Entailment Functions and Reasoning under Inconsistency

Extended Abstract

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ABSTRACT

This study proposes an intuitive and flexible framework for defining a large variety of paraconsistent entailment relations. We first introduce a notion named entailment function (EF) that is used for associating a value called entailment degree to every pair of belief base and formula. Then, we introduce our EF-based framework for defining paraconsistent entailment relations. Finally, we discuss a connection between the notion of entailment function and that of inconsistency measure.

KEYWORDS

Paraconsistency; Entailment; Reasoning under inconsistency; Inconsistency Measures

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1 INTRODUCTION

We are interested in this work in reasoning under inconsistency using paraconsistent entailment relations. An entailment relation for a formal logic is said to be paraconsistent if it does not satisfy the principle of explosion. The literature is rich of paraconsistent logics that have been defined in different ways, such as many-valued approaches (e.g. see [2, 5, 7, 13]), and the approaches based on the notion of maximal consistent subset (e.g. [3, 4, 14].

We here propose an intuitive and flexible framework for defining a large variety of paraconsistent entailment relations. it is obtained by using a notion named *entailment function*, which is used to associate a value, called *entailment degree*, to every pair of belief base and formula. Intuitively, the entailment degree of a belief base B and a formula ϕ can be seen as the truth value of the fact "B entails ϕ". In particular, 0 represents the impossibility to be a true conclusion of a belief base, while 1 represents the necessity to be a true conclusion of a belief base. Then, we introduce our framework for defining paraconsistent entailment relations. Our approach consists simply in using an entailment degree threshold for selecting informative conclusions. Finally, we discuss an interesting connection between the notion of entailment function and that of inconsistency measure (IM) (e.g. see [10]). In particular, we introduce an approach for defining IM-based entailment functions. Note that we change postulates for the entailment functions to take into account that most of the proposals for IMes that have been made in the literature use the set $\mathbb{R}^+$.

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2 BELIEF BASES

We consider in this work that every piece of information is represented using classical propositional logic. A belief base is generally defined as a finite set of formulas. We use in this work a generalization of this notion by dividing a belief base into two parts.

Definition 2.1 (Belief base). A *belief base* is an ordered pair $\langle\Gamma, \Delta\rangle$ where Γ and Δ are finite sets of formulas and $\Gamma \nvdash \bot$. The set Γ is called the *necessary part* and Δ the *possible part*.

A belief base $\langle\Gamma, \Delta\rangle$ is said to be *inconsistent* if $\Gamma\cup\Delta \vdash \bot$. Moreover, we generalize the classical entailment relation $\vdash$ to the belief bases as follows: $\langle\Gamma, \Delta\rangle \vdash \phi$ iff $\Gamma \cup \Delta \vdash \phi$.

Definition 2.2 (NMCS). Given a belief base $B = \langle\Gamma, \Delta\rangle$, a set M of formulas is an NMCS of B if (i) M is a maximal consistent subset of $\Gamma \cup \Delta$ and (ii) $\Gamma \subseteq M$.

Definition 2.3 (Free Formula). Given a belief base $B = \langle\Gamma, \Delta\rangle$, a formula $\phi \in \Gamma \cup \Delta$ is said to be *free* in B if $\phi \in \bigcap_{M\in \mathsf{NMC}(B)} M$.

We use $\mathsf{NMC}(B)$ and $\mathsf{Free}(B)$ to denote respectively the set of all the NMCSes of B and the set of all the free formulas of B.

3 ENTAILMENT FUNCTIONS

An entailment function associates an *entailment degree* to a formula w.r.t. a belief base. An entailment degree of a formula according to a belief base can be interpreted as a truth value associated to the fact that this formula is entailed by the considered belief base. Formally speaking, an *entailment function* E is defined as a function that maps a belief base and a formula onto a value in $[0, 1]$.

Definition 3.1 (Rational Entailment Function). An entailment function E is said to be *rational* if it satisfies the following properties for every belief base $B = \langle\Gamma, \Delta\rangle$ and for all formulas ϕ, ψ:

- Necessity: if $\Gamma \vdash \phi$ then $E(B, \phi) = 1$;
- Impossibility: if $\Gamma \vdash \phi$, then $E(B, \neg\phi) = 0$;
- Consequence: if $\phi \vdash \psi$ then $E(B, \phi) \leq E(B, \psi)$;
- Consistency: if $\Gamma \cup \Delta \nvdash \bot$, then if $\Gamma \cup \Delta \vdash \phi$ then $E(B, \phi) = 1$ else $E(B, \phi) = 0$.

