On the Performance of Stable Outcomes in Modified Fractional Hedonic Games with Egalitarian Social Welfare

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ABSTRACT

In this paper we consider *modified fractional hedonic games*, that are coalition formation games defined over an undirected edge-weighted graph $G = (N, E, w)$, where N is the set of agents and for any edge $\{u, v\} \in E$, $w_{u,v} = w_{v,u}$ reflects how much agents u and v benefit from belonging to the same coalition. More specifically, given a coalition structure, i.e., a partition of the agents into coalitions, the utility of an agent u is given by the sum of $w_{u,v}$ over all other agents v belonging to the same coalition of u averaged over all other members of that coalition, i.e., excluding herself.

We focus on common stability notions: we are interested in strong Nash stable, Nash stable and core stable outcomes. In [18], the existence of these natural outcomes for modified fractional hedonic games is completely characterized; moreover, many tight or asymptotically tight results on their performance are shown for the classical utilitarian social welfare function, that is defined as the sum of all agents' utilities.

Motivated by the fact that an outcome with an high utilitarian social welfare could be extremely harsh for some agents, we provide a comprehensive analysis on the performance of strong Nash stable, Nash stable and core stable outcomes for modified fractional hedonic games under the egalitarian social welfare function, that is defined as the minimum among all agents' utilities.

KEYWORDS

Coalition Formation Games; Hedonic Games; Nash; Core; Egalitarian social welfare; Price of Anarchy; Price of Stability

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1 INTRODUCTION

Hedonic games, introduced in [13], model the formation of coalitions of agents. They are games in which agents have preferences over the set of all possible agent coalitions, and the utility of an agent depends on the composition of the coalition she belongs to. A significant amount of research considered the study of many classes of hedonic games and characterized various solutions concepts like strong Nash stability, core stability and Nash stability (see [5] for a survey on the topic).

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While the standard model of hedonic games assumes that agents' preferences over coalitions are ordinal, there are several prominent classes of hedonic games where agents assign cardinal utilities to coalitions. *Modified fractional hedonic games* (MFHG), introduced in [23], constitute a natural and succinctly representable class of hedonic games (together with additively separable hedonic games [3] and fractional hedonic games [2]). An instance of MFHG can be modeled by means of a weighted undirected graph $G = (N, E, w)$, where nodes in N represent the agents, and the weight $w_{u,v} = w_{v,u}$ of an edge $\{u, v\} \in E$ represents how much agents u and v benefit from belonging to the same coalition. Given a coalition structure, i.e., a partition of the agents into coalitions, the utility of an agent u is given by the sum of $w_{u,v}$ over all other agents v belonging to the same coalition of u averaged over all other members of that coalition, i.e., excluding herself. MFHG model natural behavioral dynamics in social environments. Even when defined on undirected unweighted bipartite graphs, they suitably model a basic economic scenario referred to in [1] as Bakers and Millers, where each agent can be considered as a buyer or a seller. There are only edges connecting buyers and sellers and every agent sees the others of the same type as market competitors. Each agent prefers to be situated in a group (market) with a small number of competitors, that is, each buyer wants to be in a group with many sellers and few other buyers, thus maximizing their ratio, in order to decrease the price of the good. On the other hand, a seller wants to be situated in a group maximizing the number of buyers against the number of sellers, in order to be able to increase the price of the good and gain a higher profit. Moreover, MFHG can model other realistic scenarios: for instance, politicians may want to be in a party that maximizes the fraction of like-minded members, or, for example, people may want to be with an as large as possible fraction of people of the same ethnic or social group.

The existence and the performance of natural stable outcomes like Nash, strong Nash, and core stable outcomes for MFHGs have been studied in [18]. In particular, they show that the existence of strong Nash equilibria is guaranteed only for unweighted graphs, while Nash equilibria and core stable outcomes always exist. Moreover, they show tight or asymptotically tight results on the performance of strong Nash stable, Nash stable and core stable outcomes for the classical utilitarian social welfare, that is defined as the sum of all agents' utilities.

Motivated by the fact that an outcome with an high utilitarian social welfare could be extremely harsh for some agents (see [20] for a nice discussion about utilitarian and egalitarian social welfare), we provide a comprehensive analysis on the performance of strong Nash stable, Nash stable and core stable outcomes for MFHG under the egalitarian social welfare, that is defined as the minimum among all agents' utilities.

1.1 Our Results

We first provide, in Section 3, some preliminary results. In particular, we perform tight and asymptotically tight analyses on the performance of Nash, strong Nash and core outcomes by means of the widely used notions of price of anarchy (resp. strong price of anarchy and core price of anarchy), and price of stability (resp. strong price of stability and core price of stability) for MFHG mainly defined on weighted graphs.

Our main technical results are then presented in Section 4 where we show tight results on the strong price of stability (and thus on the core price of stability and price of stability) for MFHG defined on undirected unweighted bipartite graphs (we recall that when MFHG are defined on undirected unweighted bipartite graphs, they suitably model a basic economic scenario referred to in [1] as Bakers and Millers). In particular, we first give a nice constructive theorem which transforms any given coalition structure $\overline{C}$ with all the coalitions isomorphic to star graphs into another coalition structure C which is strong Nash stable and such that $\mathrm{SW}(C) \geq \mathrm{SW}(\overline{C})$. We then show that for MFHG defined on undirected unweighted bipartite graphs there always exists an optimum coalition structure with coalitions isomorphic to star graphs. Combined with the previous constructive theorem we obtain that the strong price of stability (and thus the price of stability and the core price of stability) is 1 for this graph topology. We finally show that the strong price of stability of undirected unweighted graphs with degree at most 2 is also 1.

Due to space limitations, some proofs are omitted.

1.2 Related Work

Hedonic games have been introduced by Dréze and Greenberg [13], and further investigated in [3, 6, 7, 9, 22, 24].

MFHG have been introduced by Olsen [23] who considers unweighted undirected graphs and investigates computational issues concerning the problem of computing a Nash stable outcome different than the trivial one where all the agents are in the same coalition. The author proves that the problem is NP-hard when we require that a coalition must contain a given subset of the agents, and that it is polynomial solvable for any connected graph containing at least four nodes. Kaklamanis et al. [17] show that the price of stability is 1 for unweighted graphs under the utilitarian social welfare. Monaco et al. [18] show that the existence of strong Nash equilibria is guaranteed only for unweighted graphs, while core stable outcomes always exist. Moreover, they show tight or asymptotically tight results on the performance of strong Nash stable, Nash stable and core stable outcomes for the classical utilitarian social welfare. Finally, Elkind et al. [14] study the set of Pareto optimal outcomes for MFHGs still for the utilitarian social welfare.

