Stability in FEN-Hedonic Games for Single-Player Deviations

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ABSTRACT

Hedonic games model how players form coalitions based on their preferences about the coalitions they can join. Lang et al. [17] introduced FEN-hedonic games where each player partitions the other players into friends, enemies, and neutral players and ranks her friends and enemies. They then use bipolar responsive extensions to derive preferences over coalitions, and since such preferences can be incomplete, they consider possible and necessary stability for various stability notions and study the related verification and existence problems in terms of computational complexity. However, in their complexity analysis they left a number of cases open. We settle several of these open problems for stability concepts based on single-player deviations: We show that possible verification can be solved in polynomial time for Nash stability, individual stability, and contractually individual stability. Yet, necessary existence is an NP-complete problem for individual stability while possible existence is easy for contractually individual stability.

KEYWORDS

hedonic game; stability; coalition formation; complexity

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1 INTRODUCTION

Hedonic games, originally proposed by Banerjee et al. [6] and independently by Bogomolnaia and Jackson [7], are cooperative games where players have preferences over the coalitions they can join. Which coalition structure will form in the end can be modeled via various stability concepts, including Nash stability, core stability, individual stability, and contractually individual stability (see, e.g., the book chapters by Aziz and Savani [4] and Elkind and Rothe [14] or the papers by Bogomolnaia and Jackson [7] and Aziz et al. [1, 3]).

A central issue in hedonic games is how they can be represented efficiently. Woeginger [23] and Lang et al. [17] survey various approaches from the literature and discuss their pros and cons, including the *individually rational encoding* and the *anonymous encoding* due to Ballester [5], the *additive encoding* [3, 21, 22, 24], the *"friends and enemies" encoding* due to Dimitrov et al. [13, 21], the *singleton encoding* used by Cechlárová et al. [9–11], *hedonic coalition nets* due to Elkind and Wooldridge [15], and *fractional hedonic games* due to Aziz et al. [2]. To overcome various issues with these former representations, Lang et al. [17] proposed the notion of FEN-hedonic game where players divide the other players into friends, enemies, and neutral players and rank their friends and their enemies (so-called *weak rankings with double threshold*). To derive preferences over coalitions, they are then using the so-called *polarized responsive set extension*, which is akin to and a generalization of the Bossong–Schweigert set extension (see [8, 12]). Since such preferences can be incomplete, they further consider *possible* and *necessary stability* (inspired by the notions of possible and necessary winner in voting [16, 25]) for a variety of stability concepts (such as those mentioned above) and study the related verification and existence problems in terms of their computational complexity.

However, in their complexity analysis Lang et al. [17] left a number of cases open. The purpose of this paper is to settle some of these open questions for stability concepts based on single-player deviations. Specifically, we show that possible verification can be solved in polynomial time for Nash stability, individual stability, and contractually individual stability. On the other hand, we show that necessary existence is NP-complete for individual stability while possible existence is easy for contractually individual stability.

This paper is organized as follows. We present the needed notions from game theory (in particular, concerning hedonic games) and complexity theory in Section 2 and the formal details of FEN-hedonic games introduced by Lang et al. [17] in Section 3. In Section 4, we prove our complexity results. We conclude in Section 5 and give a brief outlook on future work.

2 PRELIMINARIES

We start by providing some basic definitions. After defining the notion of hedonic game, we will explain some notions of stability to be considered later in the paper. Finally, we will also give some basic background of complexity theory, to be used later on in Section 4 where we will present some complexity results regarding the stability of FEN-hedonic games.

2.1 Hedonic Games

Let $A = \{1, \dots, n\}$ be a set of *players* (or *agents*). Every subset C of A is called a *coalition.* Let, further, $\mathscr{A}_i = \{C \subseteq A \mid i \in C\}$ be the set of all coalitions $C \subseteq A$ containing agent $i \in A$. The *preference relation* $\succeq_i$ of player i is a complete weak order over $\mathscr{A}_i$. For any two coalitions $C, D \in \mathscr{A}_i$, we say that i *weakly prefers* C *to* D if $C \succeq_i D$; i *prefers* C *to* D (denoted by $C \succ_i D$) if $C \succeq_i D$ and not $D \succeq_i C$; and i *is indifferent between* C *and* D (denoted by $C \sim_i D$) if $C \succeq_i D$ and $D \succeq_i C$. As usual, we also define $C \preceq_i D \Leftrightarrow D \succeq_i C$ and $C \prec_i D \Leftrightarrow D \succ_i C$.

A *hedonic game* is defined as a pair $(A, \succeq)$, where $A = \{1, \dots, n\}$ is a set of *players* and $\succeq = (\succeq_1, \dots, \succeq_n)$ is a profile of preference relations where each relation $\succeq_i$ gives the preference order of player i.

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A *coalition structure* Γ for a hedonic game $(A, \succeq)$ is a partition of A into disjoint coalitions, where $\Gamma(i)$ denotes the coalition in Γ containing i. We denote the set of all possible coalition structures for a hedonic game $(A, \succeq)$ by $\mathscr{C}_{(A,\succeq)}$. For simplicity, we omit stating the hedonic game explicitly and just write $\mathscr{C}$ if $(A, \succeq)$ is clear from the context.

2.2 Notions of Stability

There are several stability concepts in hedonic games, indicating whether some player(s) may have an incentive to deviate from a given coalition structure. In the literature one distinguishes between different types of stability concepts that concern the deviation of single players or of groups of players or the comparison of coalition structures; we here will focus on concepts concerning single-player deviation only. While most of these concepts have already been intensively studied for common hedonic games, we will apply them to FEN-hedonic games introduced by Lang et al. [17] (see also [20]) and formally defined here in Section 3, and will study the complexity of the related decision problems in Section 4.

The stability concepts concerning single-player deviation are based on the question of whether there are single players who would prefer to be in another coalition than the one assigned to them by the given coalition structure Γ.

Definition 2.1. Let $(A, \succeq)$ be a hedonic game with $A = \{1, \dots, n\}$ and $\succeq = (\succeq_1, \dots, \succeq_n)$. A coalition structure $\Gamma \in \mathscr{C}_{(A,\succeq)}$ is

(1) *perfect* if each player weakly prefers her assigned coalition to every other coalition containing her; formally:

$$(\forall i \in A)(\forall C \in \mathscr{A}_i)\big[\Gamma(i) \succeq_i C\big];$$

(2) *individually rational* if every player weakly prefers her assigned coalition to being alone:

$$(\forall i \in A)\big[\Gamma(i) \succeq_i \{i\}\big];$$

(3) *Nash stable* if no player prefers another coalition in Γ:

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\}\big];$$

(4) *individually stable* if no player prefers another coalition in Γ and could deviate to it without harming any player in that new coalition:

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\}) \\ \big[\Gamma(i) \succeq_i C \cup \{i\} \vee (\exists j \in C)[C \succ_j C \cup \{i\}]\big];$$

(5) *contractually individually stable* if no player prefers another coalition in Γ and could deviate to it without harming a player in the new or her assigned coalition:

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\} \vee \\ (\exists j \in C)[C \succ_j C \cup \{i\}] \vee (\exists k \in \Gamma(i))[\Gamma(i) \succ_k \Gamma(i) \setminus \{i\}]\big].$$

The known relations among these stability concepts are as follows (see, e.g., the book chapter by Aziz and Savani [4]): Perfectness implies Nash stability (i.e., every perfect coalition structure is Nash stable), which in turn implies individual stability, which implies both contractually individual stability and individual rationality.

