Strategyproof Facility Location for Three Agents on a Circle*

Extended Abstract

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ABSTRACT

We present two randomized facility location mechanisms on a circle that are strictly better than Random Dictator, and provide the first lower bound for randomized strategyproof facility location problems.

KEYWORDS

Facility location; Mechanism design; Randomized mechanisms

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1 INTRODUCTION

In a facility location problem, a central authority faces a set of n agents who report their location in some metric space $\langle \mathcal{X}, d\rangle$, and needs to decide where to place a facility. That is, a deterministic facility location mechanism is a function $f : \mathcal{X}^n \rightarrow \mathcal{X}$, and a randomized mechanism outputs a distribution over $\mathcal{X}$. Each agent i wants the facility to be placed as close as possible to her own location a_i, that is, to minimize $d(a_i, f(\boldsymbol{a}))$ in expectation. The challenge is to design a *strategyproof* mechanism f, such that reporting the truthful location is a weakly dominant strategy for every agent; and the social cost ($SC(\boldsymbol{a}, x) := \sum_{i\leq n} d(a_i, x)$) is minimized for $x = f(\boldsymbol{a})$.

In 2009, the agenda of approximation mechanisms without money was made explicit in a paper by Procaccia and Tennenholtz [12], who used facility location as their primary domain of demonstration due to its simplicity. A mechanism f has λ-approximation if $SC(f(\boldsymbol{a})) \leq \lambda SC(\boldsymbol{a}, x)$ for all $\boldsymbol{a}$ and $x \in \mathcal{X}$.

For *deterministic mechanisms* it is known that if $\langle \mathcal{X}, d\rangle$ is a line (or a tree) then the Median mechanism is both strategyproof and optimal [10]. In contrast, on a circle, every deterministic strategyproof mechanism is dictatorial, and its approximation ratio is at least linear in n [4, 13].

W.r.t. randomized mechanisms, much less is known. There are no lower bounds, and an upper bound of $2 - \frac{2}{n}$ is obtained by the trivial *Random Dictator* (RD) mechanism [1]. More variations of the single facility problem were studied in [2, 5, 8, 9]. See [7] Section 5.3 for a recent overview of approximation results for a single facility.

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In this paper we improve both upper and lower bounds, focusing on the case of three agents on a circle.

2 PROPORTIONAL CIRCLE DISTANCE MECHANISM

Note that when an odd number of agents are placed on a circle, each agent is facing an arc (see Figure 1). We denote by L_i the arc facing to agent i.

Definition 2.1. The *Proportional Circle Distance (PCD) mechanism* assigns the facility to each location a_i w.p. $\frac{L_i}{\sum_{j\leq n} L_j}$.

Theorem 2.2. *PCD is strategyproof for any odd n.*

In the limit, PCD obtains an approximation ratio no better than 2, just like RD. However for three agents, RD obtains $2 - \frac{2}{n} = \frac{4}{3}$, and PCD is substantially better.

Proposition 2.3. *For $n = 3$ agents, the PCD mechanism has an approximation ratio of $\frac{5}{4} = 1.25$, and this is tight.*

The PCD mechanism for three agents can also be extended to general metric spaces (see full version). While it maintains strategyproofness, note that it is a *peaks-only* mechanism: it always places the facility on one of the peaks a_i. Unfortunately, no peaks-only mechanism can do better than RD on general graphs: consider a star graph, where agents are on the leafs.

3 QUADRATIC CIRCLE DISTANCE MECHANISM

Since the optimal location with 3 agents is always the peak facing the longest arc, to improve the approximation ratio we must put more weight on peaks facing long arcs (at least in the "bad" instances).

Definition 3.1. The *q-Quadratic Circle Distance (q-QCD) mechanism* assigns the facility to a_i w.p. *proportional to* $s_i = \max\{(L_i)^2, q^2\}$.

That is, q puts a lower bound on the probability that each agent is selected.

Theorem 3.2. *For $n = 3$ agents, The $\frac{1}{4}$-QCD mechanism is strategyproof.*

Proposition 3.3. *For $n = 3$ agents, The $\frac{1}{4}$-QCD mechanism has an approximation ratio of $\frac{7}{6} \cong 1.166$, and this is tight.*

It is an open question whether the QCD mechanism can be extended to more agents and/or to more general topologies.

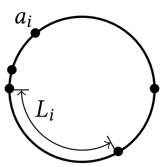

Figure 1

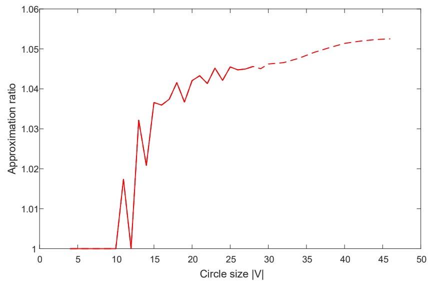

Figure 2

metric space	Any	Circle
RD	1.333 (from [1])	1.333 (from [1])
PCD	1.333	1.25
$\frac{1}{4}$-QCD	-	1.166
best UB	1.333	1.166
LB (peaks-only)	1.333	1.0523
LB	1.0833	1.0456

Table 1: A summary of approximation bounds for 3-agent randomized mechanisms.