Fractional hedonic games, where the utility of an agent u is divided by the size of the coalition she belongs to (including herself), have been introduced by Aziz et al. [2]. They prove that the core can be empty for games played on general graphs and that it is not empty for games played on some classes of undirected and unweighted graphs (that is, graphs with degree at most 2, multipartite complete graphs, bipartite graphs admitting a perfect matching and regular bipartite graphs). Brandl et al. [10] study the existence of core and individual stability in fractional hedonic games and the computational complexity of deciding whether a core and individual stable partition exists in a given fractional hedonic game. Bilò et al. [8] initiated the study of Nash stable outcomes for fractional hedonic games and study their existence, complexity and performance for general and specific graph topologies for the utilitarian social welfare. In particolar, they show a lower bound of 1.0025 and an upper bound of 1.0294 to the price of stability for simple symmetric fractional hedonic games (where the valuations can be only 0 or 1) played on unweighted bipartite graphs. Other stability concepts applied to fractional hedonic games are discussed in [10, 25]. Aziz et al. [4] consider the computational complexity of computing welfare maximizing partitions (not necessarily stable) for fractional hedonic games, while in [15] the authors consider the online scenario. Finally, strategyproof mechanisms for fractional hedonic games have been proposed in [16]. We remark that MFHG and fractional hedonic games are very similar, however, they perform differently. In fact, the core can be empty even for simple symmetric fractional hedonic games [10]. Moreover, for unweighted graphs, even 2-strong Nash stable outcomes are not guaranteed to exist for fractional hedonic games [8]. For more details about the difference between MFHG and fractional hedonic games see Section 1.3 of [18].

To the best of our knowledge, no paper considers MFHG under the egalitarian social welfare. However, in [4] the authors consider the computational complexity of computing partitions (not necessarily stable) for fractional hedonic games which maximize the egalitarian social welfare. We stress that the egalitarian social welfare has been studied in many other settings, like e.g., congestion games [12], k-cut games [11], and fair division problems [21].

2 MODEL

For an integer $n > 0$, denote with $[n]$ the set $\{1, \ldots, n\}$.

We model a coalition formation game by means of an undirected edge-weighted graph $G = (N, E, w)$; we denote with $n = |N|$ the number of its nodes. For the sake of convenience, we adopt the notation (u, v) and $w_{u,v}$ to denote the edge $\{u, v\} \in E$ and its weight $w(\{u, v\})$, respectively. Say that G is unweighted if $w_{u,v} = 1$ for every $(u, v) \in E$, and in this case we denote the graph with $G = (N, E)$. Let $\delta^u(G) = \sum_{v \in N:(u,v) \in E} w_{u,v}$ be the sum of the weights of all the edges incident to u. Moreover, let $\delta^u_{max}(G) = \max_{v \in N:(u,v) \in E} w_{u,v}$ be the maximum edge-weight incident to u. We will omit to specify (G) when clear from the context. Given a set of edges $X \subseteq E$, denote with $W(X) = \sum_{(u,v) \in X} w_{u,v}$ the total weight of edges in X. Given a subset of nodes $S \subseteq N$, $G_S = (S, E_S)$ is the subgraph of G induced by the set S, i.e., $E_S = \{(u, v) \in E : u, v \in S\}$.

Given an undirected edge-weighted graph $G = (N, E, w)$, a *modified fractional hedonic game* (MFHG) induced by G, denoted as $\mathcal{G}(G)$, is the game in which each node $u \in N$ is associated with an agent. We assume that agents are numbered from 1 to n and, for every $u \in [n]$, each agent chooses to join a certain *coalition* among n candidate ones: the strategy of agent u is an integer $j \in [n]$, meaning that agent u is selecting candidate coalition C_j. A coalition structure (also called outcome or partition) is a partition of the set of agents into n coalitions $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ such that $C_j \subseteq N$ for each $j \in [n]$, $\bigcup_{j\in[n]} C_j = N$ and $C_i \cap C_j = \emptyset$ for any $i, j \in [n]$ with $i \neq j$. Notice that, since the number of candidate coalitions is equal to the number of agents (nodes), some coalition may be empty. If $u \in C_j$, we say that agent u is a member of the coalition C_j. We denote by $C(u)$ the coalition in $\mathcal{C}$ of which agent u is a member. We say that an agent u is *isolated* in the outcome $\mathcal{C}$ when $|C(u)| = 1$. In a coalition structure $\mathcal{C}$, the utility of agent u is defined as

$$\mu_u(\mathcal{C}) = \sum_{v\in C(u)} \frac{w_{u,v}}{|C(u)| - 1} = \frac{\delta^u(G_{C(u)})}{|C(u)| - 1},$$

when $|C(u)| > 1$. Moreover, we define $\mu_u(\mathcal{C}) = 0$ when u is isolated in $\mathcal{C}$. We notice that, for any outcome $\mathcal{C}$, we have that $\mu_u(\mathcal{C}) \leq \delta^u_{max}$.

Each agent chooses the coalition she belongs to with the aim of maximizing her utility. We denote by $(\mathcal{C}, u, j)$, the new coalition structure obtained from $\mathcal{C}$ by moving agent u from $C(u)$ to C_j; formally, assuming that $C(u) = C_k$, $\mathcal{C}' = (\mathcal{C}, u, j)$ is such that $C'_k = C_k \setminus \{u\}$ and $C'_j = C_j \cup \{u\}$. An agent *deviates* if she changes the coalition she belongs to. Given an outcome $\mathcal{C}$, an *improving move* (or simply a *move*) for agent u is a deviation to any coalition C_j that strictly increases her utility, i.e., $\mu_u((\mathcal{C}, u, j)) > \mu_u(\mathcal{C})$. An agent is *stable* if she cannot perform a move. An outcome is *(pure) Nash stable* (or a *Nash equilibrium*) if every agent is stable. An *improving dynamics*, or simply a dynamics, is a sequence of moves. A game has the *finite improvement path property* if it does not admit an improvement dynamics of infinite length. Clearly, a game possessing the finite improvement path property always admits a Nash stable outcome. We denote with $\mathsf{N}(\mathcal{G}(G))$ the set of Nash stable outcomes of $\mathcal{G}(G)$.

An outcome $\mathcal{C}$ is a *k-strong Nash equilibrium* if, for each $\mathcal{C}'$ obtained from $\mathcal{C}$, when a subset of at most k agents $K \subseteq N$ (with $|K| \leq k$) jointly change (or deviate from) their strategies (not necessarily selecting the same candidate coalition), $\mu_u(\mathcal{C}) \geq \mu_u(\mathcal{C}')$ for some u belonging to K, that is, after the joint collective deviation, there always exists an agent in the set of deviating ones who does not improve her utility. We denote with $k{-}\mathsf{SN}(\mathcal{G}(G))$ the set of k-strong Nash stable outcomes of $\mathcal{G}(G)$. We simply say that an outcome $\mathcal{C}$ is a strong Nash equilibrium if $\mathcal{C}$ is an n-strong Nash equilibrium. It is easy to see that, for any graph G and any $k \geq 2$, $k{-}\mathsf{SN}(\mathcal{G}(G)) \subseteq (k-1){-}\mathsf{SN}(\mathcal{G}(G))$, while the vice versa does not in general hold. Clearly, $1{-}\mathsf{SN}(\mathcal{G}(G)) = \mathsf{N}(\mathcal{G}(G))$.