2.3 Notions from Complexity Theory

The reader is assumed to be familiar with the basic notions of complexity theory; in particular, with the complexity classes P and NP, the notion of polynomial-time many-one reducibility and the notions of NP-hardness and NP-completeness. Recall that a (decision) problem X polynomial-time many-one reduces to a (decision) problem Y (both encoded as sets of strings over some alphabet representing the yes-instances of the problems) if there exists a polynomial-time computable function f such that for each instance x, $x \in X$ if and only if $f(x) \in Y$. Further, Y is NP-hard if every NP problem polynomial-time many-one reduces to Y, and Y is NP-complete if Y is NP-hard and in NP. For more background on computational complexity, we refer to the textbooks by Papadimitriou [18] and Rothe [19].

3 FEN-HEDONIC GAMES

In this section we present the definition of FEN-hedonic games which were first introduced by Lang et al. [17]. The letters FEN stand for "friends, enemies, and neutral players" to reflect the basic assumption of FEN-hedonic games, which unlike the "friends and enemies" encoding of hedonic games due to Dimitrov et al. [13] (see also Sung and Dimitrov [21]) also have neutral players. In more detail, in FEN-hedonic games players don't submit complete preferences over coalitions (which would be too costly) or ordinal rankings over players. Instead, they submit so-called *weak rankings with double threshold*, which partition the players into friends, enemies, and neutral players with friends and enemies being ranked, whereas neutral players are not ranked. In this representation different models of expressing the players' preferences were combined in order to handle some crucial problems, as will be briefly explained next (see [17] for more details).

3.1 Former Models

Eliciting the players' preferences has always been a main issue in game theory and the related area of computational social choice. When preferences over sets are required, there is always the problem that these preferences have exponential size. The same holds in our case: Hedonic games require complete orders over all coalitions containing a given agent and there are exponentially many of those coalitions in the number of agents. Hence, asking players to specify their entire rankings would result in preference representations of exponential size.

Another issue is that it might be considered an unrealistic assumption that agents have complete preferences over all coalitions. They may not be willing or not even be able to state a complete ranking. Some authors tried to handle this issue by asking the players only for a small part of their preferences (e.g., in the *individually rational encoding* due to Ballester [5]), but this brings up another problem: Eliciting only partial preferences always comes along with a loss of expressivity.

To address these issues, Lang et al. [17] considered some existing preference representations in hedonic games and then defined their new representation of *weak rankings with double thresholds.* By doing so, they combined aspects of various former concepts and tried to avoid all issues—exponential size of the requested preference representations, too harsh or too unrealistic assumptions, and the loss of expressivity. An interesting discussion of how to approach the above issues can be found in Section 1 of their paper [17] (see also the book chapter by Aziz and Savani [4] for a more detailed treatise).

3.2 Weak Rankings with Double Threshold

Let $A = \{1, \dots, n\}$ be a set of players. A *weak ranking with double thresholds* for an agent $i \in A$ is denoted by $\trianglerighteq_i^{+0-}$. It is obtained by a partition of the remaining players $A \setminus \{i\}$ into friends of i, enemies of i, and neutral players that i is indifferent about. Additionally, player i has to submit a ranking of her friends as well as of her enemies. We write $\trianglerighteq_i^{+0-} = (\trianglerighteq_i^{+} | A_i^0 | \trianglerighteq_i^{-})$ or $\trianglerighteq_i^{+0-} = (\trianglerighteq_i^{+} | j_1 \cdots j_k | \trianglerighteq_i^{-})$, where $\trianglerighteq_i^{+}$ is a weak order over the set A_i^+ of i's friends, $\trianglerighteq_i^{-}$ a weak order over the set A_i^- of i's enemies, and $A_i^0 = \{j_1, \dots, j_k\}$ is the set of i's neutral players.

Every player i is assumed to prefer her friends to her neutral players and her neutral players to her enemies. The *weak ranking with double thresholds* $\trianglerighteq_i^{+0-}$ therefore induces a weak order $\trianglerighteq_i$, defined as follows (extracted from [17]): $\trianglerighteq_i$ coincides with $\trianglerighteq_i^{+}$ on A_i^+; $f \triangleright_i j$ for each $f \in A_i^+$ and $j \in A_i^0$; $j_1 \sim_i j_2 \sim_i \cdots \sim_i j_k$ for $A_i^0 = \{j_1, j_2, \dots, j_k\}$; $j \triangleright_i e$ for each $j \in A_i^0$ and $e \in A_i^-$; and $\trianglerighteq_i$ coincides with $\trianglerighteq_i^{-}$ on A_i^-.

Example 3.1. Let $A = \{1, \dots, 10\}$. Some possible *weak rankings with double thresholds* are

$$\trianglerighteq_1^{+0-} = (2 \triangleright_1^{+} 3 \sim_1^{+} 4 \quad | \; 5\ 6\ 7 \quad | \; 8 \sim_1^{-} 9 \triangleright_1^{-} 10 \quad),$$

$$\trianglerighteq_2^{+0-} = (6 \sim_2^{+} 7 \sim_2^{+} 8 \sim_2^{+} 9 \quad | \; 3\ 4\ 5\ 10 \quad | \; 1 \quad),$$

$$\trianglerighteq_3^{+0-} = (\quad | \; 5\ 6\ 7\ 8\ 9\ 10 \quad | \; 1 \triangleright_3^{-} 2 \triangleright_3^{-} 4 \quad).$$

The weak orders $\trianglerighteq_1$, $\trianglerighteq_2$, and $\trianglerighteq_3$ induced by $\trianglerighteq_1^{+0-}$, $\trianglerighteq_2^{+0-}$, and $\trianglerighteq_3^{+0-}$ are then given by

$$2 \triangleright_1 3 \sim_1 4 \triangleright_1 5 \sim_1 6 \sim_1 7 \triangleright_1 8 \sim_1 9 \triangleright_1 10,$$
$$6 \sim_2 7 \sim_2 8 \sim_2 9 \triangleright_2 3 \sim_2 4 \sim_2 5 \sim_2 10 \triangleright_2 1, \quad \text{and}$$
$$5 \sim_3 6 \sim_3 7 \sim_3 8 \sim_3 9 \sim_3 10 \triangleright_3 1 \triangleright_3 2 \triangleright_3 4.$$

3.3 FEN-Hedonic Games

A *FEN-hedonic game* is a pair $(A, \trianglerighteq^{+0-})$ consisting of a set of agents $A = \{1, \dots, n\}$ and a profile $\trianglerighteq^{+0-} = \left(\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-}\right)$ of preferences, where $\trianglerighteq_i^{+0-}$ is a *weak ranking with double threshold* for agent $i \in A$. Again, a coalition structure for a *FEN-hedonic game* $(A, \trianglerighteq^{+0-})$ is a partition of A into disjoint coalitions and we denote the set of all possible coalition structures by $\mathscr{C}_{(A, \trianglerighteq^{+0-})}$.

