Fair Resource Sharing and Dorm Assignment

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ABSTRACT

In this work, we study the fair resource sharing problem, where a set of resources needs to be shared by a set of agents. Each agent is unit-demand and each resource can serve a limited number of agents. The agents have (heterogeneous) preferences for the resources, and preferences for other agents with whom they share the resources. Our definition of fairness is mainly captured by *envy-freeness*. Due to the fact that an envy-free assignment may not exist even in simple settings, we propose a way to relax the definition: *Pareto envy-freeness*, where an assignment is Pareto envy-free if for any two agents i and j, agent i does not envy agent j for her received resource or the set of agents she shares the resource with. We study to what extent Pareto envy-free assignments exist. Particularly, we are interested in a typical model, *dorm assignment problem*, where a number of students need to be accommodated to the dorms with the same capacity and the students' preferences for dorm-mates are binary. We show that when the capacities of the dorms are 2, a Pareto envy-free assignment always exists and can be found in polynomial time; however, if the capacities increase to 3, Pareto envy-freeness cannot be guaranteed any more.

KEYWORDS

fair division, resource sharing, dorm assignment

1 INTRODUCTION

We consider a general resource sharing framework in this work, where a set of heterogeneous resources needs to be fairly shared among a set of unit-demand agents. Besides the preferences that each agent has for the resources, the preferences for other agents with whom she shares the resource are also taken into consideration. Each resource has a fixed capacity, which is the maximum number of agents it can serve simultaneously. The fairness concept we focus on in this work is the *envy-free* assignments [23], where every agent does not want to exchange with any other agent. Fair resource sharing happens a lot in reality, such as dorm assignment, project collaborating, public transportation, and more. Particularly, *dorm assignment* is a canonical problem faced by many schools,

where students need to be accommodated to different dorms. Different students may have different preferences for the dorms based on each dorm's size, location, furniture, cooking condition, and others. Moreover, each student also prefers sharing a dorm with her/his friends, lab mates, or someone with whom she/he has similar timetable. Then our problem is to assign the students to the dorms in an envy-free way.

Arguably, dorm assignment problem is originated from the *stable roommate problem* [25, 31], where $2m$ students need to be assigned to m rooms in a stable way. An assignment is *stable* if there is no pair of agents who want to exchange their rooms. There exist other characterizations of stability in various matching settings, like exchange stability [15] and popular matchings [10]. In this line of work, people do not consider the agents' value for the rooms. However, in real-word scenarios, due to the heterogeneity of the rooms, the agents' values toward rooms are non-negligible.

Dorm assignment problem has also been studied in the market settings, such as [16, 29, 35]. In a market, the dorms have (different) prices (or rent) that needs to be shared between the agents assigned to them. Here each agent's utility (i.e., net value) for an assignment is the difference between her value and her rent share. As shown by [35], if each room has capacity 1, then there is an assignment and a price profile, such that the matching between the agents and rooms is envy-free. When the capacities of the dorms increase to 2, it is shown in [16] that although individual envy-freeness cannot be guaranteed, a direct application of the result in [35] ensures *room envy-freeness* by treating every pair of dormmates as a whole. That is, by moving the agents in a dorm together to another dorm, their total utility cannot be increased.

Our work is different from these lines of researches in two perspectives. First, monetary transfers are not allowed in our model. This is because discriminations are strictly prohibited in many settings, like political elections, organ donations, as well as school affairs (including dorm assignments). Accordingly, money cannot be used as a medium of compensation. Second, different from [16], we care about the utility of each individual, even when the capacities of the dorms are greater than one. Therefore, in this work, we aim at characterizing to what extent the individual fairness can be guaranteed when monetary transfers are not permitted.

1.1 Main Results

Our contributions are three-fold. First, we adapt envy-freeness, as well as proportionality, to the resource sharing problem and study their properties. Interestingly, unlike the classic fair resource allocation domain, these two solution concepts in general are not compatible in our model. Instead, an envy-free assignment can guarantee a 2-approximation of proportionality if the capacity of each resource is at least 2, and the bound is tight.

Second, as envy-freeness is hard to be satisfied, we relax it to *Pareto envy-freeness (PEF)*. Informally, an assignment is called PEF if for any two agents i and j, either i does not envy j with respect to her resource, or with respect to the agents with whom j shares her resource. Similarly, we also define *Pareto proportionality (PPROP)*, which ensures that for any agent i, either she is assigned to one of her top 50% resources or shares a resource with at least a proportional number of friends. We show that for the dorm assignment problem with capacity 2, a PEF assignment must also be PPROP.

Finally, we move to study the existence of PEF assignments. We show that for the dorm assignment problem with capacity 2, a PEF assignment (thus PPROP) is guaranteed to exist and can be found in polynomial time. However, if the capacity for the dorms is increased to 3, PEF assignments cannot be guaranteed any more.

We regard our main technical contribution as the existence of PEF assignments for the dorm sharing problem with capacity 2. We first use Gallai-Edmonds Theorem [33] to identify a Tutte set and decompose all maximum matchings. For the agents that can be matched by any maximum matching excluding the ones in Tutte set, we assign them to dorms such that they do not envy other agents' dorm mates. For the other agents, by utilizing Hall's Theorem [27], we show that they can be paired such that either both agents have similar preferences for dorms or they are friends with each other. Thus all agents can be assigned to dorms in a PEF way.

1.2 Other Related Works

Fair allocation has been extensively studied in many settings, such as cake cutting [7, 12, 20], indivisible goods allocation [13, 14, 32], and indivisible chores allocation [6, 8, 28]. Although envy-free and proportional allocations always exist in cake cutting problem, they may not exist when the items are indivisible. Accordingly, people study relaxed but more realistic fairness notions, such as envy-freeness up to one item [32] where existence is guaranteed, and maximin share fairness [13] where constant approximations are guaranteed. Our work also falls under the domain of relaxing fairness notions toward existence; however, in our model the resources need to be shared instead of being partitioned among agents.

Our work is also related to *fair rent division* [1, 3, 4, 24], which studies fair ways to assign rooms to agents and divide the rent among them. With monetary transfers, an envy-free solution is always guaranteed, thus works like [3, 24] focus on identifying and finding the "best" envy-free solutions. Two major differences between our work and theirs are (1) in our model, monetary transfers are not allowed, and (2) besides the values for the rooms, we also consider each agent's external values for her roommates.