We say that a coalition $T \subseteq N$ *strongly blocks* an outcome $\mathcal{C}$, if each agent $u \in T$ strictly prefers T, i.e., strictly improve her utility with respect to her current coalition $C(u)$. An outcome that does not admit a strongly blocking coalition is called *core stable* and is said to be in the *core*. We denote with $\mathsf{CR}(\mathcal{G}(G))$ the core of $\mathcal{G}(G)$. We notice that for any graph G we have $n{-}\mathsf{SN}(\mathcal{G}(G)) \subseteq \mathsf{CR}(\mathcal{G}(G))$.

The *egalitarian social welfare* of a coalition structure $\mathcal{C}$ is the minimum of the agents' utilities, i.e.,

$$\mathsf{SW}(\mathcal{C}) = \min_{u\in N} \mu_u(\mathcal{C}).$$

Given a game $\mathcal{G}(G)$, an *optimum* coalition structure $\mathcal{C}^*(\mathcal{G}(G))$ is one that maximizes the social welfare of $\mathcal{G}(G)$. The *price of anarchy* (resp. *strong price of anarchy* and *core price of anarchy*) of a modified fractional hedonic game $\mathcal{G}(G)$ is defined as the worst-case ratio between the social welfare of a social optimum outcome and that of a Nash equilibrium (resp. core). Formally, for any $k = 1, \ldots, n$, $\mathsf{PoA}(\mathcal{G}(G)) = \max_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$ (resp. $k{-}\mathsf{SPoA}(\mathcal{G}(G)) = \max_{\mathcal{C}\in k-\mathsf{SN}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$ and $\mathsf{CPoA}(\mathcal{G}(G)) = \max_{\mathcal{C}\in\mathsf{CR}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$). Analogously, the *price of stability* (resp. *strong price of stability* and *core price of stability*) of $\mathcal{G}(G)$ is defined as the best-case ratio between the social welfare of a social optimum outcome and that of a Nash equilibrium (resp. strong Nash equilibrium and core). Formally, for any $k = 1, \ldots, n$, $\mathsf{PoS}(\mathcal{G}(G)) = \min_{\mathcal{C}\in\mathsf{N}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$

(resp. $k{-}\mathsf{SPoS}(\mathcal{G}(G)) = \min_{\mathcal{C}\in k-\mathsf{SN}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$ and $\mathsf{CPoS}(\mathcal{G}(G)) = \min_{\mathcal{C}\in\mathsf{CR}(\mathcal{G}(G))} \frac{\mathsf{SW}(\mathcal{C}^*(\mathcal{G}(G)))}{\mathsf{SW}(\mathcal{C})}$). Clearly, for any game $\mathcal{G}(G)$ it holds that $1 \leq \mathsf{PoS}(\mathcal{G}(G)) \leq \mathsf{PoA}(\mathcal{G}(G))$ (resp. $1 \leq k{-}\mathsf{SPoS}(\mathcal{G}(G)) \leq k{-}\mathsf{SPoA}(\mathcal{G}(G))$ and $1 \leq \mathsf{CPoS}(\mathcal{G}(G)) \leq \mathsf{CPoA}(\mathcal{G}(G))$).

3 PRELIMINARY RESULTS

In this section we provide preliminary results on the performance of MFHG. Our main technical results will be provided in Section 4.

3.1 Nash Stable Outcomes

We first provide tight results on the performance of Nash stable outcomes for MFHG. It is known that (see Theorem 4.1 in [18]) there exists a graph G containing edges with negative weights such that $\mathcal{G}(G)$ admits no Nash stable outcome, therefore we focus on graphs with positive weights.

3.1.1 Price of Anarchy. We start by analyzing the price of anarchy of MFHG. On the one hand, it is possible to prove that the price of anarchy is at least $n - 1$ even for unweighted paths.

Theorem 3.1. *There exists an unweighted path G such that* $\mathsf{PoA}(\mathcal{G}(G)) \geq n - 1$.

On the other hand, it is possible to show that the price of anarchy is at most $n - 1$ for any weighted graph, closing in a tight way the bound provided in Theorem 3.1.

Theorem 3.2. *For any weighted graph G,* $\mathsf{PoA}(\mathcal{G}(G)) \leq n - 1$.

3.1.2 Price of Stability. In this section we analyze the price of stability of MFHG. We show that it is at least $n - 1$ for weighted trees. It is worth noticing that, since an upper bound to the price of anarchy is also an upper bound to the price of stability, Theorem

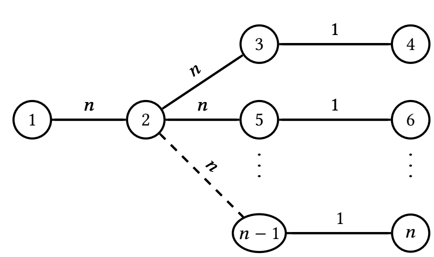

Figure 1: The tree G used in the proof of Theorem 3.3.

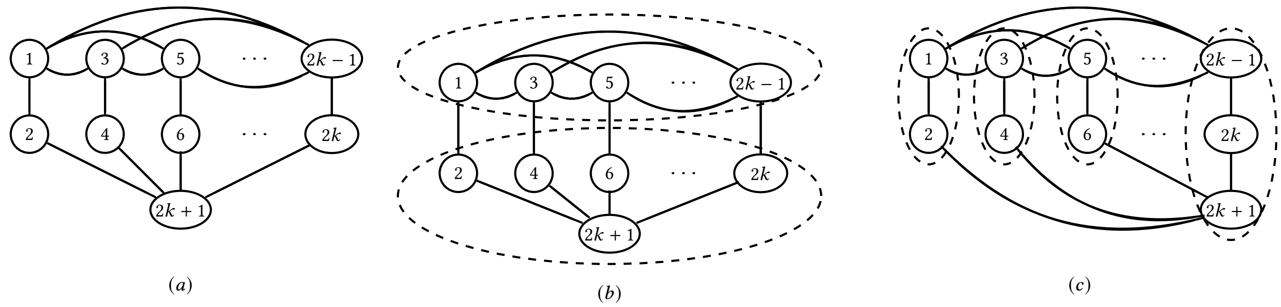

Figure 2: (a) The graph G used in the proof of Theorem 3.6; (b) Coalition structure C; (c) Coalition structure C'.

3.3 provides a tight lower bound to the upper bound proved in Theorem 3.2.