So far, we got preferences over players, but what we require are preferences over coalitions. To obtain these, *set extensions* are used. As Lang et al. [17] do, we will make use of the *polarized responsive extension*, which is defined as follows:

3.3.1 Polarized responsive extension. The *extended preference* $\succeq_i^{+0-}$ for a weak ranking with double threshold $\trianglerighteq_i^{+0-}$ is defined as follows. For any two coalitions $X, Y \in \mathscr{A}_i$, $X \succeq_i^{+0-} Y$ if and only if the following two conditions hold:

(1) There is an injective function $\sigma : Y \cap A_i^+ \to X \cap A_i^+$ such that for every $y \in Y \cap A_i^+$, we have $\sigma(y) \trianglerighteq_i y$.
(2) There is an injective function $\theta : X \cap A_i^- \to Y \cap A_i^-$ such that for every $x \in X \cap A_i^-$, we have $x \trianglerighteq_i \theta(x)$.

As usual, we define $X \succ_i^{+0-} Y \Leftrightarrow X \succeq_i^{+0-} Y \wedge \neg Y \succeq_i^{+0-} X$; $X \sim_i^{+0-} Y \Leftrightarrow X \succeq_i^{+0-} Y \wedge Y \succeq_i^{+0-} X$; $X \preceq_i^{+0-} Y \Leftrightarrow Y \succeq_i^{+0-} X$; and $X \prec_i^{+0-} Y \Leftrightarrow Y \succ_i^{+0-} X$.

3.3.2 Explanation and incompleteness. Let us explain the above definition of the responsive extension principle in some more detail. A coalition X is preferred to a coalition Y by player i if, firstly, for each friend f_Y of i in Y there is a friend f_X of i in X that is at least as preferred by i as f_Y (i.e., $f_X \trianglerighteq_i f_Y$) and, secondly, for each enemy e_X of i in X there is an enemy e_Y of i in Y that is at least as disliked by i as e_X (i.e., $e_X \trianglerighteq_i e_Y$). Additionally, these friends in X and enemies in Y both have to be chosen pairwise distinctly. Thus this condition also implies that there have to be at least as many of i's friends in X as in Y and at most as many enemies of i in X as in Y.

A coalition becomes more preferred by adding a friend and less preferred by adding an enemy. Moreover, exchanging a friend for a neutral player or an enemy makes a coalition less preferred, while the opposite operation makes it more preferred. Similarly, exchanging a neutral player for an enemy makes a coalition less preferred and the opposite operation makes it more preferred. When a friend is replaced by a better (i.e., preferred) friend, the new coalition is preferred to the old one. Likewise, by replacing an enemy by a more preferred enemy a coalition becomes more preferred. But there are also some coalitions that are incomparable according to the responsive extension. For instance, when two players, a friend and an enemy, are added to or removed from a coalition, it is not specified by the responsive extension principle which coalition, the new or the old one, is preferred—they remain incomparable. Another case of incomparable coalitions is described below.

Example 3.2. Consider the FEN-hedonic game

$$(A, \trianglerighteq^{+0-}) = (\{1, 2, 3, 4\}, (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_4^{+0-}))$$

with

$$\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \triangleright_1 4 \quad | \quad | \quad) \text{ and}$$
$$\trianglerighteq_2^{+0-} = (\quad | \; 1\ 3 \quad | \; 4 \quad).$$

Then the polarized responsive extension $\succeq_1^{+0-}$ of $\trianglerighteq_1^{+0-}$ is incomplete (or partial). In Figure 1, an arrow from a coalition C to a coalition D means that C is preferred to D.

As can be seen in Figure 1, coalitions $\{1, 3, 4\}$ and $\{1, 2\}$ are incomparable with respect to $\succeq_1^{+0-}$. By contrast, the polarized responsive extension $\succeq_2^{+0-}$ of $\trianglerighteq_2^{+0-}$ is complete:

$$\{2\} \sim_2^{+0-} \{1, 2\} \sim_2^{+0-} \{2, 3\} \sim_2^{+0-} \{1, 2, 3\} \succ_2^{+0-} \{2, 4\}$$
$$\sim_2^{+0-} \{1, 2, 4\} \sim_2^{+0-} \{2, 3, 4\} \sim_2^{+0-} \{1, 2, 3, 4\}.$$

3.3.3 Extension to complete preferences. Since the polarized responsive extension can lead to incomplete preferences by leaving the relation between some coalitions undecided, we consider the following definition of *extensions to complete preferences.*

Definition 3.3. A complete relation $\succeq_i$ over $\mathscr{A}_i$ *extends a (possibly incomplete) relation* $\succeq_i^{+0-}$ if $C \succ_i^{+0-} D$ implies $C \succ_i D$ and $C \sim_i^{+0-} D$ implies $C \sim_i D$ for all coalitions $C, D \in \mathscr{A}_i$. We define $Ext(\succeq_i^{+0-})$ as the set of all complete relations extending $\succeq_i^{+0-}$.

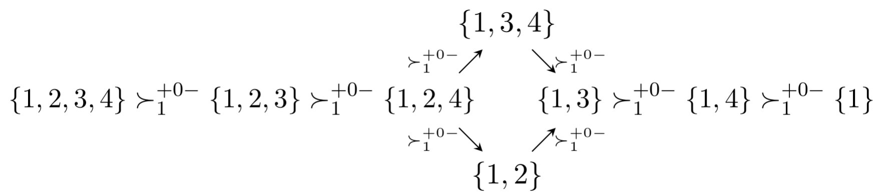

Figure 1: The polarized responsive extension $\succeq_1^{+0-}$ of player 1's preference $\trianglerighteq_1^{+0-}$ in Example 3.2

Example 3.4. The partial preference relation $\succeq_1^{+0-}$ shown in Figure 1 can be extended to either of the following two complete relations: $\{1,2,3,4\} \succ_1 \{1,2,3\} \succ_1 \{1,2,4\} \succ_1 \{1,3,4\} \succ_1 \{1,2\} \succ_1 \{1,3\} \succ_1 \{1,4\} \succ_1 \{1\}$ and $\{1,2,3,4\} \succ_1' \{1,2,3\} \succ_1' \{1,2,4\} \succ_1' \{1,2\} \succ_1' \{1,3,4\} \succ_1' \{1,3\} \succ_1' \{1,4\} \succ_1' \{1\}$. Hence, $Ext(\succeq_1^{+0-}) = \{\succeq_1, \succeq_1'\}$.

3.4 Stability in FEN-Hedonic Games

We will now focus on the stability concepts introduced in Section 2.2 and apply them to FEN-hedonic games. Given a FEN-hedonic game H and a stability concept α, we consider the *verification* question (*Does a given coalition structure* Γ *satisfy* α *in the FEN-hedonic game* H*?*) and the *existence* question (*Does there exist a coalition structure that satisfies* α *in* H*?*).