Another related topic studied in the literature is the *fair house assignment problem* initiated by [30, 34]. Some recent achievements can be found in [2, 9, 26]. The models considered in those papers are special cases of our paper where the capacities of the resources are one and the external values between the agents are zero.

Some game theoretic models are also related to our work. For example, in *hedonic games* [5, 11, 22] and *group activity selection problem* [18, 19, 21], the agents form coalitions, and have preferences over the coalitions they might join. The preference of a coalition is determined by the members in the coalition or its size for anonymous games. Clearly, hedonic games do not consider the

values of the coalitions (i.e., "resources"), and group activity selection problem mainly studies the approval preferences. Another fundamental difference to our model is that in these game-theoretic settings, the agents always have an option to quit the game while in our problems, all the agents need to be served. Moreover, in their models, the results are mainly focused on the stable assignments, while the fairness is largely overlooked.

2 RESOURCE SHARING FRAMEWORK

In the resource sharing problem, a set of resources M needs to be assigned to a set of agents N. The number of agents is n and the size of the resource is m, i.e., $|N| = n$ and $|M| = m$. Each resource $j \in M$ has a capacity constraint $c_j \geq 1$, which means the maximum number of agents resource j can serve simultaneously is at most c_j. Throughout the paper, we assume that the supply meets the demand, that is, $n = \sum_{j \in M} c_j$. Each agent $i \in N$ has a *value* $v_{ij} \geq 0$ for each resource $j \in M$ and each agent is unit-demand; that is, for any subset $S \subseteq M$ of resources, agent i's value for the subset is the maximum value in the subset, i.e., $v_i(S) = \max_{j \in S} v_{ij}$. We abuse the notions a bit by using $v = (v_1, \cdots, v_n)$ to denote the valuation profile of the agents, where $v_i = (v_{i1}, \ldots, v_{im})$.

A feasible assignment (or allocation) requires that every agent is assigned one resource and each resource j is assigned to (at most) c_j agents. Formally, denoting an assignment by $x = (x_{ij})_{i \in N, j \in M}$ where $x_{ij} \in \{0, 1\}$, X is feasible if for all $i \in N, j \in M$, $\sum_{i \in N} x_{ij} \leq c_j$ and $\sum_{j \in M} x_{ij} = 1$. Given an assignment X, let $X_j = \{i \in N | x_{ij} = 1\}$ be the set of agents that are assigned to resource j. Finally, let $X = (X_1, \ldots, X_m)$.

Each agent $i \in N$ has a preference over other agents $N \setminus \{i\}$, indicating her willingness for whom she shares a resource with, which is called *external values* or *externalities*. Formally, agent i gains external value $e_{ii'} \in \mathbb{R}$ if agent i shares a resource with agnet i'. Except in Section 6, we assume $e_{ii'} \geq 0$. Let $e = (e_i)_{i \in N}$ be the externality profile where $e_i = (e_{ii'})_{i' \neq i}$. For convenience, we also denote $e_i = (e_{ii'})_{i' \in N}$ where $e_{ii} = 0$. We use $I = (N, M, v, e)$ to denote an instance of the resource sharing problem.

For any assignment X, let j_i be the resource that is assigned to agent i, i.e., $x_{ij_i} = 1$. Then agent i's value for the resource given assignment X is denoted by

$$v_i(X) = \sum_{j \in M} v_{ij} \cdot x_{ij} = v_{ij_i}$$

and her external value is denoted by

$$e_i(X) = \sum_{j \in M} \sum_{i' \neq i} e_{ii'} \cdot x_{i'j} \cdot x_{ij} = \sum_{i' \in X_{j_i} \setminus \{i\}} e_{ii'}.$$

It is assumed that the agent's utility is additive over the values for the resources and the externalities. Thus the total utility of agent i for assignment X is defined by the sum of the values, i.e.,

$$u_i(X) = v_i(X) + e_i(X) = v_{ij_i} + \sum_{i' \in X_{j_i} \setminus \{i\}} e_{ii'}.$$

When it is clear that $i \in X_{j_i}$, we also denote $v_i(X_{j_i}) = v_i(X)$, $e_i(X_{j_i}) = e_i(X)$, and $u_i(X_{j_i}) = u_i(X)$.

A typical case of the resource sharing problem is called the *canonical model*, when all the following requirements are satisfied.

(1) all dorms have the same capacity $c \geq 1$;

(2) the external values are binary, i.e., $e_{ii'} \in \{0, 1\}$ for any i, i';

(3) the external values are symmetric, i.e. $e_{ii'} = e_{i'i}$ for any i, i'.

Thus in the canonical model, the externalities among the agents can be described as an undirected and unweighted graph $G = (N, E)$ where each node represents an agent and every edge $e = (i, i') \in E$ between two agents i and i' means they are *friends* with each other and would like to share a resource together. We refer such a graph as the *externality graph*. Particularly, when capacity $c = 2$, the model is referred as *dorm assignment problem*, where m dorms need to be shared among $n = 2m$ students.

3 ENVY-FREENESS AND PROPORTIONALITY

Arguably, in fair allocation literature, the most widely studied fairness criteria are *envy-freeness* and *proportionality*. In this section, we first adapt these two definitions to our framework.

3.1 Definitions

Roughly, an envy-free assignment guarantees each agent does not want to exchange her assignment with any other agent.

Definition 3.1. An assignment X is called *envy-free (EF)* if $u_i(X) \geq u_i(X^{ii'})$ for every pair of agents i and i', where $X^{ii'}$ is the resulting assignment by exchanging the resources assigned to agents i and i'.

Next we adapt proportionality to the resource sharing problem. Intuitively, we can view the proportional value as the expected value of the agents if we randomly assign them to the resources. This concept coincides with the traditional definition of proportionality in the classical fair allocation setting. In the resource sharing problem, an agent i's average value for the resources is $\frac{1}{m} \sum_{j \in M} v_{ij}$. For the average external value, note that the average externality for a single agent is $\frac{1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'}$, and the average number of agents sharing a resource is $\bar{c} = \frac{1}{m} \sum_j c_j - 1$. Accordingly, agent i's proportional value in our model is defined as

$$\text{PROP}_i = \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{\bar{c} - 1}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'}.$$

Definition 3.2. An assignment X is called α-*approximate proportional (α-PROP)* if $u_i(X) \geq \alpha \cdot \text{PROP}_i$ for every agent $i \in N$. If $\alpha = 1$, X is called *proportional (PROP)*.

