Fairness and Efficiency in Facility Location Problems with Continuous Demands

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ABSTRACT

In the facility location problem with continuous demands, where customers are continuously distributed on an area, a planner wants to locate the facilities and allocate the customers to their closest facilities under the proximity rule. In this work, we focus on the fairness and system efficiency from the facility's perspective. Each facility is assumed to have a preference (represented as valuation function) over the subsets of customers. For the fairness of facilities, we provide approximation guarantees for the proportionality and envy-freeness. For the efficiency, we study the utilitarian and egalitarian social welfare. In addition, we are interested in quantifying the possible trade-offs between meeting the fairness criteria and maximizing social welfare, measured by the price of fairness.

KEYWORDS

Facility location; Continuous demands; Fairness; Efficiency

ACM Reference Format:

Chenhao Wang and Mengqi Zhang. 2021. Fairness and Efficiency in Facility Location Problems with Continuous Demands. In *Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021), Online, May 3–7, 2021*, IFAAMAS, 9 pages.

1 INTRODUCTION

Facility location problems (FLPs) and their variants [27, 28, 33, 34] have been extensively studied in many fields. In the classic FLPs, given a set of locations of customers and a set of facilities, we are to locate these facilities to serve the customers. When the number of customers on each edge of a network is large (for example, customers of ATMs and shopping malls), treating each customer as a single point will make the problem intractable, even for small instances. A common alternative approach in the literature is considering a single point on each edge as the representative for an entire edge and assigning all customer demands to that point. However, it has a main drawback [17] that each demand point (in fact, corresponding to an edge) is allocated to a single facility, indicating that all customers on the edge must go to this facility to receive service, which violates the assumption that each customer is assigned to the closest facility.

To overcome this difficulty, an alternative assumption [2, 15, 38] is that customers are continuously distributed on an edge or an area, rather than concentrated at discrete points. In other words, each edge or area has a known demand density that depends on its location. This "distributed" demand could also be used to represent random occurrences of demand from within areas. This paper focuses on the facility location problem with contiguous demands (FLCD), in a heterogeneous geographical space.

While one branch of research on FLPs focuses on optimizing a system objective regarding the customers, e.g., minimizing the total/maximum distance of customers to their closest facilities [13, 20], another branch of researchers investigate the FLPs from the perspective of facilities, where commercial facilities operate in a competitive environment [14, 29]. In this paper, we study the FLCD from the facility's perspective. Each facility is assumed to have a preference or valuation function over the customers the facility can serve, that is, serving a preferable subset of customers will give a higher utility to the facility. A manager wants to locate the facilities reflecting these preferences. This models many scenarios in real life, for example, shopping malls or grocery stores have a preference over the residents or customers with different consumer demands [9, 10], and a college/school has a preference over the students with different educational backgrounds when recruiting students [1, 19].

We adopt the deterministic proximity rule proposed by Hotelling [25], i.e., each customer patronizes the closest facility, which implies that all competing facilities are equally attractive, and the total buying power concentrated at a demand point is spent at the same facility (the "all or nothing" assumption) [10].

Precisely, we want to locate a set of n facilities, and determine an allocation of customers, which are contiguously distributed on a line, to the facilities. Clearly, once the facilities are located, the allocation is uniquely determined, by the proximity rule. We are particularly interested in the fairness among these competing facilities as well as the system efficiency (maximization of utilitarian or egalitarian social welfare). Merely maximizing social welfare may result in an unbalanced allocation of customers to facilities, which is perceived as unfair, while pursuing the fairness too much would lead to a large efficiency loss. So there is a tradeoff between the fairness and efficiency. The problem studied combines facility location and fair division [11], and may provide a new dimension for the study of these two types of problems.

1.1 Main Results

We study the facility location problem with continuous demands, where customers are uniformly distributed on a line segment $[0, 1]$. An algorithm outputs a location profile of n facilities along with an allocation of customers among facilities. We are interested in *valid allocation* of customers, which admits a feasible location profile of facilities such that every customer is assigned to his closest facility

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Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021), U. Endriss, A. Nowé, F. Dignum, A. Lomuscio (eds.), May 3–7, 2021, Online.

Table 1: Multiplicative approximation guarantees on fairness for our setting and contiguous cake cutting.