To answer these questions, we first need to specify when a coalition structure satisfies a stability concept in a FEN-hedonic game. The crucial point here is that the preferences over coalitions which are obtained by the polarized responsive extension might be incomplete. This problem was already discussed by Lang et al. [17]. In order to handle incomplete preferences they decided to leave all incomparabilities open and consider the set of all possible extensions. Motivated by the notions of possible and necessary winner in voting [16, 25], they distinguished whether a stability concept is *possibly* or *necessarily* satisfied (i.e., for *some* or for *all* extensions).

Definition 3.5 (possible and necessary stability, Lang et al. [17]). Given a FEN-hedonic game $H = (A, \trianglerighteq^{+0-})$ with $A = \{1, \ldots, n\}$ and $\trianglerighteq^{+0-} = (\trianglerighteq_1^{+0-}, \ldots, \trianglerighteq_n^{+0-})$, a stability concept α, and a coalition structure $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$, we define:

(1) Γ *possibly satisfies* α if there exists a profile $P = (\succeq_1, \ldots, \succeq_n)$ in $\bigtimes_{i=1}^n Ext(\succeq_i^{+0-})$ such that Γ fulfills α in the hedonic game (A, P).
(2) Γ *necessarily satisfies* α if for all profiles $P = (\succeq_1, \ldots, \succeq_n)$ in $\bigtimes_{i=1}^n Ext(\succeq_i^{+0-})$ it holds that Γ fulfills α in the hedonic game (A, P).

Given these definitions, we can now focus on the corresponding verification and existence problems as indicated in the questions above. The two definitions of possible and necessary stability lead to two problems for each, one related to the verification and the other to the existence question.

For a stability concept α (such as those presented in Definition 2.1), the verification problems are defined as follows:

Possible-α-Verification (PαV)

Given: A FEN-hedonic game $(A, \trianglerighteq^{+0-})$ and a coalition structure $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$.
Question: Does Γ possibly satisfy α?

Necessary-α-Verification (NαV)

Given: A FEN-hedonic game $(A, \trianglerighteq^{+0-})$ and a coalition structure $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$.
Question: Does Γ necessarily satisfy α?

For a stability concept α, the existence problems are defined as follows:

Possible-α-Existence (PαE)

Given: A FEN-hedonic game $(A, \trianglerighteq^{+0-})$.
Question: Does there exist a coalition structure $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$ that possibly satisfies α?

Necessary-α-Existence (NαE)

Given: A FEN-hedonic game $(A, \trianglerighteq^{+0-})$.
Question: Does there exist a coalition structure $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$ that necessarily satisfies α?

Notice that a yes-instance for NαV (NαE) is always a yes-instance for PαV (PαE) because necessary stability implies possible stability. Furthermore, it holds that if (H, Γ) is a yes-instance for NαV (PαV) then H is a yes-instance for NαE (PαE). Summarized it holds that: $(H, \Gamma) \in$ NαV $\Rightarrow (H, \Gamma) \in$ PαV; $H \in$ NαE $\Rightarrow H \in$ PαE; $(H, \Gamma) \in$ NαV $\Rightarrow H \in$ NαE; and $(H, \Gamma) \in$ PαV $\Rightarrow H \in$ PαE.

4 COMPLEXITY RESULTS

In this section, we present our complexity results for some of the problems NαV, NαE, PαV, and PαE that were left open by Lang et al. [17].

4.1 Overview of Results

Table 1 gives an overview of results regarding the complexity of the problems NαV, NαE, PαV, and PαE for the stability concepts α presented in Definition 2.1. Known results are due to Lang et al. [17] and our new results are indicated by the theorem or proposition they are stated in.

4.2 Some Useful Observations

We now make some useful observations that are easy to verify; the first one is Proposition 6 in the paper by Lang et al. [17].

Table 1: Overview of complexity results

α	Verification		Existence	
	Possible	Necessary	Possible	Necessary
perfectness	in P [17]	in P [17]	in P [17]	in P [17]
individual rationality	in P [17]	in P [17]	in P [17]	in P [17]
individual stability	in P (Thm 4.5)	in P [17]	in NP [17]	NP-complete (Thm 4.9)
contractually individual stability	in P (Thm 4.6)	in P [17]	in P (Thm 4.7)	in NP [17]
Nash stability	in P (Thm 4.4)	in P [17]	NP-complete [17]	NP-complete [17]

Observation 4.1. *Given a weak ranking with double threshold* $\trianglerighteq_i^{+0-}$ *for an agent* $i \in A$ *and two coalitions* $C, D \subseteq \mathscr{A}_i$, *the relation* $C \succeq_i^{+0-} D$ *can be checked in polynomial time.*

Observation 4.2. *For a FEN-hedonic game* $(A, \trianglerighteq^{+0-})$, *where* $\trianglerighteq^{+0-} = (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-})$, *a coalition* $C \subseteq A$, *two players* $i, j \in A$ *with* $j \in C, i \notin C$, *and any extension* $\succeq_j \in Ext(\succeq_j^{+0-})$, *it holds that* $C \succ_j C \cup \{i\} \Leftrightarrow i \in A_j^-$.

Observation 4.3. *For a FEN-hedonic game* $(A, \trianglerighteq^{+0-})$, *where* $\trianglerighteq^{+0-} = (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-})$, *a coalition* $C \subseteq A$, *two players* $i, k \in C$, *and any extension* $\succeq_k \in Ext(\succeq_k^{+0-})$, *we have* $C \succ_k C \setminus \{i\} \Leftrightarrow i \in A_k^+$.

4.3 Possible Verification Is Easy

We show that possible verification is easy for Nash stability, individual stability, and contractually individual stability.

Theorem 4.4. Possible-Nash-Stability-Verification *is in* P.

Proof. Given a FEN-hedonic game $(A, \trianglerighteq^{+0-})$ with a set of agents $A = \{1, \dots, n\}$ and a profile $\trianglerighteq^{+0-} = (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-})$ and a coalition structure $\Gamma = \{C_1, \dots, C_k\}, k \geq 1$, it is possible to determine whether Γ is possibly Nash stable in polynomial time. This can be done by Algorithm 1.