3.2 The Relationship between EF and PROP

It is well known that in the classic resource allocation setting, envy-freeness implies proportionality. But an interesting observation for the resource sharing problem is that they are not compatible.

First it is not hard to see that a PROP assignment does not have guarantee for EFness as PROPness only cares the average utility instead of the exact assignment of the others.

Next we note that if there is a resource with capacity 1, an EF assignment does not have any guarantee for PROPness either. Consider the following instance with 2 resources and 4 agents. Let the capacities of the two resources be $c_1 = 1, c_2 = 3$. Suppose agent 1 has value 0 for both resources, and external value 0 for agents 2 and 3, and external value $T \gg 0$ for agent 4. Accordingly, allocating agent 4 to resource 1 and agents 1, 2, 3 to resource 2 is envy-free for agent 1. To see this, note that the utility of agent 1 under such an

assignment is 0 and by exchanging with agent 4, agent 1's utility cannot be increased. However,

$$\text{PROP}_1 = \frac{\bar{c} - 1}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'} = \frac{T}{3} \gg 0.$$

Thus we conclude that an EF assignment cannot guarantee any approximation of PROPness if some resource's capacity is 1.

Fortunately, if all resources have capacities at least 2, we are able to show that envy-freeness implies $(1 - \frac{1}{c_{\min}})$-approximate proportionality, where $c_{\min} = \min_j c_j$ and $1 - \frac{1}{c_{\min}} \geq \frac{1}{2}$.

THEOREM 3.3. *When $c_j \geq 2$ for all j, any EF assignment is $(1 - \frac{1}{c_{\min}})$-approximate PROP.*

PROOF. Let X be an arbitrary EF assignment and i be an arbitrary agent, where j_i is the resource that is assigned to i. Then

$$u_i(X) = v_{ij_i} + \sum_{i' \in X_{j_i} \setminus \{i\}} e_{ii'} \qquad (1)$$

First, for any $j \neq j_i$ and any $i' \in X_j$,

$$u_i(X) \geq u_i(X^{ii}) = v_{ij} + \sum_{l \in X_j \setminus \{i'\}} e_{il}. \qquad (2)$$

Sum Inequality 2 for all $i' \in X_j$ and divide by c_j,

$$u_i(X) \geq v_{ij} + \sum_{l \in X_j} e_{ik} - \frac{1}{c_j} \cdot \sum_{l \in X_j} e_{il}$$

$$\geq v_{ij} + \frac{c_j - 1}{c_j} \sum_{l \in X_j} e_{il} \geq v_{ij} + \frac{c_{\min} - 1}{c_{\min}} \sum_{l \in X_j} e_{il}, \quad (3)$$

where the last inequality is because $c_j \geq c_{\min}$ for all j. Next, sum all Inequalities 1 and 3 for each resource j and divide by m,

$$u_i(X) \geq \frac{1}{m} \sum_j v_{ij} + \frac{c_{\min} - 1}{m \cdot c_{\min}} \sum_j \sum_{l \in X_j} e_{il}. \qquad (4)$$

On the other hand, by the definition of proportionality,

$$\text{PROP}_i = \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{\bar{c} - 1}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$

$$= \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{m} \frac{\sum_j (c_j - 1)}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$

$$\leq \frac{1}{m} \sum_{j \in M} v_{ij} + \frac{1}{m} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$

$$\leq \frac{c_{\min}}{c_{\min} - 1} \cdot u_i(X),$$

where the first inequality is because $\sum_j (c_j - 1) \leq n - 1$.

Finally, since the above analysis holds for every agent i, X is $(1 - \frac{1}{c_{\min}})$-approximate PROP. □

Actually, the analysis in Theorem 3.3 is asymptotically tight using the following example.

Example 3.4. Let $c \geq 2$ be any integer and $m \gg c$ be a sufficiently large integer. Consider the following instance, where m resources need to be assigned to $n = (c-1)m^2 + cm - c + 1$ agents. The capacity of the first resource is $c_1 = (c - 1)m^2 + 1$ and the capacity of any resource $j > 1$ is $c_j = c$. Note that $\sum_{j=1}^{m} c_j = (c-1)m^2 + 1 + c(m-1) =$

n. Let $X = (X_1, \cdots, X_m)$ be an allocation where agent 1 is assigned to resource 1, i.e., $1 \in X_1$. Let agent 1's external values be $e_{1k} = 1$ if $k \in X_1$; and $e_{1k} = m^2$ otherwise. Assume agent 1 does not have values for the resources.

As $u_1(X) = \sum_{k \in X_1 \setminus \{1\}} e_{1k} = (c-1)m^2$, which means that by exchanging agent 1 with any agent in resource $j > 1$, agent 1's utility cannot be increased. Hence allocation X is envy-free. Then we compute agent 1's proportional utility. By the design of the instance,

$$\bar{c} = \frac{1}{m} \sum_j c_j - 1 = \frac{(c-1)m^2 + 1 + c(m-1)}{m} - 1$$
$$= \frac{(c-1)m^2 + (c-1)m - c + 1}{m},$$

and

$$\sum_{k \in N \setminus \{1\}} e_{1k} = (c-1)m^2 + cm^2 \cdot (m-1) = (cm-1)m^2.$$

Thus,

$$\lim_{m \to \infty} \frac{\mathrm{PROP}_i}{u_1(X)} = \lim_{m \to \infty} \frac{\bar{c} - 1}{(n-1)(c-1)m^2} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$
$$= \lim_{m \to \infty} \frac{((c-1)m^2 + (c-1)m - c + 1) \cdot (cm-1)m^2}{m \cdot ((c-1)m^2 + cm - c) \cdot (c-1)m^2}$$
$$= \frac{c}{c-1}.$$

We further note that the an EF assignment may not be PROP even if all the resources have the same capacity. Due to space limit, we omit such an example here, but in the following we show that the approximation ratio for PROPness can be improved.

