotelling-Downs model.* It assumes a market of infinitely many clients which are distributed evenly on a line and finitely many firms which want to open a facility and which strategically select a specific facility location in the market to sell their service. Every client wants to obtain the offered service and selects the nearest facility to get it. The utility of the firms is proportional to the number of clients visiting their facility. Thus, the location decision of a firm depends on the facility locations of all its competitors as well as on the anticipated behavior of the clients. This two-stage setting is challenging to analyze but at the same time yields a plausible prediction of real-world phenomena.

One such phenomenon is known as the *principle of minimum differentiation* [5, 13] and it states that competing firms selling the same service tend to co-locate their facilities instead of spreading them evenly along the market. This can be readily observed in practice, e.g., stores of different fast-food chains or consumer goods shops are often located right next to each other. For the original version where clients simply select the nearest facility, Eaton and Lipsey [13] proved in a seminal paper that for $n \neq 3$ competing firms the Hotelling-Downs model has pure subgame perfect equilibria which respect the principle of minimum differentiation.

However, the original Hotelling-Downs model is overly simple. A more realistic variant, where the cost function of a client is a linear combination of distance and waiting time, was proposed by Kohlberg [25] and will be the focus of our attention. Kohlberg's model is especially interesting, since it can be interpreted as an interpolation of two extreme models: the Hotelling-Downs model, where clients select the nearest facility and classical *Selfish Load Balancing* [39], where clients select the least congested facility.

For Kohlberg's model it is known that no pure subgame perfect equilibria exist where the facility locations are pairwise different. Furthermore, in a recent paper Peters et al. [31] show for up to six facilities that pure equilibria exist if and only if there is an even number of facilities and the clients' cost function is tilted heavily

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towards preferring less congested facilities. Moreover, in sharp contrast to the principle of minimum differentiation, they show that in these unique equilibria only two facilities are co-located.

In this paper we re-establish the principle of minimum differentiation for Kohlberg's model by considering *approximate pure subgame perfect equilibria*. We believe that in contrast to studying exact subgame perfect equilibria, investigating approximate subgame perfect equilibria yields more reliable predictions since the study of exact equilibria assumes actors who radically change their current strategy even if they can improve only by a tiny margin. In the real world this is not true, as many actors only move out of their "comfort zone" if a significant improvement can be realized. This threshold behavior can naturally be modeled via a suitably chosen approximation factor. Furthermore, approximate equilibria are the only hope for a plausible prediction for many variants of the Hotelling-Downs model where exact equilibria do not exist. To the best of our knowledge, approximate equilibria have not been studied before in the realm of Location Analysis.

Related Work. The Hotelling-Downs model was also analyzed for non-linear markets, e.g., on graphs [19, 20, 30], fixed locations on a circle [34], finite sets of locations [27, 28], and optimal interval division [36]. Moreover, many facility location games are variants of the Hotelling-Downs model and there is a rich body of work analyzing them [7, 11, 15, 18, 21, 33, 38] and Vornoi games [2, 3, 12].

In our model facilities offer their service for the same price. Models where facilities can also strategically set the price have been considered [1, 8, 9, 22, 23, 26, 29]. Other recent work investigates different client attraction functions instead of simply using the distance to the facilities [4, 16, 35].

Using agent-based simulations for variants of the Hotelling-Downs model seems to be a quite novel approach. We could find only the recent work of van Leeuwen & Lijesen [37] in which the authors claim to present the first such approach. They study a multi-stage variant with pricing which is different from our setting.

2 OUR CONTRIBUTION

We study approximate pure subgame perfect equilibria in Kohlberg's model of spatial competition with negative network externalities in which n facility players strategically select a location in a linear market. Our slightly reformulated model has a parameter $0 \leq \alpha \leq 1$, where $\alpha = 0$ yields the original Hotelling-Downs model, i.e., clients select the nearest facility, and where $\alpha = 1$ yields classical selfish load balancing, i.e., clients select the least congested facility.

First, we study the case $n = 3$, which for $\alpha = 0$ is the famous unique case of the Hotelling-Downs model where exact equilibria do not exist. We show that for all α an approximate subgame perfect equilibrium exists with approximation factor $\rho \leq 1.2808$. Moreover, for $\alpha = 0$ we show that this bound is tight.

Next, we consider the facility placement which is socially optimal for the clients and analyze its approximation factor, i.e., we answer the question how tolerant the facility players have to be to accept the social optimum placement for the clients. For this placement, in which the facilities are uniformly distributed along the linear market, we derive exact analytical results for $4 \leq n \leq 10$. Building on this and on a conjecture specifying the facility which has the best improving deviation, we generalize our results to $n \geq 4$. We find that the obtained approximation factor ρ approaches 1.5 for low α which implies that in these cases facility players must be very tolerant to support these client optimal placements (cf. Fig. 1).

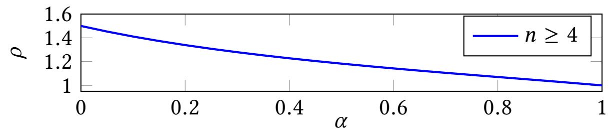

Figure 1: Results for client-optimal facility placement.

We contrast this by our main contribution, which is a study of a facility placement proposed by Eaton & Lipsey [13] (cf. Fig. 2) from an approximation perspective. This placement supports the principle of minimum differentiation since all but at most one facility are co-located with another facility and at the same time it is an exact equilibrium for both extreme cases of the model, i.e., for $\alpha = 0$ and $\alpha = 1$. We provide analytical proofs that for these placements $\rho \leq 1.0866$ holds for $4 \leq n \leq 10$. Also, based on another conjecture, we show that for arbitrary even $n \geq 10$ we get $\rho \approx 1.08$ (cf. Fig. 3).

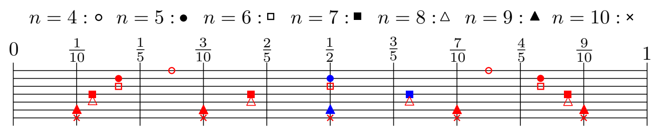

Figure 2: Facility placements from [13] for $4 \leq n \leq 10$. Co-located facilities are red, single facilities are colored blue.

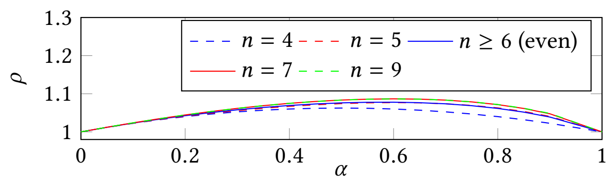

Figure 3: Results for facility placements from [13].

Our conjectures used for proving the general results are based on the analytical results for $n \leq 10$ and on extensive agent-based simulations of a discretized variant of the model. It turns out that these simulations yield reliable predictions for the original model and we also use them for providing promising results for the general case with odd n. In particular, we demonstrate that empirically we have $\rho \approx 1.08$ for arbitrary $n \geq 10$.

Last but not least, we show that the facility placements proposed by Eaton & Lipsey [13] are also socially good for the clients. We compare their social cost with the cost of the social optimum placement and prove a low ratio for all α.

All omitted details can be found in the full version [17].

3 CONCLUSION

We demonstrate that for Kohlberg's model facility placements exist which adhere to the principle of minimum differentiation, are close to stability in the sense that facilities can only improve their utility by at most 8% by deviating and these placements are also socially beneficial for all clients. This remarkble contrast to the results of Peters et al. [31] indicates that studying approximate equilibria may yield more realistic results than solely focusing on exact equilibria. Moreover, investigating approximate equilibria may also lead to new insights for other models in the realm of Location Analysis.

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