Algorithm 1: PNSV

```
Data: A FEN-hedonic game (A, (⊵_1^{+0-}, ..., ⊵_n^{+0-})) and a
      coalition structure Γ.
Result: "YES" if Γ is possibly Nash stable; "NO" otherwise.
1 for i ∈ A do
2     for C ∈ Γ ∪ {∅} do
3         if Γ(i) ≺_i^{+0-} C ∪ {i} then
4             output "NO";
5 output "YES";
```

A coalition structure Γ is possibly Nash stable if there is a profile $\succeq = (\succeq_1, \dots, \succeq_n) \in \bigtimes_{i=1}^n Ext(\succeq_i^{+0-})$ such that $(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\}\big]$. Therefore, we just need to check if we can extend $\succeq^{+0-}$ in such a way that $\Gamma(i) \succeq_i C \cup \{i\}$ holds for every $i \in A$ and $C \in \Gamma \cup \emptyset$. Hence, we iterate all $i \in A$ and $C \in \Gamma \cup \{\emptyset\}$. There are four cases possible: (1) $\Gamma(i) \succ_i^{+0-} C \cup \{i\}$, (2) $\Gamma(i) \sim_i^{+0-} C \cup \{i\}$, (3) $\Gamma(i) \prec_i^{+0-} C \cup \{i\}$, or (4) the relation between $\Gamma(i)$ and $C \cup \{i\}$ is undecided. In cases (1) and (2), Nash stability is not violated, so the algorithm just continues with the next iteration. If $\Gamma(i) \prec_i^{+0-} C \cup \{i\}$, this is clearly violating Nash stability and "NO" is output. If the relation between $\Gamma(i)$ and $C \cup \{i\}$ is undecided, then it is possible to set $\Gamma(i) \succ_i C \cup \{i\}$ in the extension $\succeq_i$ of $\succeq_i^{+0-}$ such that Nash stability is not violated. Accordingly, the algorithm does nothing in this case and continues with the next iteration: We set $\Gamma(i) \succ_i C \cup \{i\}$ for all $C \in \Gamma \cup \{\emptyset\}$ where the relation between $\Gamma(i)$ and $C \cup \{i\}$ is undecided. If "NO" is not output at any moment then $\Gamma(i) \prec_i^{+0-} C \cup \{i\}$ is never the case and "YES" is output because Γ is possibly Nash stable. The outer for-loop (line 1) runs $|A| = n$ times and the inner for-loop (line 2) $|\Gamma \cup \{\emptyset\}| = k + 1 \leq n + 1$ times. The relation between $\Gamma(i)$ and $C \cup \{i\}$ (line 3) can be checked in polynomial time by Observation 4.1, so the whole algorithm works in polynomial time. □

Theorem 4.5. Possible-Individual-Stability-Verification *is in* P.

Proof. Algotithm 2 solves the problem in polynomial time.

Algorithm 2: PISV

```
Data: A FEN-hedonic game (A, (⊵_1^{+0-}, ..., ⊵_n^{+0-})) and a
      coalition structure Γ.
Result: "YES" if Γ is possibly individually stable; "NO"
        otherwise.
for i ∈ A do
    for C ∈ Γ ∪ {∅} do
        if Γ(i) ≺_i^{+0-} C ∪ {i} then
            found ⟵ false;
            for j ∈ C do
                if i ∈ A_j^- then
                    found ⟵ true;
            if ¬found then
                output "NO";
output "YES";
```

A coalition structure Γ is possibly individually stable if there is a profile $\succeq = (\succeq_1, \dots, \succeq_n) \in \bigtimes_{i=1}^n Ext(\succeq_i^{+0-})$ such that

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\}) \big[\Gamma(i) \succeq_i C \cup \{i\} \vee (\exists j \in C)[C \succ_j C \cup \{i\}]\big].$$

By Observation 4.2, this is equivalent to

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\} \vee (\exists j \in C)[i \in A_j^-]\big].$$

Therefore, we just need to check if we can extend $\succeq^{+0-}$ in such a way that the latter condition holds. Hence, we iterate all $i \in A$ and $C \in \Gamma \cup \{\emptyset\}$. First, we check if $\Gamma(i) \succeq_i C \cup \{i\}$ is not possibly true. This is the case if $\Gamma(i) \prec_i^{+0-} C \cup \{i\}$. If this does hold then $(\exists j \in C)[i \in A_j^-]$ has to be true in order for the whole equation to be true. And this is checked in lines 5 to 9 of the algorithm. If "NO" was not output during any iteration then the condition displayed above has to be true for at least one extended profile. Thus Γ is possibly individually stable and "YES" is output. Again, it is easy to see that the algorithm runs in polynomial time. ◻

Theorem 4.6. Possible-Contractually-Individual-Stability-Verification *is in* P.

Proof. Algorithm 3 solves the problem in polynomial time.

Algorithm 3: PCISV

Data: A FEN-hedonic game $(A, (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-}))$ and a coalition structure Γ.
Result: "YES" if Γ is possibly contractually individually stable; "NO" otherwise.

1 **for** $i \in A$ **do**
2   *skiprest* ⟵ false;
3   **for** $k \in \Gamma(i) \setminus \{i\}$ **do**
4   **if** $i \in A_k^+$ **then**
5   *skiprest* ⟵ true;
6   **if** ¬*skiprest* **then**
7   **for** $C \in \Gamma \cup \{\emptyset\}$ **do**
8   **if** $\Gamma(i) \prec_i^{+0-} C \cup \{i\}$ **then**
9   *found* ⟵ false;
10   **for** $j \in C$ **do**
11   **if** $i \in A_j^-$ **then**
12   *found* ⟵ true;
13   **if** ¬*found* **then**
14   output "NO";
15 output "YES";

A coalition structure Γ is possibly contractually individually stable if there is a profile $\succeq = (\succeq_1, \dots, \succeq_n) \in \bigtimes_{i=1}^n Ext(\succeq_i^{+0-})$ such that

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\} \lor$$
$$(\exists j \in C)[C \succ_j C \cup \{i\}] \lor (\exists k \in \Gamma(i))[\Gamma(i) \succ_k \Gamma(i) \setminus \{i\}]\big].$$

By Observations 4.2 and 4.3, this is equivalent to

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\} \lor$$
$$(\exists j \in C)[i \in A_j^-] \lor (\exists k \in \Gamma(i))[i \in A_k^+]\big],$$

which in turn is equivalent to

$$(\forall i \in A)\Big[(\exists k \in \Gamma(i))[i \in A_k^+] \lor$$
$$(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i C \cup \{i\} \lor (\exists j \in C)[i \in A_j^-]\big]\Big].$$

Hence, Algorithm 3 checks if this condition possibly holds and answers accordingly. Again, it is easy to see that the algorithm runs in polynomial time. ◻

4.4 The Existence Problems

We will now turn to the existence problems for individual stability and contractually individual stability. We first show that possible existence is easy for contractually individual stability. Afterwards, we will show that necessary existence is NP-complete for individual stability.

Theorem 4.7. Possible-Contractually-Individual-Stability-Existence *(for short,* PCISE*) is in* P.

Proof. There always exists a coalition structure that is possibly contractually individually stable. This can be shown by a simple proof.