COROLLARY 3.5. *When all agents have the same capacity $c \geq 2$ for all j, any EF assignment is $(1 - \frac{1}{n})$-approximate PROP.*

PROOF. Essentially, the proof shares the same idea with Lemma 3.3 and we only mention the differences in the following. Let X be an arbitrary envy-free allocation, and i be an arbitrary agent. Similar to Inequality 3, we have

$$u_i(X) \geq v_{ij} + \sum_{l \in X_j} e_{ik} - \frac{1}{c} \cdot \sum_{l \in X_j} e_{il} \geq v_{ij} + \frac{c-1}{c} \sum_{l \in X_j} e_{il}.$$

Summing over above inequalities for each resource j and divide by m, we have

$$u_i(X) \geq \frac{1}{m} \sum_j v_{ij} + \frac{1}{m} \cdot \frac{c-1}{c} \sum_j \sum_{i' \in X_j} e_{ii'}$$
$$= \frac{1}{m} \sum_j v_{ij} + \frac{1}{m} \cdot \frac{c-1}{c} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$
$$= \frac{1}{m} \sum_j v_{ij} + \frac{c-1}{n} \sum_{i' \in N \setminus \{i\}} e_{ii'}$$
$$\geq \frac{n-1}{n} \left(\frac{1}{m} \sum_j v_{ij} + \frac{c-1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \right)$$
$$= \frac{n-1}{n} \mathrm{PROP}_i.$$

Hence assignment X is $(1 - \frac{1}{n})$-approximate PROP. □

At the end of this section, we observe that envy-freeness and proportionality are very strong fairness requirements such that there exist a number of instances where every assignment is neither proportional nor envy-free. A simple example is when there are two resources both with capacities 1 which need to be assigned to two agents. Assume both agents have value 1 for resource 1 and value 0 for resource 2, and all the external values are 0. In this instance, any agent assigned to resource 2 is not proportional and significantly envies the other agent.

4 PARETO ENVY-FREENESS AND PROPORTIONALITY

As the we have discussed that envy-freeness, as well as proportionality, is a strong fairness notion, in this section, we give up the additivity of the utilities and view it as a two-dimensional vector. Roughly, we relax the requirement to be there is one dimension that is satisfied.

4.1 Pareto Envy-freeness

Definition 4.1. An assignment X is called *Pareto-envy-free (PEF)* if for every pair of agents i and i', at least one of the following two inequalities holds

(1) $v_i(X) \geq v_i(X^{ii'})$; or
(2) $e_i(X) \geq e_i(X^{ii'})$,

where $X^{ii'}$ represents the assignment by exchanging the resources assigned to agents i and i'.

From the definition, it is not hard to see that envy-freeness implies Pareto-envy-freeness. Intuitively, a PEF assignment requires that for any two agents, one cannot be worse off than the other agent with respect to both the values for the resources assigned to them and the external values for agents sharing the same resources with them. This definition of PEF also makes sense when the agents only have ordinal preference for the resources or when the values for the resources and the externalities are not comparable.

We also note that if some agent i does not have any friends, i.e., $\sum_{i' \in X_{j_i} \setminus \{i\}} e_{ii'} = 0$, then requirement (2) of PEF is trivially satisfied.

As we will see in the later sections, PEF assignments cannot always be guaranteed. Then one may consider to further relax the requirement (2) as

$$e_i(X) \geq e_i((X_{j_{i'}}^{ii'} \setminus S, X_{-j_{i'}}^{ii'}))$$

for some $S \subseteq X_{j_{i'}}^{ii'} \setminus \{i\}$ with cardinality at most k. This is often called *envy-free up to k items* in the literature [13]. However, including PEF itself, all such relaxations are still very demanding requirements. Consider the following instance with 2 resources and n agents where n is sufficiently large. All agents have identical values: 2 for resource 1 and 1 for resource 2; and identical external values: every agent has external value 1 for any other agent. The capacities of resources 1 and 2 are $n-1$ and 1 respectively. Thus for an arbitrary assignment X, one of the agents, say i, has to be assigned to resource 2, such that $v_i(X) = 1$ and $e_i(X) = 0$. However, by exchanging with any agent $i' \neq i$ that is assigned to resource 1 and letting $X^{ii'}$ be the resulting assignment, $e_i(X^{ii'}) = n-2$, which means that agent i always has lower external value for her assigned resource than the

other one even after all but one agents are removed from the other resource. One reason for the above impossibility is the unbalanced capacities. Thus in the future sections, we will mainly consider the canonical model, where all the resources have the same capacity, and study to what extent a PEF assignment exists.

4.2 Pareto Proportionality

Next, we provide a similar approach to relax the requirements of proportionality for the canonical model.

Definition 4.2. An assignment X for a canonical instance is called *Pareto-proportional (PPROP)* if for every agent i, at least one of the following two inequalities holds

(1) $\mu_i(X) \geq \frac{1}{2}m$; or
(2) $e_i(X) \geq \lfloor \frac{c-1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \rceil$,

where $\mu_i(X) = |\{j \in M \mid v_{ij_i} \geq v_{ij}\}|$ represents the number of resources that are worse off for agent i comparing to her assigned resource j_i, and $\lfloor x \rceil$ represents rounding x to the nearest integer, i.e. $\lfloor x \rceil = \lceil x \rceil$ if $x \geq \lfloor x \rfloor + \frac{1}{2}$ and $\lfloor x \rceil = \lfloor x \rfloor$ otherwise.

Intuitively, a PPROP assignment requires that for any agent, either she is assigned to one of her top 50% resources or shares a resource with a number of her friends that is no fewer than the rounded proportionality, i.e., $\lfloor \frac{c-1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \rceil$. Note that for the first requirement, we do not use the cardinal definition of proportionality for the value of the resources. This is to balance some extreme situations such as when there is exactly one resource that is super valuable for all agents, then the cardinal version of the definition cannot be satisfied. Accordingly, PROP does not imply PPROP any more. In the following theorem, we show the connection between PEF and PPROP in the special case of $c = 2$ or $m = 2$.

THEOREM 4.3. *For any canonical instance with $c = 2$ or $m = 2$, a PEF assignment is also PPROP.*

PROOF. Let $X = (X_1, \cdots, X_m)$ be an arbitrary PEF assignment. We show that for an arbitrarily agent i, X is also PPROP for her.

We first consider the situation when $c = 2$ (but arbitrary m) and distinguish between the following two cases.