Theorem 3.3. *For any even number $n \geq 4$, there exists a weighted tree G with n nodes such that* $\mathrm{PoS}(\mathcal{G}(G)) \geq n-1$.

Proof. Let G be the tree graph with an even number of nodes n depicted in Figure 1, in which the set of edges is $E = E_1 \cup E_2 \cup E_3$, with $E_1 = \{\{1,2\}\}$, $E_2 = \{\{2,i\} | i = 3,5,\ldots,n-1\}$ and $E_3 = \{\{i,i+1\} | i = 3,5,\ldots,n-1\}$. The weights of edges in E_1 and E_2 are equal to n, while the weights of edges in E_3 are equal to 1.

We claim that the coalition structure in which all the agents belong to the same coalition is the unique Nash stable outcome.

Indeed, in any Nash equilibrium, all the agents $1,2$ and $3,5,\ldots,n-1$, which are endpoints of at least an edge of weight n, want to stay in the coalition where is agent 2, because in this way their utility is at least $\frac{n}{n-1} > 1$, while it is at most 1 if they are not with agent 2.

Notice also that, in any Nash equilibrium, agents $4,6,\ldots,n$ want to stay in the same coalition of agents $3,5,\ldots,n-1$, respectively, because only in this way they can get a positive utility.

It follows that the unique Nash stable outcome C is the one in which all agents form a unique coalition. In this case, the social welfare is $\mathrm{SW}(C) = \frac{1}{n-1}$, because $\mu_4(C) = \mu_6(C) = \ldots = \mu_n(C) = \frac{1}{n-1}$.

Consider outcome C' in which there is a coalition for every edge of the perfect matching $\{\{i,i+1\} | i = 1,3,5,\ldots,n-1\}$. Since $\mu_1(C') = \mu_2(C') = n$ and $\mu_3(C') = \mu_4(C') = \ldots = \mu_n(C') = 1$, it holds that $\mathrm{SW}(C') = 1$.

It follows that, for an optimal outcome C^*, $\mathrm{SW}(C^*) \geq \mathrm{SW}(C') = 1$. Hence, the claim follows. □

3.2 Core Stable Outcomes

We now provide tight results on the performance of core stable outcomes for MFHG.

3.2.1 Core Price of Anarchy. In this section we analyze the core price of anarchy of MFHG. In particular, it is possible to show that the price of anarchy is unbounded even for unweighted paths of 3 nodes.

Theorem 3.4. *There exists an unweighted path G such that* $\mathrm{CPoA}(\mathcal{G}(G))$ *is unbounded.*

3.2.2 Core Price of Stability. In this section we analyze the core price of stability of MFHG. In particular, it is possible to show that the price of stability is unbounded for weighted paths of 3 nodes.

Theorem 3.5. *There exists a weighted path G such that* $\mathrm{CPoS}(\mathcal{G}(G))$ *is unbounded.*

3.3 Strong Nash Stable Outcomes

We now consider strong Nash stable outcomes for MFHG. We know from [18] that the existence of strong Nash equilibria is guaranteed only for unweighted graphs. Moreover, there exists a star graph G containing only non-negative edge-weights such that $\mathcal{G}(G)$ admits no 2-strong Nash stable outcome [18]. Therefore, we only consider unweighted graphs.

We now show that the strong price of anarchy is at least $\frac{n+1}{4}$.

Theorem 3.6. *For any $k > 0$, there exists an unweighted graph G with $n \geq k$ nodes such that* $n\text{−}\mathrm{SPoA}(\mathcal{G}(G)) \geq \frac{n+1}{4}$.

Proof. Let us consider the graph $G = (N,E)$ depicted in Figure 2.a with $n = 2k+1$ agents, in which $N = N_1 \cup N_2 \cup N_3$ and $E = E_1 \cup E_2 \cup E_3$. In particular, $N_1 = \{1,3,\ldots,2k-1\}$ is the set containing the agents in the first upper layer, $N_2 = \{2,4,\ldots,2k\}$ is the set containing the agents in the second layer and $N_3 = \{2k+1\}$ contains the unique agent in the third layer; moreover, $E_1 = \{\{u,v\} | u,v \in N_1\}$ induces a clique among all agents in N_1, $E_2 = \{\{u,u+1\} | u \in N_1\}$ contains edges connecting each node in N_1 to the corresponding one in N_2 and $E_3 = \{\{u,2k+1\} | u \in N_2\}$ contains edges connecting each node in N_2 to node $2k+1$.

On the one hand, we claim that the outcome $C = \{N_1, N_2 \cup N_3, \emptyset, \ldots, \emptyset\}$, depicted in Figure 2.b, is a strong Nash equilibrium. Indeed, any agent belonging to the coalition N_1 gets utility 1, which is the maximum one can get. It implies that agents in N_1 do not have any interest on deviating. Also the single agent $2k+1$ of third layer of G gets utility 1 in C. Therefore, the only agents that can have an incentive to move from C are the ones in N_2. However, in order to improve their utility, any agent of N_2 should form a new coalition together with an agent of N_1. It follows that C is a strong Nash equilibrium. Notice that $\mathrm{SW}(C) = \frac{1}{k+1} = \frac{2}{n+1}$.

On the other hand, the coalition structure $C' = \{\{1,2\},\{3,4\},\ldots,\{2k-3,2k-2\},\{2k-1,2k,2k+1\},\emptyset,\ldots,\emptyset\}$, depicted in Figure 2.c and composed by k non-empty coalitions, has social welfare $\mathrm{SW}(C') = \frac{1}{2}$, because $\mu_{2k-1}(C') = \mu_{2k+1}(C') = \frac{1}{2}$, while the utilities of all other nodes are equal to 1. It implies that the optimal coalition structure C^* is such that $\mathrm{SW}(C^*) \geq \mathrm{SW}(C') = \frac{1}{2}$. It follows that $n-\mathrm{SPoA}(\mathcal{G}(G)) \geq \frac{\frac{1}{2}}{\frac{2}{n+1}} = \frac{n+1}{4}$. □

We notice that an asymptotically matching upper bound is given in Theorem 3.2 since a strong Nash stable coalition structure is also Nash stable.

4 MAIN RESULTS

In this section we provide our main technical results, claiming that optimal performances can be obtained even by strong Nash outcomes in the case of bipartite unweighted graphs. We first need the following additional definition.

Definition 4.1. Given a graph G, a *star*-coalition structure is a coalition structure $C = \{C_1, \ldots, C_n\}$ in which, for $i \in [n]$, every non-empty C_i is such that $G(C_i)$ is isomorphic to a star graph, i.e., $G(C_i) = (C_i, E_i)$ such that there exist (i) a node $u \in C_i$, called *center*, with degree $|C_i| - 1$ and (ii) $|C_i| - 1$ nodes with degree 1 and connected to node u, called *leaves*. Finally, let $L^{\geq 3}(C)$ be the set containing all leaves belonging to all coalitions C of C such that $|C| \geq 3$.