Necessity states that if a formula is entailed by the necessary part of a belief base then it is necessarily entailed by the latter. Similarly, Impossibility states that if a formula is entailed by the necessary part then its negation cannot be entailed by the latter. Necessity and Impossibility come from the fact that we consider as known all the formulas occurring in the necessary part of any belief base. Consequence states that a logically weaker formula cannot be less true. We use this postulate to capture the fact that the logical consequences of a formula are at least as informative as this formula. Consistency guarantees that the classical reasoning is preserved in the case of consistent belief bases.

One can also consider the following additional rationality postulates for entailment functions that formalize interesting aspects. For all belief base $B = \langle \Gamma, \Delta \rangle$ and for all two formulas ϕ, ψ:

- FreeForm : if $\mathsf{Free}(B) \vdash \phi$ and $\mathsf{Free}(B) \nvdash \psi$ then $E(B, \phi) \geq E(B, \psi)$;
- ENMCS (resp. S − ENMCS) : if (*i*) $\exists M \in \mathsf{NMC}(B)$ s.t. $M \vdash \phi$ and (*ii*) $\forall M \in \mathsf{NMC}(B)$ we have $M \nvdash \psi$, then $E(B, \phi) \geq E(B, \psi)$ (resp. $E(B, \phi) > E(B, \psi)$);
- ANMCS (resp. S − ANMCS) : if (*i*) $\forall M \in \mathsf{NMC}(B)$ we have $M \vdash \phi$ and (*ii*) $\exists M \in \mathsf{NMC}(B)$ s.t. $M \nvdash \psi$, then $E(B, \phi) \geq E(B, \psi)$ (resp. $E(B, \phi) > E(B, \psi)$).

Example 3.2. We define in this example an entailment function, denoted E_{NMC}, that is based on the use of NMCSes:

$$E_{\mathsf{NMC}}(\langle \Gamma, \Delta \rangle, \phi) = \frac{|\{M \in \mathsf{NMC}(B) \mid M \vdash \phi\}|}{|\mathsf{NMC}(\langle \Gamma, \Delta \rangle)|}$$

For instance, we have $E_{\mathsf{NMC}}(\langle \{\neg p \vee \neg q\}, \{p, \neg p\} \rangle, \neg q) = \frac{1}{2}$, since there are two NMCSes $M_1 = \{\neg p \vee \neg q, p\}$ and $M_2 = \{\neg p \vee \neg q, \neg p\}$ and only M_1 entails $\neg q$.

Proposition 3.3. *E_{NMC} is a rational entailment function that satisfies the following properties:* FreeForm, S − ANMCS, S − ENMCS.

4 PARACONSISTENT ENTAILMENT RELATIONS

We here propose an approach for defining paraconsistent entailment relations, which is based on the use of entailment functions. The main idea consists in using an entailment degree threshold for selecting informative formulas.

Definition 4.1 (EF-based Entailment Relation). Given an entailment function E and a value $v \in [0, 1]$, we define the entailment relation $\vdash_E^{v,\geq}$ (resp. $\vdash_E^{v,>}$) as follows: $B \vdash_E^{v} \phi$ iff $E(B, \phi) \geq v$ (resp. $E(B, \phi) > v$).

Proposition 4.2. *Let E be a rational entailment function and $v \in]0, 1]$ and $v' \in [0, 1[$. Then, the following properties holds for all belief base $B = \langle \Gamma, \Delta \rangle$ and $\vdash_E \in \{\vdash_E^{v,\geq}, \vdash_E^{v',>}\}$:*

1. *$\forall \phi$, if $\Gamma \vdash \phi$, then we have $B \vdash_E \phi$ and $B \nvdash_E \neg\phi$;*
2. *if $\Gamma \cup \Delta \nvdash \bot$, then $\forall \phi$, $B \vdash_E \phi$ iff $\Gamma \cup \Delta \vdash \phi$;*
3. *$\forall \phi, \psi$, if $B \vdash_E \phi$ and $\phi \vdash \psi$, then $B \vdash_E \psi$.*

An EF-based entailment relation may lead to contradictory formulas from a same belief base. To avoid this situation, one can require the following postulate: $E(B, \phi \wedge \psi) = min(E(B, \phi), E(B, \psi))$ (Conjunction), which allows to get adjunction property.