Fairness	Our problem		Contiguous cake cutting [3]	
	Lower	Upper	Lower	Upper
Proportionality	$\frac{1}{2}$	$\frac{3}{4}$	1	1
Envy-freeness	$\frac{1}{12+o(1)}$	$\frac{1}{2}$	$\frac{1}{3+o(1)}$	1

Table 2: Our results on the price of fairness

Price of fairness	Best price				Worst price			
	Utilitarian		Egalitarian		Utilitarian		Egalitarian	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Proportionality	$\frac{\sqrt{n}}{2}$	$n-1+\frac{1}{n}$	1	1	$n-\frac{1}{n}$	n	n	n
Envy-freeness	$\frac{\sqrt{n}}{2}$	$\frac{\sqrt{n}}{2}+1-o(1)$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{\sqrt{n}}{2}$	$\frac{\sqrt{n}}{2}+1-o(1)$	$\frac{n}{2}$	n

in the allocation. Three aspects of the facility location problem are investigated in this paper from the perspective of facilities: fairness, efficiency, and the trade-off between fairness and efficiency.

We note that our problem is closely related to the fair division problem of *contiguous cake cutting* (also called cake cutting with connected pieces) [23, 35, 36], in which a single heterogeneous good, modeled as $[0, 1]$, must be fairly divided among multiple parties (facilities in our terminology) such that the bundle for each facility is contiguous, i.e., a single interval. Clearly a valid allocation in our problem satisfies the contiguous constraint. The difference is that, in our problem, we have an additional constraint that every point in $[0, 1]$ must be assigned to the closest facility. So the solution space in our problem is more restricted.

Fairness (Section 3): we assess the existence and approximability of fair valid allocations with respect to proportionality and envy-freeness. A ρ-proportional allocation means that every facility has utility at least $\frac{\rho}{n}$, and a ρ-envy-free allocation means that everyone's envy is bounded by a multiplicative factor of ρ. Upper bounds (nonexistence results) and lower bounds (existence results) on the multiplicative approximation guarantees are derived. We prove that, the existence of a $(\frac{3}{4}+\epsilon)$-proportional (resp. $(\frac{1}{2}+\epsilon)$-envy-free) valid allocation is not guaranteed for any $\epsilon > 0$, and there is an efficient algorithm that returns a $\frac{1}{2}$-proportional (resp. $\frac{1}{12+o(1)}$-envy-free) allocation. Table 1 compares the results on multiplicative approximation of fairness in our problem and contiguous cake cutting, which shows that it is more difficult to approximate both fairness criteria in our problem.

Efficiency (Section 4): we study the problem of finding valid allocations that maximize the utilitarian and egalitarian social welfare, where the utilitarian welfare is the total utility of all facilities and egalitarian welfare is the minimum utility of facilities. We first prove that the problem of maximizing either type of social welfare is NP-hard, even if the valuation functions are piecewise-uniform. Then we show that there is a $(4 + o(1))$-approximation for the utilitarian social welfare.

Trade-off between fairness and efficiency (Section 5): we use the concept of price of fairness to measure the efficiency loss under a fairness constraint. We calculate the (best or worst) price of fairness for both utilitarian and egalitarian social welfare, which is defined as the ratio of the maximum possible social welfare over that of a (best or worst) fair allocation. For the best price of fairness, several results for contiguous cake cutting problem in [4] are applicable to our problem, since we can find feasible location profiles of facilities in the constructed instances such that each contiguous allocation is valid. For the worst price of fairness, we are the first to derive lower bounds and upper bounds for both types of social welfare. Table 2 summarizes our results on the price of fairness.

1.2 Related Work

Our work is grounded on a string of fruitful research for facility location and fair division problems. We note that in a closely related work [12], it studies the discrete version where one wants to locate facilities to serve a group of items (customers) located on a path graph under the proximity rule, taking both fairness and efficiency into consideration.

FLPs with continuous demands. While classic FLPs studied in the literature often assume that the customers are located on the nodes of a network, to better model realistic problems, many researchers [2, 7, 15–17, 31] consider the FLPs with continuously distributed demands. Drezener *et al.* [17] initially consider the single-facility FLP where demands are uniformly distributed in an area or areas. They present in [15] the exact calculation of the optimal location by double integration. Arkat and Jafari [2] study a single-facility FLP in which demands are uniformly distributed along the network edges. Using the concept of distributed demands, Golabi *et al.* [22] propose an FLP in humanitarian relief logistics using UAV drones. Shavarani *et al.* [32] extend the previous model by considering two levels of services.