We start by showing in Lemma 4.2 that, for MFHG defined on undirected unweighted graph and given a *star*−coalition structure $\overline{C}$, it is possible to compute another coalition structure C which is a strong Nash equilibrium and such that $\mathrm{SW}(C) \geq \mathrm{SW}(\overline{C})$. This result will be useful for providing an upper bound to the strong price of stability.

Lemma 4.2. *Given a graph $G = (N, E)$ and a star-coalition structure $\overline{C}$ for $\mathcal{G}(G)$, it is possible to compute an outcome $C \in n-\mathrm{SN}(\mathcal{G}(G))$ such that* $\mathrm{SW}(C) \geq \mathrm{SW}(\overline{C})$.

Proof. Given that any coalition composed by only one agent has utility zero, without loss of generality we assume that for any $\overline{C} \in \overline{C}$, $|\overline{C}| \geq 2$.

Consider Algorithm 1 described below.

Algorithm 1 It takes as input a *star*-coalition structure $\overline{C}$ and returns a strong Nash equilibrium.

1: $C^0 \leftarrow \overline{C}$
2: $i \leftarrow 0$
3: **while** there exist $x, y \in L^{\geq 3}(C^i)$ such that $\{x, y\} \in E$ **or** C^i is not Nash stable **do**
4: $\quad i \leftarrow i + 1$ ▷ Beginning of step i
5: $\quad$ **if** there exist $x, y \in L^{\geq 3}(C^{i-1})$ such that $\{x, y\} \in E$ **then**
6: $\quad\quad C^i \leftarrow C^{i-1} \setminus \{C^{i-1}(x), C^{i-1}(y)\} \cup \{\{x, y\}\} \cup \{C^{i-1}(x) \setminus \{x\}\} \cup \{C^{i-1}(y) \setminus \{y\}\}$
7: $\quad$ **else** ▷ C^{i-1} is not Nash stable
8: $\quad\quad$ Let u be an agent with an improving move to C_j^{i-1}
9: $\quad\quad C^i \leftarrow (C^{i-1}, u, j)$
10: $\quad$ **end if**
11: **end while**
12: **return** C^i

Let C be the coalition structure returned by Algorithm 1. We first show that Algorithm 1 is guaranteed to terminate. Moreover, for any $i \geq 0$, let $\vec{x}^i$ be the vector obtained by listing $\mu_u(C^i)$ (for all $u \in N$) in non-decreasing order. Notice that the first component of $\vec{x}^i$ is $x_1^i = \min_{u \in N} \mu_u(C^i)$ and therefore it holds that $x_1^i = \mathrm{SW}(C^i)$.

As usual, given two n-dimensional vectors $\vec{y}$ and $\vec{y}'$, we say that the first one is greater than the second one for the lexicographical order (and we write $\vec{y} \succ \vec{y}'$) if $y_j > y_j'$ for the first component j for which y_j and y_j' differ.

Since the set of possible vectors is finite, in order to guarantee the termination of Algorithm 1 it suffices to prove that, for any $i \geq 1$, $\vec{x}^i \succ \vec{x}^{i-1}$, i.e., vectors $\vec{x}^i$ always lexicographically increase after each step of the algorithm.

In order to prove this property for any $i \geq 1$, we consider two disjoint cases, depending on the *if* condition at line 5 of Algorithm 1:

- If there exist $x, y \in L^{\geq 3}(C^{i-1})$ such that $\{x, y\} \in E$, then the algorithm obtains a new coalition structure C^i by removing leaves x and y from their coalitions in C^{i-1} and by putting them together in a new coalition. Since $x, y \in L^{\geq 3}(C^{i-1})$, for all agents $u \in C^{i-1}(x) \cup C^{i-1}(y) \setminus \{x, y\}$ it holds that $\mu_u(C^{i-1}) \leq \mu_u(C^i)$. Moreover, it clearly holds that $\mu_x(C^{i-1}) < \mu_x(C^i)$ and $\mu_y(C^{i-1}) < \mu_y(C^i)$. Therefore, for any agent $u \in N$ it holds that $\mu_u(C^{i-1}) \leq \mu_u(C^i)$ and at least for an agent, say x, $\mu_x(C^{i-1}) < \mu_x(C^i)$: it follows that $\vec{x}^i \succ \vec{x}^{i-1}$.
- Otherwise, it holds that (i) there exists no couple $x, y \in L^{\geq 3}(C^{i-1})$ such that $\{x, y\} \in E$ and (ii) C^{i-1} is not Nash stable, i.e., there exists an agent u with an improving move towards a coalition C_j^{i-1}.
  First of all, notice that, in outcome C^{i-1}, the centers of the stars and both nodes belonging to coalitions of cardinality 2 do not deviate from their strategy, because their utility in C^{i-1} is equal to 1, that is the best they can obtain. Therefore, it holds that $u \in L^{\geq 3}(C^{i-1})$.
  Moreover, since there exists no couple $x, y \in L^{\geq 3}(C^{i-1})$ such that $\{x, y\} \in E$, in $E_{C_j^i}$ no edge between u and $v \in L^{\geq 3}(C^{i-1})$ can exist, i.e., node u only has an edge towards the center of C_j^i (notice that, if $|C_j^i| = 3$, then u can have an edge towards any, but not both, nodes in C_j^{i-1}, say node $w \in C_j^{i-1}$: node w becomes in this way the center of coalition C_j^i).
  We have to consider all agents $v \in N$ such that $\mu_v(C^i) \neq \mu_v(C^{i-1})$: clearly, these agents belong either to coalition $C^{i-1}(u)$ or to coalition C_j^{i-1}. Let x be any agent in coalition $C^{i-1}(u)$ such that $x \neq u$ and $x \neq w$, where w is the center of coalition $C^{i-1}(u)$. Since $|C^{i-1}(x)| = 1 + |C^i(x)|$, it holds that $\mu_x(C^i) > \mu_x(C^{i-1})$; moreover, $\mu_w(C^i) = \mu_w(C^{i-1}) = 1$ for the center w and $\mu_u(C^i) > \mu_u(C^{i-1})$ because agent u performs a Nash improving move.
  It remains to deal with any agent $v \in C_j^{i-1}$. Again, it holds that $\mu_w(C^i) = \mu_w(C^{i-1}) = 1$ for the center w of coalition C_j^i. Consider now any other node $v \in C_j^{i-1} \setminus \{w\}$. Since $|C_j^{i-1}| < |C_j^i|$, node v is lowering her utility, i.e., $\mu_v(C^i) < \mu_v(C^{i-1})$. Given that u performs a Nash improving move, it holds that $\mu_x(C^{i-1}) = \mu_u(C^{i-1}) < \mu_u(C^i) = \mu_v(C^i)$, i.e., $\mu_x(C^{i-1}) < \mu_v(C^i)$; moreover, since agent x is improving

her utility as an effect of the fact that $|C^{i-1}(x)| = 1 + |C^i(x)|$ and both utilities of agents x and v are of the form $\frac{1}{\alpha-1}$ (where α is the cardinality of the coalition they belong to), it follows that, when considering $\mu_x(C^i)$ instead of $\mu_x(C^{i-1})$, strict inequality in $\mu_x(C^{i-1}) < \mu_v(C^i)$ can become non-strict, but cannot be reversed: it holds that $\mu_x(C^i) \leq \mu_v(C^i)$. This means that the nodes for which the utility decreases have a utility at least equal, in $\vec{x}^i$ to the one of another node with increased utility, and therefore it still holds that $\vec{x}^i > \vec{x}^{i-1}$.