Consider any FEN-hedonic game $(A, \trianglerighteq^{+0-})$ and assume, for the sake of contradiction, that all coalition structures $\Gamma \in \mathscr{C}_{(A, \trianglerighteq^{+0-})}$ are not possibly contractually individually stable. Then, starting with an arbitrary coalition structure Γ_1, there always has to be a coalition structure Γ_{i+1} which is preferred to Γ_i by at least one player, namely the player who wants to deviate to another coalition, and is weakly preferred by all other players. Since all these coalition structures have to be pairwise distinct, there have to be infinitely many coalition structures. This is a contradiction because $\mathscr{C}_{(A, \trianglerighteq^{+0-})}$ is always finite. Hence, the answer for the decision problem PCISE is always yes, so it trivially is in P. ◻

We will now show that deciding whether, given a FEN-hedonic game, there exists a necessarily individually stable coalition structure is NP-complete. To this end, Construction 4.8 is needed, and we briefly explain the ideas behind this construction. We will provide a polynomial-time many-one reduction from Necessary-Nash-Stability-Existence (for short, NNSE), which is NP-complete by a result of Lang et al. [17]. We take a FEN-hedonic game H that is an instance of NNSE and construct another FEN-hedonic game H', which is an instance of Necessary-Individual-Stability-Existence, such that there exists a necessarily individually stable coalition structure for H' if and only if there exists a necessarily Nash stable coalition structure for H.

In the upcoming construction, we define so-called *clone players* which have the same preferences as the original players (from H) but unlike the original players are not the enemy of any other player. By this trick we eliminate the possibility that other players can prevent the deviation of a clone player. Furthermore, we include so-called *structure players* to ensure that every necessarily individually stable coalition structure has to satisfy a certain form. Finally, so-called *friend* and *enemy players* help the structure players to fulfill their purpose.

Construction 4.8. *Let* $H = (A, \trianglerighteq^{+0-})$ *be a FEN-hedonic game, where* $A = \{1, \dots, n\}$, $\trianglerighteq^{+0-} = (\trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-})$, *and for each* $i \in A$, $\trianglerighteq_i^{+0-} = (\trianglerighteq_i^+ | A_i^0 | \trianglerighteq_i^-)$ *with* $\trianglerighteq_i^+$ *being the weak order over the set of* i*'s friends* A_i^+ *and* $\trianglerighteq_i^-$ *the weak order over the set of* i*'s enemies* A_i^-. *We now construct a FEN-hedonic game* H' *in polynomial time. Let* $H' = (A', \trianglerighteq^{+0-\prime})$ *be a FEN-hedonic game with*

$$A' = A \cup Clone \cup Structure \cup Friend_A \cup Friend_B \cup Enemy,$$

Clone = $\{c_1, ..., c_n\}$, *Structure* = $\{s_1, ..., s_n\}$, *Friend*$_A$ = $\{a_1, ..., a_n\}$, *Friend*$_B$ = $\{b_1, ..., b_n\}$, *and Enemy* = $\{e_1, ..., e_n\}$. *For* $1 \le i \le n$, *let*

$$\trianglerighteq_i^{+0-\prime} = \left(\,\middle|\, A' \setminus \{i\} \,\middle|\, \right),$$
$$\trianglerighteq_{c_i}^{+0-\prime} = \left(\trianglerighteq_i^+ \,\middle|\, A' \setminus (A_i^+ \cup A_i^- \cup \{c_i\}) \,\middle|\, \trianglerighteq_i^-\right),$$
$$\trianglerighteq_{s_i}^{+0-\prime} = \left(i \sim_{s_i} c_i \triangleright_{s_i} a_i \sim_{s_i} b_i \,\middle|\, A' \setminus \{i, c_i, s_i, a_i, b_i, e_i\} \,\middle|\, e_i\right),$$
$$\trianglerighteq_{a_i}^{+0-\prime} = \left(b_i \,\middle|\, A' \setminus \{a_i, b_i\} \,\middle|\, \right),$$
$$\trianglerighteq_{b_i}^{+0-\prime} = \left(e_i \,\middle|\, A' \setminus \{b_i, e_i\} \,\middle|\, \right),$$
$$\trianglerighteq_{e_i}^{+0-\prime} = \left(\,\middle|\, A' \setminus (A \cup \{e_i\}) \,\middle|\, 1 \sim_{e_i} \cdots \sim_{e_i} n\right).$$

This construction can obviously be done in polynomial time.

Theorem 4.9. Necessary-Individual-Stability-Existence *(for short,* NISE*) is* NP*-complete.*

Proof. To see that NISE is in NP, let the FEN-hedonic game $H = (A, \trianglerighteq^{+0-})$ be a given instance. We nondeterministically guess a coalition structure $\Gamma \in \mathscr{C}_{(A,\trianglerighteq^{+0-})}$ that might be a solution for this instance. Then we check whether Γ indeed is a solution, i.e., whether Γ necessarily satisfies individual stability. This is possible in polynomial time by a result of Lang et al. [17].

We show NP-hardness of NISE by providing a polynomial-time many-one reduction from NNSE. To do so, we consider the FEN-hedonic games H and H' as defined in Construction 4.8, where H is considered to be an instance of NNSE and H' an instance of NISE. Obviously, the construction of H' can be done in polynomial time. We will now show that

$$H \in \text{NNSE} \iff H' \in \text{NISE}.$$

From left to right, assume that $H \in$ NNSE. This means that there exists a coalition structure $\Gamma \in \mathscr{C}_{(A,\trianglerighteq^{+0-})}$ such that for every extended profile $P = (\succeq_1, ..., \succeq_n) \in \bigtimes_{i=1}^n \text{Ext}(\succeq_i^{+0-})$, it holds that $(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\left[\Gamma(i) \succeq_i C \cup \{i\}\right]$. Since this relation holds for every extended profile, we have

$$(\forall i \in A)(\forall C \in \Gamma \cup \{\emptyset\})\left[\Gamma(i) \succeq_i^{+0-} C \cup \{i\}\right]. \tag{1}$$

We will now show that $H' \in$ NISE, i.e., that there is a coalition structure $\Gamma' \in \mathscr{C}_{(A',\trianglerighteq^{+0-\prime})}$ such that

$$(\forall i \in A')(\forall C' \in \Gamma' \cup \{\emptyset\})\big[\Gamma'(i) \succeq_i^{+0-\prime} C' \cup \{i\} \vee (\exists j \in C')[i \in A_j^{-\prime}]\big]. \tag{2}$$

We consider the coalition structure $\Gamma' = \{D_C, E_C \mid C \in \Gamma\}$ with $D_C = \{j, c_j, s_j \mid j \in C\}$ and $E_C = \{a_j, b_j, e_j \mid j \in C\}$. It then holds for all $i \in A$ that $\Gamma'(i) = \Gamma'(c_i) = \Gamma'(s_i) = \{j, c_j, s_j \mid j \in \Gamma(i)\}$ and $\Gamma'(a_i) = \Gamma'(b_i) = \Gamma'(e_i) = \{a_j, b_j, e_j \mid j \in \Gamma(i)\}$.

We will now show that (2) holds for all players in $A' = A \cup$ *Clone* $\cup$ *Structure* $\cup$ *Friend*$_A$ $\cup$ *Friend*$_B$ $\cup$ *Enemy*. First, consider the players $i \in A$. It holds that $\Gamma'(i) \succeq_i^{+0-\prime} C' \cup \{i\}$ for all $C' \in \Gamma' \cup \{\emptyset\}$ because i doesn't have any friends or enemies and therefore is indifferent between any two coalitions. Hence, (2) is satisfied for all $i \in A$ and all $C' \in \Gamma' \cup \{\emptyset\}$.