4 LOWER BOUNDS VIA LINEAR PROGRAMMING

Typically, a first step in proving lower bounds, is to characterize the set of relevant mechanisms.

Conjecture 4.1. *For any n, the best strategyproof mechanism on a circle is peaks-only.*

We can prove a somewhat weaker result:

Proposition 4.2. *For any n, the optimal strategyproof mechanism w.l.o.g. only places the facility either on peaks, or on points antipodal to peaks.*

For a given graph (V, E), finding the optimal randomized strategyproof mechanism for three agents can be written as a simple linear optimization program as follows. There are $|V|^4 + 1$ variables: $(p_{\boldsymbol{a},z})_{\boldsymbol{a} \in V^3, z \in V}$, where $p_{\boldsymbol{a},z} = f_{\boldsymbol{a}}(z)$ is the probability that the facility is placed on z in profile $\boldsymbol{a}$; and $\lambda \in \mathbb{R}$ which is the approximation factor. The optimization goal is simply to minimize λ. There are four types of constraints:

(1) Feasibility constraints: $p_{\boldsymbol{a},z} \geq 0$ for all $\boldsymbol{a} \in V^3, z \in V$;
(2) Probability constraints: $\sum_{z \in V} p_{\boldsymbol{a},z} = 1$ for all $\boldsymbol{a} \in V^3$;
(3) Incentive constraints: For every profile $\boldsymbol{a} \in V^3$, any agent $i \in \{1, 2, 3\}$, and any alternative location $a'_i \in V$, set:
$$\sum_{z \in V} d(z, a_i) p_{\boldsymbol{a},z} \leq \sum_{z \in V} d(z, a_i) p_{(\boldsymbol{a}_{-i}, a'_i), z};$$
(4) Approximation constraints: For every profile $\boldsymbol{a} \in V^3$, set:
$$\sum_{i \in \{1,2,3\}} \sum_{z \in V} d(z, a_i) p_{\boldsymbol{a},z} \leq \lambda \cdot \min_{z \in V} \sum_{i \in \{1,2,3\}} d(z, a_i).$$

In total, we get a bit more than $3|V|^4$ linear constraints. This is feasible for small graphs with commercial solvers, especially such that handle well sparse constraint matrices (we used Matlab's linprog function).

Lemma 4.3. *For any strategyproof [peaks-only] mechanism f on the circle, there is a neutral and anonymous strategyproof [peaks-only] g, such that* $\max_{\boldsymbol{a}} SC(\boldsymbol{a}, g(\boldsymbol{a})) \leq \max_{\boldsymbol{a}'} SC(\boldsymbol{a}, f(\boldsymbol{a}'))$.

It is well known that mechanism design problems for finite domains can be written as linear programs [3]. Automated mechanism design had also been applied to facility location problems, for one or more facilities on a line [6, 11]. Due to the specifics of the problems they considered, they used advanced machine learning techniques rather than linear programming.

Theorem 4.4. *There is no strategyproof mechanism for circle graphs whose approximation ratio is better than* 1.0456. *If we add the peaks-only requirement, the lower bound is* 1.0523.

To prove the theorem, we coded two linear programs: one that computes the optimal mechanism, and one that computes the optimal peaks-only mechanism. The number of variables for a circle with M vertices is M^4 (or $3M^3$ for peaks-only mechanisms) so we applied some improvements based on symmetries. This enables us to solve the obtained program for all mechanism on circles up to $M = 28$, and the program for peaks-only mechanisms for circles up to $M = 44$. We note that the worst-case approximation bounds in both programs are the same for any $|V| \leq 28$, which supports Conjecture 4.1 above on peaks-only mechanisms, but leaves the proof as a challenge. The worst-case approximation ratios of the optimal mechanism for finite circles are shown in Figure 2.

Allowing for general graphs, we were able to push the lower bound a bit higher, to $\frac{13}{12} \cong 1.0833$ (see full version).

5 SUMMARY

Table 1 summarizes our results, and puts them in the context of known bounds. In the full version, we also provide results for deterministic mechanisms in the 2-dimensional plane. We leave many open questions for future research. In particular, whether the QCD mechanism can be generalized for more agents, and whether there are classes of graphs that are inherently more difficult than circles. The most important question is whether the upper bound of $\frac{4}{3}$ ($2 - \frac{2}{n}$ for general n) is tight for general graphs/metric spaces. Also, while we improved the upper bound for 3 agents on a circle from $\frac{4}{3}$ to $\frac{7}{6}$, and the lower bound from 1 to the bounds in Theorem 4.4, there is still a non-negligible gap.

Acknowledgments

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