Therefore, at the end of this process, we obtain another star-coalition structure C being a Nash equilibrium; moreover, since, for any $i \geq 0$, $x_1^i = \mathrm{SW}(C^i)$, and, for any $i \geq 1$, $\vec{x}^i > \vec{x}^{i-1}$, it directly follows that $\mathrm{SW}(C) \geq \mathrm{SW}(\overline{C})$.

In order to prove the Lemma, it remains to show that outcome C is also a strong Nash equilibrium.

Assume, by way of contradiction, that C is not a strong Nash equilibrium, and assume that $K \subseteq N$ is one of the smallest (i.e., of minimum cardinality) set of agents with a profitable joint deviation. For any $u \in K$, let j_u be the index of the coalition u selects in the joint deviation leading to outcome C'.

Notice that, for every $C \in C$, the center of coalition C (if $|C| \geq 3$), and both nodes of the coalition (if $|C| = 2$) are such that their utility is equal to 1, that is the best possible these agents can achieve. Moreover, by Algorithm 1, in coalition structure C there exists no couple $x, y \in L^{\geq 3}(C)$ such that $\{x, y\} \in E$. It follows that the set K of agents can only contain agents in $L^{\geq 3}(C)$ and each of these agents $u \in K$ only has an edge towards a node of her candidate coalition j_u: In fact, if $|C_{j_u}| = 2$, agent u cannot have two edges because a bipartite graph does not contain any cycle of 3 nodes, and, if $|C_{j_u}| \geq 3$, u has an edge only towards the center of star C_{j_u}.

Let us consider the directed weighted graph $F = (C, E_F, w_F)$ in which the nodes are the coalitions in C and there is a directed arc (C_i, C_j) of weight x between two coalitions if, in the considered deviation of agents in K, x agents move from star coalition C_i to star coalition C_j.

Notice that F is a directed acyclic graph. In fact, until a cycle of weight at least 1 exists, it is possible to obtain a set $K' \subset K$ in which we remove, starting from set K, an agent for each component involved in the cycle. As it can be easily verified, since the final cardinality of each involved coalition is unchanged, if K is a set of agents possessing a joint deviation, also all agents in K' have a joint deviation, consisting in the same selection of coalitions as in the original deviation. Since $|K'| < |K|$ we obtain a contradiction to the fact that K is one of the smallest set of agents with a profitable joint deviation.

Therefore, there is in F a coalition C_ℓ without outgoing arcs. Consider now any agent $u \in K$ such that $j_u = \ell$. Since agent u benefits from the joint deviation, it hods that $\mu_u(C) < \mu_u(C')$. Consider now coalition structure $C'' = (C, u, \ell)$. Since C_ℓ can have only ingoing arcs in F, it holds that $|C'_\ell| \geq |C''_\ell|$ and therefore $\mu_u(C') \leq \mu_u(C'')$. Hence, we obtain $\mu_u(C) < \mu_u(C'')$, i.e., agent u also posses a (Nash) improving move in coalition structure C: a contradiction to the fact that C is a Nash equilibrium. □

We now show that, if the input graph is bipartite, then there always exists a star-coalition structure being also optimal.

Lemma 4.3. *For any unweighted bipartite graph $G = (N, E)$, there exists a star-coalition structure C for $\mathcal{G}(G)$ such that, for any $u \in N$, it holds that $\mu_u(C) \geq \frac{\delta^u(G)}{|N|-1}$.*

Proof. Let $G = (A \cup B, E)$, with E such that for every edge $\{u, x\} \in E$ it holds that $u \in A$ and $x \in B$. We also refer to nodes in A (respectively, in B) as nodes in the left (respectively, right) hand side of G.

Consider Algorithm 2, that is composed by three phases: phase 1 is the while loop of lines 3–6, phase 2 the while loop of lines 7–13 and phase 3 the while loop of lines 14–17. For any $k = 1, 2, 3$, let i_k be the value of variable i at the end of phase k.

Algorithm 2 It takes as input a bipartite graph $G = (A \cup B, E)$ and returns a *star*-coalition structure.

1: $C^0 \leftarrow \{\{1\}, \{2\}, \dots, \{n\}\}$
2: $i \leftarrow 0$
3: **while** there exists $x \in B$ with a Nash improving move selecting coalition $C^i_j = C^i(u)$, with $u \in A$ **do**
4: $\quad i \leftarrow i + 1$ ▷ Beginning of step i, phase 1
5: $\quad C^i \leftarrow (C^{i-1}, x, j)$
6: **end while**
7: **while** there exist $u, v_1, \dots, v_\ell, w \in A$ and $y, z_1, \dots, z_{\ell+1} \in B$ such that $|C^i(u)| = 1$
   and for any $t \in [\ell]$, $\{v_t, z_t\} \in C^i$
   and $\{z_{\ell+1}, w, y\} \in C^i$
   and for any $t \in [\ell]$, $\{v_t, z_{t+1}\} \in E$
   and $\{u, z\} \in E$
   do
8: $\quad i \leftarrow i + 1$ ▷ Beginning of step i, phase 2
9: $\quad C^i \leftarrow C^{i-1} \setminus \{\{w, z_{\ell+1}, y\}\} \cup \{\{u, z_1\}, \{w, y\}\}$
10: $\quad$ **for** $t = 1$ to ℓ **do**
11: $\quad\quad C^i \leftarrow C^i \setminus \{\{v_t, z_t\}\} \cup \{\{v_t, z_{t+1}\}\}$
12: $\quad$ **end for**
13: **end while**
14: **while** there exists $u \in A$ with a Nash improving move selecting coalition $C^i_j = C^i(x)$, with $x \in B$ **do**
15: $\quad i \leftarrow i + 1$ ▷ Beginning of step i, phase 3
16: $\quad C^i \leftarrow (C^{i-1}, u, j)$
17: **end while**
18: **return** C^i

In the first phase of Algorithm 2, agents in B move according to a Nash dynamics in which they can select all coalitions containing an agent $u \in A$. We now show that the while loop at lines 3–6 terminates.