Next, consider the clone players c_i. For all $D_C = \{j, c_j, s_j \mid j \in C\} \in \Gamma'$, it holds that $\Gamma'(c_i) = \{j, c_j, s_j \mid j \in \Gamma(i)\} \succeq_{c_i}^{+0-\prime} \{j, c_j, s_j \mid j \in C\} \cup \{c_i\} = D_C \cup \{c_i\}$ if and only if $\Gamma(i) \cup \{c_i\} \succeq_{c_i}^{+0-\prime} C \cup \{c_i\}$ because c_i is neutral to all other clone players and all structure players. This in turn is equivalent to $\Gamma(i) \setminus \{i\} \cup \{c_i\} \succeq_{c_i}^{+0-\prime} C \cup \{c_i\}$ because c_i is neutral to i. Since c_i has the same friends, order over friends, enemies, and order over enemies as player i has in H, the last preference relation is equivalent to $\Gamma(i) \succeq_i^{+0-} C \cup \{i\}$, which holds by assumption, see Equation (1). Hence, (2) is satisfied for all c_i and $D_C \in \Gamma'$.

Now, consider all $E_C = \{a_j, b_j, e_j \mid j \in C\} \in \Gamma'$. Again, $\Gamma'(c_i) = \{j, c_j, s_j \mid j \in \Gamma(i)\} \succeq_{c_i}^{+0-\prime} \{a_j, b_j, e_j \mid j \in C\} \cup \{c_i\} = E_C \cup \{c_i\}$ is equivalent to $\Gamma(i) \setminus \{i\} \cup \{c_i\} \succeq_{c_i}^{+0-\prime} \{c_i\}$ by removing all neutral players. This is equivalent to $\Gamma(i) \succeq_i^{+0-} \{i\}$, which holds by Equation (1). It is easy to see that the same argumentation is possible for the empty coalition $\emptyset$. Hence, (2) is satisfied for all $c_i \in$ *Clone* and all $C' \in \Gamma' \cup \{\emptyset\}$.

We now turn to the structure players s_i. $\Gamma'(s_i) = \{j, c_j, s_j \mid j \in \Gamma(i)\}$ contains s_i's two best friends, i and c_i, and no enemy. Every other coalition can only contain at most two other friends of s_i, namely a_i and b_i, which are ranked lower than i and c_i. Hence, s_i prefers $\Gamma'(s_i)$ to every other coalition in $\Gamma' \cup \{\emptyset\}$ and (2) is satisfied for all $s_i \in$ *Structure*.

For all $a_i \in$ *Friend*$_A$, it holds that $\Gamma'(a_i) = \{a_j, b_j, e_j \mid j \in \Gamma(i)\} \succeq_{a_i}^{+0-\prime} C' \cup \{a_i\}$ for every coalition $C' \in \Gamma' \cup \{\emptyset\}$ because $\Gamma'(a_i)$ contains b_i (a_i's only friend) and no enemies. Therefore, (2) holds for all $a_i \in$ *Friend*$_A$. Analogously, (2) also holds for all $b_i \in$ *Friend*$_B$. Finally, consider the enemy players e_i. Since e_i has no friends and $\Gamma'(e_i) = \{a_j, b_j, e_j \mid j \in \Gamma(i)\}$ doesn't contain any enemies of e_i, it holds that $\Gamma'(e_i) \succeq_{e_i}^{+0-\prime} C' \cup \{e_i\}$ for every $C' \in \Gamma' \cup \{\emptyset\}$. So, (2) also holds for all $e_i \in$ *Enemy*.

Thus (2) is satisfied for all players in A' and all $C' \in \Gamma' \cup \{\emptyset\}$, which means that Γ' is necessarily individually stable for H' and $H' \in$ NISE.

From right to left, assume that $H' \in$ NISE. Then, there is a $\Gamma' \in \mathscr{C}_{(A',\trianglerighteq^{+0-\prime})}$ such that (2) holds. Consider such a coalition structure Γ'. We will now show that Γ' necessarily needs to be of the following form because (2) couldn't hold otherwise:

$$\Gamma' = \{D_C \mid C \in \Gamma\} \cup \{E_C \mid C \in \Delta\} \text{ for some partitions } \Gamma \text{ and } \Delta \text{ of } A, \text{ where } D_C = \{j, c_j, s_j \mid j \in C\} \text{ and } E_C = \{a_j, b_j, e_j \mid j \in C\}.$$

Now consider any $i \in A$. First, note that none of c_i, s_i, a_i, and b_i are the enemy of any other player, which is why the first part of (2) has to hold for them, i.e., $\Gamma'(p) \succeq_p^{+0-\prime} C' \cup \{p\}$ for $p \in \{c_i, s_i, a_i, b_i\}$ and all $C' \in \Gamma' \cup \{\emptyset\}$. Furthermore, for player e_i and coalition $C' = \emptyset$, we have $\Gamma'(e_i) \succeq_{e_i}^{+0-\prime} \{e_i\}$ because there is no player in $\emptyset$ who could see e_i as an enemy.

Since a_i doesn't want to deviate from $\Gamma'(a_i)$, a_i has to be together with b_i because b_i is a_i's only friend and a_i has no enemies. Otherwise, a_i would always prefer the coalition containing b_i. For an analogous reason, b_i has to be together with e_i. Furthermore, i can't be in the same coalition as e_i because i is an enemy of e_i and e_i would rather be alone otherwise. Hence, we already know that $\{a_i, b_i, e_i\} \subseteq E$ and $\{i\} \subseteq D$ for some $D, E \in \Gamma'$ with $D \neq E$.

There remain ten cases for the allocation of s_i and c_i. By excluding nine of these cases, it will follow that $s_i, c_i \in D$. Recall that $\Gamma'(s_i) \succeq_{s_i}^{+0-\prime} C' \cup \{s_i\}$ holds for all $C' \in \Gamma' \cup \{\emptyset\}$. All of the nine cases presented in the following imply that this is not true for at least one coalition $C' \in \Gamma' \cup \{\emptyset\}$. Hence, they can't hold. For an overview of the cases, see Table 2.

Table 2: Ten cases for the allocation of s_i and c_i and why nine of them cannot hold.