Case 1. If $\sum_{i' \in N \setminus \{i\}} e_{ii'} < m = \frac{n}{2}$, then

$$\lfloor \frac{c-1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \rceil \leq \lfloor \frac{1}{n-1} \cdot (\frac{n}{2} - 1) \rceil = 0.$$

Thus the second requirement of PPROP always holds with respect to agent i.

Case 2. If $\frac{n}{2} = m \leq \sum_{i' \in N \setminus \{i\}} e_{ii'} \leq 2m - 1$, then

$$\lfloor \frac{c-1}{n-1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \rceil = 1.$$

Hence, as long as agent i shares a resource with a friend, X is PPROP to her. If agent i does not have a friend in X_{j_i}, then all her friends would occupy at least $\lceil \frac{m}{2} \rceil$ resources. Since assignment X is PEF to agent i, X_{j_i} should be better than those $\lceil \frac{m}{2} \rceil$ resources for her. That is the first requirement of PPROP is satisfied with respect to i.

Next we turn to the situation when $m = 2$ (but arbitrary c) by considering the following three cases.

Case 1. If agent i gets her preferred resource, the first requirement of PPROP is satisfied with respect to i.

Case 2. If all $\frac{n}{2}$ agents in X_{3-j_i} are the friends of agent i, then all the other $\frac{n}{2} - 1$ agents in X_{j_i} except i must also be the friends of agent i as allocation X is PEF. That is (recall that n is even), $\sum_{i' \in N \setminus \{i\}} e_{ii'} = n - 1$,

$$e_i(X) = \frac{n}{2} - 1,$$

and

$$\left\lfloor \frac{\frac{n}{2} - 1}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \right\rceil = \frac{n}{2} - 1.$$

Thus the second requirement of PPROP is satisfied.

Case 3. If there is at least one agent in X_{3-j_i} that is not the friend of agent i, then the number of friends for agent i in resource j_i must be at least the number of friends in resource $3 - j_i$. That is

$$e_i(X) \geq \left\lceil \frac{1}{2} \sum_{i' \in N \setminus \{i\}} e_{ii'} \right\rceil \geq \left\lfloor \frac{\frac{n}{2} - 1}{n - 1} \sum_{i' \in N \setminus \{i\}} e_{ii'} \right\rceil,$$

and the second requirement of PPROP is satisfied.

In conclusion, as i is an arbitrary agent, assignment X must also be PPROP. □

Note that Theorem 4.3 does not hold when $c \geq 3$ and $m \geq 3$. Consider the following instance with $m = 3$ and $c = 3$. Suppose four agents $\{1, 2, 3, 4\}$ are friends with each other, two agents $\{5, 6\}$ are friends with each other, and three agents $\{7, 8, 9\}$ are friends with each other. But none in $\{1, 2, 3, 4\}$ or $\{5, 6\}$ or $\{7, 8, 9\}$ is friend with any agent that is not in her set. Suppose all nine agents have identical resource preference with resource 1 being the best, resource 2 the second and resource 3 the worst. We claim that assignment $X = (X_1, X_2, X_3)$ with $X_1 = \{7, 8, 9\}$, $X_2 = \{4, 5, 6\}$ and $X_3 = \{1, 2, 3\}$ is PEF. Since every agent in $\{7, 8, 9\}$ gets her best resource, X is PEF to them. Since every agent in $\{5, 6\}$ shares a resource with her unique friend, X is PEF to them. Since every agent in $\{1, 2, 3\}$ shares a resource with two of her three friends, X is PEF to them. Since agent 4 gets a better resource than any of her friends in $\{1, 2, 3\}$, X is PEF to her. However, assignment X is not PPROP to agent 4: she does not get her top 1 resource or share a resource with at least 1 of her friends.

5 FAIR DORM ASSIGNMENT PROBLEM

In this section, we study to what extent PEF assignments can be guaranteed to exist for canonical models. As we will see, for dorm assignment problem (where every dorm has capacity 2), a PEF assignment (and of course PPROP) always exists and can be found in polynomial time; however, if the dorm capacity is increased to 3, PEF assignments cannot be guaranteed.

5.1 More Notations

Before introducing our main result, we provide additional notations to describe our algorithm. Let $G = (V, E)$ be an arbitrary externality graph. A matching of G is called *perfect* if it covers all nodes of G, and *nearly perfect* if it covers all but one nodes. Graph G is called *factor-critical* if after removing any node of G, the remaining graph contains a perfect matching.

For any set of nodes $A \subseteq V$, denote by $G \setminus A$ the induced subgraph of $V \setminus A$ in G. $G \setminus A$ may be composed of one or more connected
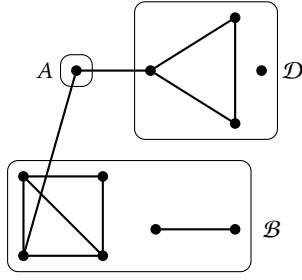
Figure 1: An illustration of Gallai-Edmonds decomposition.

components (or components for short). A component of $G \setminus A$ is called *even* (or *odd*) if it contains even (or odd) number of nodes. Denote by $\mathcal{B} = \mathcal{B}(A)$ and $\mathcal{D} = \mathcal{D}(A)$ the set of all even and odd components in $G \setminus A$, respectively. We use $V(\mathcal{B})$ and $V(\mathcal{D})$ to represent the set of nodes in $\mathcal{B}$ and $\mathcal{D}$. A set $A \subseteq V$ is called a *Tutte set* if each maximum matching $\mathcal{M}$ of G can be decomposed as

$$\mathcal{M} = \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{D}} \cup \mathcal{M}_{A,\mathcal{D}},$$

where $\mathcal{M}_{\mathcal{B}}$ is the set of perfect matchings for each even component $B \in \mathcal{B}$, $\mathcal{M}_{\mathcal{D}}$ is the set of nearly perfect matchings in each odd component $D \in \mathcal{D}$, and $\mathcal{M}_{A,\mathcal{D}}$ is a matching which matches every node in A to a node in some odd component in $\mathcal{D}$.