Indeed, we notice that the game played by agents belonging to B is equivalent to a *singleton congestion game with identical latency functions* (CGI) in which we also have a set of resources, i.e., a Nash equilibrium (respectively, an improving move) in this new game is also a Nash equilibrium (respectively, an improving move) in our game and vice versa. In a CGI, agent's strategy consists of a resource. The delay of a resource is given by the number of agents choosing it, and the cost that each agent aims at minimizing is the delay of her selected resource. In particular, the set of agents is B and the set of resources is A. In fact, in our game every agent aims at minimizing the cardinality of the star coalition (centered in a node of A) she belongs to. It is well known [19] that CGI are potential games and that any dynamics in a potential game leads to a Nash equilibrium.

Clearly, outcome C^{i_1} is a *star*-coalition structure by the way it is constructed. Let us analyze the utility of the agents at the end of Phase 1.

For any $u \in A$ such that $|C^{i_1}(u)| > 1$, since each node in B choosing the same coalition of u in an improving move possesses an edge towards u, it holds that $\mu_u(C^{i_1}) = 1 \geq \frac{\delta^u(G)}{n-1}$.

For any $x \in B$, consider the set of nodes $A_x \subseteq A$ such that, for any $u \in A_x$, $\{u, x\} \in E$; clearly, $|A_x| = \delta^x(G)$. Notice that x can choose as strategy one of the $\delta^x(G)$ coalitions containing nodes in A_x; moreover, there must exist at least a node in $v \in A_x$ such that $|C^{i_1}(v) \setminus \{x\}| \leq 1 + \frac{|B|-1}{|A_x|} \leq 1 + \frac{n-|A_x|-1}{|A_x|} = 1 + \frac{n-\delta^x(G)-1}{\delta^x(G)} = \frac{n-1}{\delta^x(G)}$, because otherwise we would obtain $\sum_{u \in A_x} |C^{i_1}(u) \setminus \{x\}| > n-1$: a contradiction because $\sum_{u \in A_x} |C^{i_1}(u) \setminus \{x\}| \leq n-1$ given that (i) C^{i_1} is a partition of N, (ii) for any $u, v \in A_x$ it holds that $C^{i_1}(u) \neq C^{i_1}(v)$ and (iii) $\bigcup_{u \in A_x} \left(C^{i_1}(u) \setminus \{x\}\right) \subseteq N \setminus \{x\}$. Since x does not have an improving move selecting a coalition containing any agent in A, it follows that her utility at the end of phase 1 has to be at least the one she would experience in coalition $C^{i_1}(v)$:

$$\mu_x(C^{i_1}) \geq \frac{1}{1 + \frac{n-\delta^x(G)-1}{\delta^x(G)} + 1 - 1} = \frac{\delta^x(G)}{n-1}.$$

It remains to deal with any $u \in A$ such that $|C^{i_1}(u)| = 1$: in fact, it holds that $\mu_u(C^{i_1}) = 0$. To this aim, we perform phases 2 and 3 of Algorithm 2.

The following property will be useful in the remainder of the proof.

Property 4.1. *At the end of phase 1, all nodes $u \in A$ belonging to a coalition of size k cannot have edges towards coalitions of C^{i_1} of size greater than $k + 1$.*

In fact, otherwise, if an edge $\{u, v\}$ between node u and a node v such that $|C^{i_1}(v)| \geq k + 2$ exists, then agent v would have an improving move in C^{i_1} towards coalition $C^{i_1}(u)$: a contradiction to the fact that phase 1 is terminated. This concludes the proof of Property 4.1.

For any $i \geq i_1$, let $N_1^i \subset A$ be the set containing all agents u that are isolated in C^i, i.e., such that $|C^i(u)| = 1$.

Before considering the phase 2 of Algorithm 2 we give the following definitions.

Definition 4.4. An *augmenting path* in a coalition structure C^i, for any $i \geq i_1$, is a path of G that starts from an isolated agent $u \in N_1^i$ and where the last but one node is a left hand side agent $v \in A$ such that $\mu_v(C^i) = 1$ and $|C^i(v)| \geq 3$.

Definition 4.5. A *semi-augmenting path* in a coalition structure C^i, for any $i \geq i_1$, is a path of G that starts from an isolated agent $u \in N_1^i$ and terminates to a left hand side agents $v \in A$ of a coalition of size 2, i.e., such that $|C^i(v)| = 2$.

Roughly speaking, phase 2 *normalizes* outcome C^{i_1} so that all augmenting paths in G are removed by decreasing the number of isolated agents (see Figure 3).

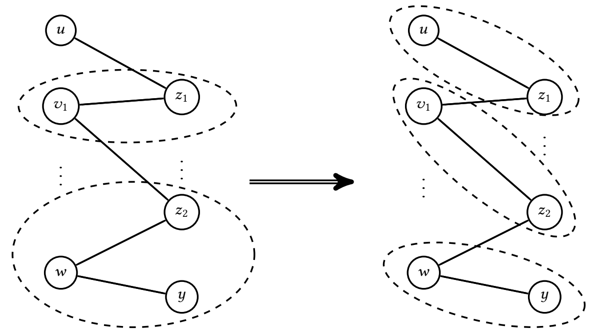

Figure 3: An example of augmenting path with $\ell = 1$ and the corresponding transformation made in phase 2 of Algorithm 2.

Notice that at each step $i > i_1$ of phase 2, $|N_1^i| < |N_1^{i-1}|$, because a node in N_1^{i-1} becomes part of a coalition of size 2: it directly follows that the while loop of lines 7–13 terminates.

Moreover, for any $i > i_1$, let N_2^i be the set containing all nodes involved in step i of the second phase. It holds that, for any $u \in N_2^i$, $\mu_u(C^i) = 1 \geq \frac{\delta_u(G)}{n-1}$, given that they are all included in coalitions of size 2 in C^i, and the utility of the other nodes remains unchanged with respect to outcome C^{i_1}.

Overall, at the end of phase 2, C^{i_2} is a *star*-coalition structure in which, for every $u \in N \setminus N_1^{i_2}$, it holds that $\mu_u(C^{i_2}) \geq \frac{\delta_u(G)}{n-1}$. Moreover, notice that the following property, derived from Property 4.1, holds, given that in phase 2 we can only decrease the size of the coalitions containing agents in B.

Property 4.2. *At the end of phase 2, all nodes $u \in A$ belonging to a coalition of size k cannot have edges towards coalitions of C^{i_2} of size greater than $k + 1$.*

In the third phase of Algorithm 2, agents in A move according to a Nash dynamics in which they can select all coalitions containing an agent $x \in B$. By exploiting the same arguments about congestion games used for the first phase, with the difference that the set of agents is A and the set of resources is B, we get that the dynamics of this phase leads to a Nash equilibrium, and therefore we are guaranteed that the while loop at lines 14–17 terminates.