	$s_i \in E$	$s_i \in D$	$s_i \in F$
$c_i \in E$	$E, D \cup \{s_i\}$ incomparable	$D, E \cup \{s_i\}$ incomparable	$F, E \cup \{s_i\}$ incomparable
$c_i \in D$	$E \prec_{s_i}^{+0-\prime} D \cup \{s_i\}$	holds	$F, E \cup \{s_i\}$ incomparable
$c_i \in F$	$E, D \cup \{s_i\}$ incomparable	$D, E \cup \{s_i\}$ incomparable	$F, E \cup \{s_i\}$ incomparable
$c_i \in G$	—	—	$F, E \cup \{s_i\}$ incomparable

- If $s_i, c_i \in E$ (i.e., $\{c_i, s_i, a_i, b_i, e_i\} \subseteq E$ and $\{i\} \subseteq D$), then $\Gamma'(s_i) = E \not\succeq_{s_i}^{+0-\prime} D \cup \{s_i\}$. E and $D \cup \{s_i\}$ are incomparable with respect to $\succeq_{s_i}^{+0-\prime}$ because E contains more friends but also more enemies than $D \cup \{s_i\}$.
- If $s_i \in E$ and $c_i \in D$ (i.e., $\{s_i, a_i, b_i, e_i\} \subseteq E$ and $\{i, c_i\} \subseteq D$), then $\Gamma'(s_i) = E \prec_{s_i}^{+0-\prime} D \cup \{s_i\}$ because $D \cup \{s_i\}$ contains the same number of friends as E, but better friends than E, and no enemies.
- If $s_i \in E$ and $c_i \in F$ for an $F \in \Gamma'$ with $D \neq F \neq E$ (i.e., $\{s_i, a_i, b_i, e_i\} \subseteq E$, $\{i\} \subseteq D$, and $\{c_i\} \subseteq F$), then $\Gamma'(s_i) = E$ and $D \cup \{s_i\}$ are incomparable again because E contains more friends but also more enemies than $D \cup \{s_i\}$. Hence, $\Gamma'(s_i) \not\succeq_{s_i}^{+0-\prime} D \cup \{s_i\}$.
- If $s_i \in F$ for an $F \in \Gamma'$ with $D \neq F \neq E$ (i.e., $\{a_i, b_i, e_i\} \subseteq E$, $\{i\} \subseteq D$, and $\{s_i\} \subseteq F$), then there remain four cases for c_i: $c_i \in E$, $c_i \in D$, $c_i \in F$, or $c_i \in G$ for a $G \in \Gamma'$ with $G \notin \{D, E, F\}$. No matter where c_i is, $\Gamma'(s_i) = F$ and $E \cup \{s_i\}$ are incomparable with respect to $\succeq_{s_i}^{+0-\prime}$ because $E \cup \{s_i\}$ contains more friends but also more enemies than F.
- If $s_i \in D$ and $c_i \in E$ (i.e., $\{c_i, a_i, b_i, e_i\} \subseteq E$ and $\{i, s_i\} \subseteq D$), then s_i is undecided concerning $\Gamma'(s_i) = D$ and $E \cup \{s_i\}$ because $E \cup \{s_i\}$ contains more friends but also more enemies than D.
- If $s_i \in D$ and $c_i \in F$ for an $F \in \Gamma'$ with $D \neq F \neq E$ (i.e., $\{a_i, b_i, e_i\} \subseteq E$, $\{i, s_i\} \subseteq D$, and $\{c_i\} \subseteq F$), then s_i again is undecided concerning $\Gamma'(s_i) = D$ and $E \cup \{s_i\}$.

The only remaining case is $s_i, c_i \in D$ (i.e., $\{a_i, b_i, e_i\} \subseteq E$ and $\{i, c_i, s_i\} \subseteq D$). Note that this case indeed fulfills $\Gamma'(s_i) \succeq_{s_i}^{+0-\prime} C' \cup \{s_i\}$ for all $C' \in \Gamma' \cup \{\emptyset\}$. Hence, for every $i \in A$, we have $\{a_i, b_i, e_i\} \subseteq E_i$ and $\{i, c_i, s_i\} \subseteq D_i$ for some $D_i, E_i \in \Gamma'$ with $D_i \neq E_i$. It furthermore holds for any $i, j \in A$ that $E_i \neq D_j$. Otherwise, we had $E_i = D_j \supseteq \{a_i, b_i, e_i, j, c_j, s_j\}$. Since j is an enemy of e_i, e_i would like to deviate to the empty coalition which is a contradiction to the assumption, see Equation (2). It follows that Γ' has the form presented above.

Finally, consider the clone players $c_i \in \mathit{Clone}$. Equation (2) also holds for c_i, i.e., we have

$$(\forall C' \in \Gamma' \cup \{\emptyset\})\big[\Gamma'(c_i) \succeq_{c_i}^{+0-\prime} C' \cup \{c_i\} \vee (\exists x \in C')[c_i \in A_x^{-\prime}]\big].$$

Since c_i is not the enemy of any other player, i.e., $c_i \notin A_x^{-\prime}$ for all $x \in A'$, it follows that

$$(\forall C' \in \Gamma' \cup \{\emptyset\})\big[\Gamma'(c_i) \succeq_{c_i}^{+0-\prime} C' \cup \{c_i\}\big]. \tag{3}$$

Recall that $\Gamma' = \{D_C \mid C \in \Gamma\} \cup \{E_C \mid C \in \Delta\}$ for some partitions Γ and Δ of A with $D_C = \{j, c_j, s_j \mid j \in C\}$ and $E_C = \{a_j, b_j, e_j \mid j \in C\}$. Furthermore, note that $\Gamma'(c_i) = D_{\Gamma(i)}$ and let $D_\emptyset = \emptyset$.

Equation (3) in particular holds for all $C' = D_C \in \Gamma' \cup \{\emptyset\}$ with $C \in \Gamma \cup \{\emptyset\}$. Hence, we have

$$(\forall C \in \Gamma \cup \{\emptyset\})\big[D_{\Gamma(i)} \succeq_{c_i}^{+0-\prime} D_C \cup \{c_i\}\big]. \tag{4}$$

Because c_i is neutral to all players $x \in A'$ with $x \notin A, x \neq c_i$, we can remove all these players from (4). With $D_C \cap A = C$, we get $(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \cup \{c_i\} \succeq_{c_i}^{+0-\prime} C \cup \{c_i\}\big]$. c_i is also neutral to i. Hence, we can remove i on the left-hand side and get

$$(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \setminus \{i\} \cup \{c_i\} \succeq_{c_i}^{+0-\prime} C \cup \{c_i\}\big]. \tag{5}$$

Finally, c_i has the same friends, order over friends, enemies, and order over enemies as player i has in H. Therefore, (5) is equivalent to $(\forall C \in \Gamma \cup \{\emptyset\})\big[\Gamma(i) \succeq_i^{+0-} C \cup \{i\}\big]$. Thus the coalition structure Γ is necessarily Nash stable for H and $H \in$ NNSE. ◻

5 CONCLUSIONS AND FUTURE WORK

We have studied the computational complexity of various stability problems based on single-player deviations in FEN-hedonic games, thus solving some related questions left open by Lang et al. [17]. An overview of our complexity results is given in Table 1.

For future work we propose to study the remaining two cases of the concepts concerning player deviation. Furthermore, we are interested in the computational complexity of stability problems that are based on groups of players deviating from their coalitions in FEN-hedonic games or on the comparison of the given coalition structure Γ with another possible coalition structure Δ. Among the former are *core stability* and *strict core stability*, while among the latter are *Pareto optimality*, *popularity*, and *strict popularity*. Lang et al. [17] established some initial results on the complexity of the related problems but, again, they also left a number of cases open.

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