Lemma 5.1 (Gallai-Edmonds Decomposition [17, 33]). *Given $G = (V, E)$, a Tutte set A can be constructed in $O(n^3)$ time such that*

(1) *all odd components $D \in \mathcal{D}$ are factor-critical;*
(2) *for any odd component $D \in \mathcal{D}$, there is a maximum matching $\mathcal{M}$ of G which does not completely cover D.*

Denote by tuple $(A, \mathcal{B}, \mathcal{D})$ a Gallai-Edmonds decomposition, which is illustrated in Figure 1. By Lemma 5.1, $|A| \leq |\mathcal{D}|$ and if $\mathcal{D} \neq \emptyset$, $|A| < |\mathcal{D}|$.

Given any graph $G = (V, E)$, for each subset $S \subseteq V$, let $\mathcal{N}(S)$ denote the neighbours of S in V, i.e.,

$$\mathcal{N}(S) = \{i \in V \setminus S \mid (i, i') \in E \text{ for some } i' \in S\}.$$

Lemma 5.2 (Hall's Theorem [27]). *For any bipartite graph $G = (L, R; E)$ with node sets L and R, and edge set E such that $|L| \leq |R|$, there exists a matching with size at least $|L|$ if and only if for any subset $S \subseteq L$, $|S| \leq |\mathcal{N}(S)|$.*

5.2 The Algorithm

Before we formally describe the algorithm, let us first discuss some intuitions. For any agent $i \in N$ in a dorm assignment problem (with capacity 2), PEFness essentially requires at least one of the following two situations: (1) agent i obtains a (weakly) better dorm with respect to i's preference than any of her friends; or (2) agent i shares an arbitrary dorm with one of her friends. Thus, one possible approach is to find a maximum matching in the externality graph. If all agents are covered by this matching, then we are free to arbitrarily assign each pair of matched agents to a dorm and the second situation could be guaranteed for all agents.

If the maximum matching does not cover all agents, we are able to assign the covered pairs of agents to some dorms only if the remaining dorms can guarantee that every uncovered agent gets a better dorm than all her friends. This is not easy to satisfy

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Algorithm 1 Algorithm CapTwo

Input: Any dorm assignment instance $\mathcal{I} = (N, M, v, e)$.
1: Let $G = (N, E)$ be the externality graph of $\mathcal{I}$.
2: Compute a Gallai-Edmonds decomposition $(A, \mathcal{B}, \mathcal{D})$ of G, and a maximum matching $\mathcal{M} = \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{A,\mathcal{D}} \cup \mathcal{M}_{\mathcal{D}}$.
3: **if** $\mathcal{D} = \emptyset$ **then**
4:     Arbitrarily assign each pair of matched agents in $\mathcal{M}$ to one dorm and go to **Output**.
5: **else**
6:     Arbitrarily assign each pair of matched agents in $\mathcal{M}_{\mathcal{B}}$ to an unassigned dorm.
7:     Let $L_1 = \{i \in V(\mathcal{D}) \mid i \text{ is covered by } \mathcal{M}_{A,\mathcal{D}}\}$, $L_2 = \{i \in V(\mathcal{D}) \mid i \text{ is not covered by } \mathcal{M}\}$ and $L = L_1 \cup L_2$. Construct a bipartite graph $G' = (A, L; E')$ where for any $a \in A$ and $l \in L$, $(a, l) \in E'$ if and only if $(a, l) \in E$.
8:     For each agent $i \in L$, let D_i be the odd component that i belongs to. Each $i \in L$ takes turns to select her *least* preferred $\frac{|D_i|-1}{2}$ dorms from all unassigned ones to accommodate each pair of matched agents (by $\mathcal{M}_{\mathcal{D}}$) in D_i.
9:     **while** there is no perfect matching between A and L in G' **do**
10:        **Case 1.** If there is a pair of nodes $\{i, i'\} \subseteq L$ that are not connected with A, assign them to an arbitrary unassigned dorm. Remove all such paired nodes from L.
11:        **Case 2.** If there exists a set of nodes $A' \subseteq A$ such that $|A'| = |\mathcal{N}(A')|$, find such a set A' and a perfect matching $\mathcal{M}'$ between A' and $\mathcal{N}(A')$. Assign each pair of matched agents (by $\mathcal{M}'$) to an arbitrary unassigned dorm. Remove all agents in $A' \cup \mathcal{N}(A')$ from A and L, and their adjacent edges from G'.
12:        **Case 3.** If there exists a set of nodes $A' \subseteq A$ such that $|A'| = |\mathcal{N}(A')| - 1$, find such a set A' and a nearly perfect matching $\mathcal{M}'$ between A' and $\mathcal{N}(A')$. Let the unmatched agent $i \in \mathcal{N}(A')$ assign these pairs of matched agents (by $\mathcal{M}'$) to i's *least* preferred $|A'|$ dorms. Remove all agents in A' from L, $\mathcal{N}(A') \setminus \{i\}$ from L, and their adjacent edges from G'.
13:        **Case 4.** If for every set of nodes $A' \subseteq A$, $|A'| \leq |\mathcal{N}(A')| - 2$, let i and i' be a pair of agents in L who have the same most preferred dorm among all remaining ones. Assign both of them to their most preferred dorm. Remove $\{i, i'\}$ from L, and their adjacent edges from G'.
14:    **end while**
15:    Find a perfect matching between A and L, and assign each pair of matched agents to an arbitrary remaining dorm.
16: **end if**
Output: An assignment of agents in N to dorms in M.

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because the agents can have arbitrary preferences over the dorms. Fortunately, the idea of Gallai-Edmonds Decomposition comes to the rescue. By leveraging the structure of a matching and carefully assigning some matched pairs to "bad" dorms for the others, we manage to obtain a PEF assignment. Our Algorithm CapTwo is described in Algorithm 1.
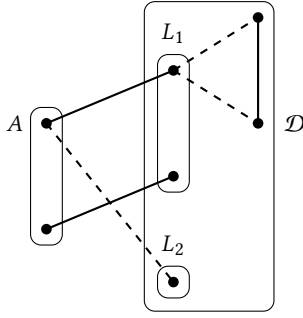
Figure 2: An illustration of the bipartite graph G' in CapTwo. The solid lines represent the edges appear in matching $\mathcal{M}$ and the dashed lines represent the edges in the original graph that do not appear in $\mathcal{M}$.