Proposition 4.3 (Adjunction). *Let E be an entailment function that satisfies* Conjunction, *$v \in [0, 1]$ and $v' \in [0, 1[$. Then, for all belief base B and formulas ϕ, ψ, if $B \vdash_E \phi$ and $B \vdash_E \psi$, then $B \vdash_E \phi \wedge \psi$ for $\vdash_E = \vdash_E^{v,\geq}, \vdash_E^{v',>}$.*

Proposition 4.4 (Non-Contradiction). *Let E be a rational entailment function that satisfies* Conjunction, *$v \in [0, 1]$ and $v' \in [0, 1[$. Then, for all belief base B and for all finite set $S \subseteq \{\phi \in \mathsf{Form} \mid B \vdash_E \phi\}$, $S \nvdash \bot$ for $\vdash_E = \vdash_E^{v,\geq}, \vdash_E^{v',>}$.*

5 INCONSISTENCY MEASURES AND ENTAILMENT FUNCTIONS

An inconsistency measure is defined as a function that associates a non negative value to each belief base [10]. It is used to quantify the amount of contradiction of a belief base. The following postulates are adaptations to our definition of belief base of postulates for inconsistency measures introduced in [10]:

- IM − Consistency: for all belief base $B = \langle \Gamma, \Delta \rangle$, $I(B) = 0$ iff $\Gamma \cup \Delta \nvdash \bot$;
- Monotonicity: for all belief base $B = \langle \Gamma, \Delta \rangle$, finite set of formulas Γ' with $\Gamma \cup \Gamma' \nvdash \bot$ and finite set of formulas Δ', $I(B) \leq I(\langle \Gamma \cup \Gamma', \Delta \cup \Delta' \rangle)$.

Most of the proposals for inconsistency measures that have been made in the literature use, instead of the interval $[0, 1]$, the set $\mathbb{R}^+$ which is possibly augmented with a greatest element denoted ∞ (e.g. [1, 6, 8–12, 15]). Thus, to associate entailment functions to such inconsistency measures, we need to reformulate Necessity and Consistency to be able to use $\mathbb{R}^+ \cup \{\infty\}$ for entailment functions instead of $[0, 1]$:

- Necessity 2: for all belief base $B = \langle \Gamma, \Delta \rangle$ and formulas ϕ, ψ, if $\Gamma \vdash \phi$ and $\Gamma \nvdash \psi$, then $E(B, \phi) \geq E(B, \psi)$.
- Consistency 2: for all belief base $B = \langle \Gamma, \Delta \rangle$ with $\Gamma \cup \Delta \nvdash \bot$ and for all formula ϕ, if $\Gamma \cup \Delta \vdash \phi$ then $E(B, \phi) > 0$; and if and $\Gamma \cup \Delta \nvdash \phi$ then $E(B, \phi) = 0$.

Let us now introduce our approach for associating an entailment function to every inconsistency measure. It consists in considering the entailment degree of a formula w.r.t. a belief base as the amount of contradiction introduced by the negation of this formula in the considered belief base. More precisely, given an inconsistency measure I, its related entailment function, denoted E_I, is defined as follows: $E_I(\langle \Gamma, \Delta \rangle, \phi) = I(\langle \Gamma, \Delta' \cup \{EQ(\Delta', \neg\phi)\} \rangle) - I(\langle \Gamma, \Delta' \rangle)$, where $\Delta' = \Delta \setminus \{\psi \in \Delta \mid \Gamma \vdash \psi \text{ or } \Gamma \vdash \neg\psi\}$ and $EQ(\Delta', \neg\phi)$ denotes any formula equivalence to $\neg\phi$ but does not belong to Δ'. It is worth noting that $EQ(\Delta', \neg\phi)$ can be computed in linear time by using the double negation law: adding the double negation $\neg\neg$ until obtaining a formula that does not belong to Δ'.

We remove from the possible part the formulas that are in $\{\psi \in \Delta \mid \Gamma \vdash \psi \text{ or } \Gamma \vdash \neg\psi\}$ because we know how to consider them using the fact that the formulas in Γ are necessary. Further, we use $EQ(\Delta', \neg\phi)$ instead of $\neg\phi$ to take into account the fact that $\neg\phi$ can be in Δ'.

Note that for all inconsistency measure I that satisfies Monotonicity, and for all belief base B and formula ϕ, we have $E_I(B, \phi) \geq 0$.

Following Definition 4.1, an EF-based entailment relation uses as a threshold a value in $[0, 1]$. It is possible to avoid the use of such a threshold using the following two approaches.

Definition 5.1 (Relative EF-based Entailment). Given an entailment function E and formula ϕ, we define the entailment relation $\vdash_E^{\phi,\geq}$ (resp. $\vdash_E^{\phi,>}$) as follows: $B \vdash_E^{\phi} \psi$ iff $E(B, \psi) \geq E(B, \phi)$ (resp. $E(B, \psi) > E(B, \phi)$).

Definition 5.2. Given an entailment function E, we define the entailment relation $\vdash_E^{\geq}$ (resp. $\vdash_E^{>}$) as follows: $B \vdash_E^{\geq} \phi$ iff $E(B, \phi) \geq E(B, \neg\phi)$ (resp. $E(B, \phi) > E(B, \neg\phi)$).

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