We now show by induction on i that, for any $i \geq i_2$, i.e., during the dynamics of the third phase, only agents in $N_1^{i_2}$ and the left hand side agents of coalitions of size 2 being the final node of a semi-augmenting path can perform an improving move. Let N_3 be the set containing all these nodes.

The base of the induction is trivially verified for $i = i_2$ (actually, in this case only agents in $N_1^{i_2}$ can perform an improving move).

Let us assume that the inductive claim holds for any $j = i_2, \ldots, i-1$: we have to prove that it also holds for i. We distinguish among three disjoint cases:

- Any agent $u \in A$ with $|C^i(u)| \geq 4$ is such that $\mu_u(C^i) = 1$ because, by Property 4.2, the agents that can move in steps $i_2, \ldots, i-1$ can not join their coalition (in fact, these agents belong, by the induction hypothesis, to coalitions of size at most 2). It follows that they can not perform an improving move at step i.

- Any agent $u \in A$ with $|C^i(u)| = 3$ is such that $\mu_u(C^i) = 1$, because at any step $j = i_2, \ldots, i-1$, the only agents that can move according to the induction hypothesis are:
  - isolated nodes. By Property 4.2, isolated nodes that can move in steps $i_2, \ldots, i-1$ can not join coalition $C^j(u)$.
  - left hand side agents of coalitions of size 2 being the final node of a semi-augmenting path. They can not join coalition $C^j(u)$ because otherwise a augmenting path would exist: a contradiction to the fact that at the end of phase 2 no augmenting path exists, given that no augmenting path can be introduced during the dynamics of phase 3. In fact, notice that, in order to introduce a new augmenting path, either a new coalition of size 2 or a new coalition of size 3 with 2 nodes in B should be added to C^{i_2}; since at lines 14–17 of Algorithm 2 only coalitions with at least 3 nodes having only one node in B can be created, it follows that no augmenting path can be introduced during the dynamics of phase 3.

  It follows that they can not perform an improving move at step i.
- Finally, consider any agent $u \in A$ with $|C^i(u)| = 2$ and such that u does not belong to any semi-augmenting path. Also in this case, it holds that they can not perform an improving move at step i because $\mu_u(C^i) = 1$. In fact, if we assume by way of contradiction that $\mu_u(C^i) < 1$, it follows that some agent $v \in A$ has joint coalition $C^j(u)$ at some step $j = i_2, \ldots, i-1$. By the induction hypothesis, v at step j can be either an isolated node or a left hand side agents of coalitions of size 2 being the final node of a semi-augmenting path. If v was an isolated node, then there exists a semi-augmenting path starting from v and ending in u: a contradiction. Otherwise, v is a left hand side agent of coalitions of size 2 being the final node of a semi-augmenting path. A longer semi-augmenting path can be obtained by adding the nodes belonging to $C^j(u)$: a contradiction.

Hence, the induction step follows.

Now, we prove that, for every node $u \in N_3$, all edges $\{u, x\} \in E$ are such that x is a right hand side agent of a coalition of size 2 in C^{i_2}. We distinguish between two cases:

- If $u \in N_1^{i_2}$, by Property 4.2 she cannot have edges towards coalitions of size greater than 2 in C^{i_2}.
- Otherwise, u is a left hand side agent of a coalition of size 2 being the final node of a semi-augmenting path. In this case, by Property 4.2 she cannot have edges towards coalitions of size greater than 3 in C^{i_2}. Moreover, if u had an edge towards a coalition of size 3 in C^{i_2}, then an augmenting path would exist at the end of phase 2: a contradiction.

It follows that each agent moving in the dynamics of phase 3 has, as possible strategies, coalitions being (also after their deviation) isomorphic to star graph, and therefore Algorithm 2 returns a star-coalition structure.

Moreover, by exploiting the same arguments used in the analysis of phase 1, it holds that at the Nash equilibrium reached at the end of phase 3, for any $u \in N_3$, $\mu_u(C^{i_3}) \geq \frac{\delta^u(G)}{n-1}$.

Notice that, among the agents not belonging to N_3, only those in $B' \subseteq B$ selected as strategies by at least one agent in N_3 (during the dynamics of phase 3) can change their utility with respect to the one they have at the end of phase 2. However, for any $x \in B'$, it holds that

$$\mu_x(C^{i_3}) = 1 \geq \frac{\delta^x(G)}{n-1}$$

because each node in A choosing the same coalition of x in an improving move possesses an edge towards x. □

By combining lemmata 4.2 and 4.3, it is possible to prove the following theorem.

Theorem 4.6. *For any unweighted bipartite graph G, n−SPoS$(\mathcal{G}(G)) = 1$.*

Proof. Consider an optimum coalition structure C^* for $\mathcal{G}(G)$. Since G is an unweighted bipartite graph, clearly, for any $C_i^* \in C^*$, also $G_{C_i^*}$ is an unweighted bipartite graph. We recall that, for any $u \in N$, $\mu_u(C^*) = \frac{\delta^u(G_{C^*(u)})}{|C^*(u)|-1}$. Therefore, by applying Lemma 4.3 to each coalition $C_i^* \in C^*$, we obtain a star-coalition structure $\overline{C}$ for $\mathcal{G}(G)$ with equal egalitarian social welfare as C^*. Finally, by applying Lemma 4.2 to $\overline{C}$ we get the claim. □

We conclude this section by showing that the strong price of stability is 1 also for graphs with maximum degree at most two, i.e., in which each node has degree at most two. Notice that such graphs can admit cycles of odd length, and therefore may not be bipartite.

Theorem 4.7. *For any unweighted graph G with maximum degree at most 2, n−SPoS$(\mathcal{G}(G)) = 1$.*

Given that a strong Nash stable is also Nash and core stable, the following result directly follows from theorems 4.6 and 4.7.

Corollary 4.8. *Let G be an unweighted graph either being bipartite or having maximum degree at most two; it holds that* PoS$(\mathcal{G}(G)) = 1$ *and* CPoS$(\mathcal{G}(G)) = 1$.

5 CONCLUSIONS

In this paper we provided a comprehensive analysis on the performance of strong Nash stable, Nash stable and core stable outcomes for MFHG under the egalitarian social welfare function.

The main left open problem is that of determining the strong price of stability, and also the price of stability and the core price of stability, for unweighted graphs which are neither bipartite nor of maximum degree at most two. To this respect, an interesting start point could be that of considering unweighted undirected triangle free graphs.

Another research direction could be that of designing truthful mechanisms for MFHG that perform well under the egalitarian social welfare function. More generally, it is worth to evaluate, under the egalitarian social welfare function, the performance of natural stable outcomes for any hedonic game in which agents assign cardinal utilities to coalitions.

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