THEOREM 5.3. *For any dorm assignment instance $\mathcal{I} = (N, M, v, e)$ with capacity 2, Algorithm* CapTwo *returns a PEF assignment.*

PROOF. As we have discussed, if $\mathcal{D} = \emptyset$ and $\mathcal{M}$ is a perfect matching, by arbitrarily assigning each pair of matched agents to an arbitrary available dorm, the requirement (2) of PEF is satisfied for every agent and the resulting assignment is PEF. Thus in the following, we assume $\mathcal{D} \neq \emptyset$.

In Step 7 of Algorithm CapTwo, L_1 is defined to be the set of nodes in $\mathcal{D}$ that are covered by $\mathcal{M}_{A, \mathcal{D}}$, and L_2 to be the set of uncovered nodes by the global maximum matching $\mathcal{M}$ in $\mathcal{D}$. Thus by Lemma 5.1, there is a one-to-one correspondence between $L = L_1 \cup L_2$ and $\mathcal{D}$, and $|A| = |L_1|$. Moreover, as $|A| < |\mathcal{D}|$, $L_2 \neq \emptyset$, i.e., there are at least two nodes in the odd components that are not covered by $\mathcal{M}$ (recall that n is even). We illustrate one possible structure of the bipartite graph G' in Figure 2.

In Step 6, by assigning all pairs of matched agents in $\mathcal{M}_{\mathcal{B}}$ to arbitrary dorms, the assignment is PEF to all agents in $V(\mathcal{B})$. Moreover, since Gallai-Edmonds decomposition guarantees that there is no edges between any odd and even components, agents in $V(\mathcal{D})$ do not envy agents in $V(\mathcal{B})$. Thus to guarantee the agents in Tutte set A do not envy the agents in $V(\mathcal{B})$ as well, it suffices to ensure that every agent in A shares a dorm with her friend.

CLAIM 5.4. *For any agent $i \in A$, she shares a dorm with one of her friends in L after the* **While** *loop or Step 15.*

We will prove Claim 5.4 and the following Claim 5.5 later.

In Step 8, for each agent $i \in L$, she selects her least preferred available dorms for the pairs of matched agents in D_i. Combining the fact that every two odd components are disjoint, this step guarantees that the dorm assigned to agent i is more preferable for i than the dorms assigned to all her friends in $V(\mathcal{D})$. Thus it suffices to prove that agents in L do not envy any agents in A either.

CLAIM 5.5. *After the* **While** *loop or Step 15, for any agent $i \in L$, either she shares a dorm with one agent in $A \cap \mathcal{N}(\{i\})$ or gets assigned a dorm that is better than any one assigned to agents in $\mathcal{N}(\{i\})$.*

Combing Claims 5.4 and 5.5, the assignment returned by Algorithm CapTwo is PEF for all agents and Theorem 5.3 holds. $\square$

Next we prove Claims 5.4 and 5.5 together by going through all the four cases in the **While** loop.

PROOF OF CLAIMS 5.4 AND 5.5. Let $G' = (A, L; E')$ be the original bipartite graph constructed in Step 7. By Lemma 5.1, the maximum matching in G' covers all nodes in A. Combining Lemma 5.2,

$$|S| \leq |\mathcal{N}(S)|, \text{ for any } S \subseteq A. \tag{5}$$

Next we go through the four cases in the **While** loop of CapTwo.

Case 1 (in Step 10) only happens when both i's and i'''s components in $\mathcal{D}$ are not connected with A in the externality graph. Thus both i and i' do not envy any nodes in A, and Claim 5.5 holds with respect to i and i'. Moreover, removing them from G' does not affect the fact that all nodes in A can still be perfectly matched in $G' \setminus \{i, i'\}$ in the next round of the **While** loop.

Case 2 (in Step 11) happens when $|A'| = |\mathcal{N}(A')|$. By Lemma 5.2 and Inequality 5, there exists a perfect matching $\mathcal{M}'$ between A' and $\mathcal{N}(A')$. Hence by assigning each pair of matched agents in $\mathcal{M}'$ to an arbitrary unassigned dorm, Claims 5.4 and 5.5 hold with respect to agents in A' and $\mathcal{N}(A')$, respectively.

Note that, for any possible maximum matching between A and L, the agents in A' can only be matched to $\mathcal{N}(A')$. Thus all agents in $A \setminus A'$ can still be perfectly matched to $L \setminus \mathcal{N}(A')$. Moreover, since A' is disjoint from $L \setminus \mathcal{N}(A')$, agents in $L \setminus \mathcal{N}(A')$ do not Pareto-envy agents in $A' \cup \mathcal{N}(A')$. Thus this step does not affect Claim 5.5 with respect to agents in $L \setminus \mathcal{N}(A')$.

Case 3 (in Step 12) happens when **Case 2** does not happen, i.e.,

$$|S| \leq |\mathcal{N}(S)| - 1 \text{ for any } S \subseteq A \tag{6}$$

and there is one $A' \subseteq A$ such that $|A'| = |\mathcal{N}(A')| - 1$. Let $\mathcal{M}'$ be the nearly perfect matching, which covers all nodes in A' and all but one agent i in $\mathcal{N}(A')$. Using the same technique with Step 8, agent i assigns her least favourite dorms to these pairs of matched agents, and eventually, i will be assigned to a dorm that is better than them. Thus both Claims 5.4 and 5.5 hold for $A' \cup \mathcal{N}(A') \setminus \{i\}$, and this step does not affect Claim 5.5 with respect to i.

Note that, by Inequality 6, for any $S \subseteq A \setminus A'$,

$$|S \cup A'| \leq |\mathcal{N}(S \cup A')| - 1 = |\mathcal{N}(S \cup A') \setminus \mathcal{N}(A')| + |\mathcal{N}(A')| - 1$$
$$= |\mathcal{N}(S) \cap (L \setminus \mathcal{N}(A'))| + |A'|,$$

where the last equality is due to A' is disjoint from $L \setminus \mathcal{N}(A')$. As $S \cap A' = \emptyset$, this implies $|S| \leq \mathcal{N}(S)$ in the remaining graph. Thus all agents in $A \setminus A'$ can still be perfectly matched to $L \setminus \mathcal{N}(A')$. Finally, similar to **Case 2**, this step does not affect Claim 5.5 with respect to agents in $L \setminus \mathcal{N}(A')$ either.

If none of **Cases 1, 2, 3** happens, it means that for any $S \subseteq A$, $|S| \leq |\mathcal{N}(S)| - 2$. Since the dorms do not have spare capacities, after Step 8, there are exactly $\frac{1}{2}(|A| + |L|)$ available dorms and

$$|L| \geq |A| + 2 > \frac{1}{2}(|A| + |L|).$$

By pigeon hole principle, there must be a pair of agents i and i' in L who have the same most preferred dorm among all remaining dorms. Note that for each of them, by the design of Steps 6, 8, and 10 to 12, this dorm is the best among all the dorms assigned to their friends. Thus, Claim 5.5 holds with respect to i and i'. After removing $\{i, i'\}$ from L, for any subset $S \subseteq A$, $|\mathcal{N}(S)|$ decreases at

most 2. Accordingly, in $G' \setminus \{i, i'\}$, Inequality 5 still holds and thus A can be perfectly matched in $G' \setminus \{i, i'\}$.

In conclusion, for each round of the **While** loop, at least one of the above four cases happens. No matter which case happens, the size of L is strictly decreased and after each round, the remaining nodes in A can still be perfectly matched by the remaining nodes in L. Eventually, Step 15 happens and there is a perfect matching between A and L. By assigning each pair of matched agents to an arbitrary remaining dorm, Claim 5.4 holds with respect to A and Claim 5.5 holds with respect to L. □

Actually, Algorithm CapTwo can be implemented in polynomial time. Due to space limit, we only include a proof sketch below.

Lemma 5.6. *For any dorm assignment instance with capacity 2, Algorithm* CapTwo *can be implemented in polynomial time.*

Proof Sketch. It suffices to show that the **While** loop in Algorithm CapTwo can be implemented efficiently. In each round, at least 2 nodes are assigned to a dorm, thus there are at most $O(n)$ rounds. Case 1 is trivial by checking the adjacency matrix of the graph. For the rest cases, by pigeon hole principle, there exists a pair of agents $\{i, i'\} \subseteq L$ who share the same most preferred dorm. If the agents in A can be fully matched to $L \setminus \{i, i'\}$, we assign $\{i, i'\}$ to their most preferred dorm by implementing Case 4. Otherwise, we implement Cases 2 or 3 by showing that a subset $A' \subseteq A$ with $\{i, i'\} \cap \mathcal{N}(A') \neq \emptyset$ and $|A'| \geq |\mathcal{N}(A')| - 1$ must exist and can be found efficiently. Moreover, after assigning dorms to A' and their neighbors according to the maximum matching, the remaining nodes in $A \setminus A'$ can be matched perfectly in future rounds. □

Combing Theorems 4.3 and 5.3, we have the following corollary.

Corollary 5.7. *For any dorm assignment problem with capacity 2, Algorithm* CapTwo *returns a PPROP assignment.*

5.3 Impossibility for $c \geq 3$

In this subsection, we show that if the capacity of the dorms increases to 3, PEF assignments may not exist.

Theorem 5.8. *There exists an instance of dorm assignment problem with capacity 3 such that no assignment is PEF.*

Proof. Consider the following instance with 3 dorms and 9 agents. The agents' external values are defined as for any $i \in \{1, 3, 5, 7\}$, agent i and $i + 1$ are friends with each other (and no more friendship between them); agent 9 does not have any friends. All agents' values for dorm $j \in \{1, 2, 3\}$ is $4 - j$.

We prove by contradiction. Suppose there is a PEF assignment, then for any agent $i \neq 9$ who is assigned to dorm 1, agent i' friend i' has to be assigned to dorm 1 as well; otherwise, agent i' will Pareto-envy the other agents living in dorm 1. Since dorm 1 can accommodate at most three agents, exactly one pair of friends can be assigned to dorm 1 and agent 9 must be assigned to dorm 1. Accordingly, all the other three pair of agents need to be assigned to dorms 2 and 3. Thus, there must be a pair of friends i_1, i_2 such that agent i_1 is assigned to dorm 2 while agent i_2 lives in dorm 3. In this case, for agent i_2, she does not share a dorm with her friend and her friend i_1 is assigned to a better dorm; thus, i_2 Pareto-envies the roommate of i_1. □

6 CONCLUSION AND DISCUSSION

6.1 Conclusion

In this work, we initiate the study of fair (capacitated) resource sharing problem. We propose envy-freeness and Pareto envy-freeness, where the latter is a relaxation of the former definition. We show that these two definitions can guarantee some degree of proportional utilities. While the existence of envy-freeness is not guaranteed, we prove that for the dorm assignment problem with capacity 2, a Pareto envy-free assignment exists and can be found in polynomial time. However, if the capacities of the dorms are increased to 3, PEF assignments may not exist.

6.2 Extension: Non-positive Preferences

In this work, we mainly focused on the case of non-negative values for the resources and non-negative externalities between the agents. However, for both definitions of PEF and PPROP (see Definitions 4.1 and 4.2), their requirements (1) only depend on the agents' ordinal preferences over the resources. Therefore, all results in Section 5 can be directly applied to the cases when the agents have negative values for the resources.

When the agents' externalities are non-positive but binary (such as enemy or non-enemy), our results for PEF apply here as well. For example, with respect to Theorem 5.3, by Definition 4.1, PEF here essentially requires for every agent i, one of the following two situations happens: (1) i obtains a better resource than any of her non-enemies; or (2) i does not share a resource with one of her enemies. Thus if the externality graph is constructed by adding an edge between a pair of agents if and only if they are not enemies with each other, Algorithm CapTwo returns a PEF assignment. Similarly, all the other results in Section 5 hold.

6.3 Future Directions

Firstly, in this work, we mainly focus on PEF assignments. As a corollary, a PPROP assignment can be guaranteed for the dorm assignment problem with capacity 2. It will be interesting to prove or disprove the existence of PPROP assignments for general models. For example, when the capacity of dorm assignment problem is 3, it is still unknown whether PPROP assignments always exist.

Secondly, the computational issues about resource sharing problem has not been discussed in this paper. For example, is it computationally tractable to decide whether a resource sharing problem admits an EF/PROP/PEF (and PPROP if it cannot be always guaranteed) assignment?

Finally, it is also interesting to adapt other fairness criteria to our setting, such as the maximin share fairness defined in [13], which is another widely studied notion in the resource allocation literature. Arguably, an even more important question might be to investigate what kind of fairness can be guaranteed for all possible instances.

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