Contiguous cake cutting. This problem is also known as cake cutting with connected pieces, in which a planner wants to divide a heterogeneous cake among the agents as fairly as possible so that everyone receives a connected piece. The seminal works of Stromquist [35, 36] show that a contiguous envy-free allocation always exists but cannot be found by a finite algorithm. Recently, Arunachaleswaran *et. al.* [3] give an efficient algorithm that returns a contiguous allocation such that each agent's envy is bounded by a multiplicative factor of 3. For the additive type of approximation, Goldberg *et. al.* [23] develop an algorithm that returns a $\frac{1}{3}$-envy-free allocation. For the problem of finding contiguous allocations that maximize the utilitarian/egalitarian welfare, Aumann *et al.* [5] prove that both problems are NP-hard when players have piecewise-constant valuations, and they provide a $8(1+(n-1)\epsilon)$-approximate algorithm for the utilitarian welfare with $\epsilon > 0$. Arunachaleswaran *et al.* [3] improves the approximation ratio to $2 + o(1)$, and further prove that maximizing the egalitarian welfare is APX-hard by a reduction from the Gap-3-SAT-5 problem. Recently, Barman and Rathi [6] study value densities of the agents that satisfy the monotone likelihood ratio property (MLRP).

Price of fairness. The price of fairness quantifies the efficiency loss of a fair allocation due to fairness constraint, by comparing the maximum possible social welfare and the social welfare induced by a fair allocation. Caragiannis *et al.* [11] initially study the price of fairness for both divisible and indivisible goods. Following this work, Heydrich and van Stee [24] consider the setting of divisible chores. Latter, Aumann and Dombb [4] focus on the contiguous allocations of divisible items, and provide tight or almost tight bounds on the price of fairness with respect to both utilitarian welfare and egalitarian welfare. Recently, Suksompong [37] complete the results by calculating bounds on the price of fairness for contiguous allocations of indivisible items.

2 PRELIMINARIES

In this section, we first introduce the model studied in this work, and then provide some preliminary results.

2.1 Model

In the FLCD, customers are uniformly distributed on a line segment, represented by the interval $[0, 1]$. We want to locate n facilities in $[0, 1]$ and allocate customers to these facilities such that each customer is assigned to his closest facility. Let $N = \{1, \dots, n\}$ be the set of facilities that need to be located. The preferences of facilities over the customers are represented by *valuation functions* $v_1, v_2, \dots, v_n$ over the intervals contained in $[0, 1]$. Formally, for each facility $i \in N$ and interval $I = [a, b] \subset [0, 1]$, facility i's valuation for interval I is $v_i(I) = v_i(a, b) \in \mathbb{R}_+$. Conforming to standard assumptions, the valuation functions are nonnegative, non-atomic (i.e., $v_i(x, x) = 0$, for any $i \in N$ and $x \in [0, 1]$, which allows us to regard two intervals to be disjoint even if they intersect exactly at an endpoint), additive (i.e., $v_i(I_1 \cup I_2) = v_i(I_1) + v_i(I_2)$, for any $i \in N$ and any two disjoint intervals $I_1, I_2 \subseteq [0, 1]$), and normalized (i.e., $v_i(0, 1) = 1$, for any $i \in N$).

We access to the valuations in the Robertson-Webb model [30], which supports oracles as *evaluation* queries, i.e., given $i \in N$ and interval I return $v_i(I)$, and *cut* queries, i.e., given $i \in N$, an initial point $x \in [0, 1]$, and value τ, return the leftmost point $y \in [x, 1]$ such that $v_i(x, y) = \tau$. We are particularly interested in a well-studied class in which valuations are induced by density functions: $v_i(a, b) = \int_a^b f_i(x)dx$, where $f_i : [0, 1] \to \mathbb{R}_+ \cup \{0\}$ is the *valuation density function* of facility i.

We say that a valuation function v_i is *piecewise-constant* if $[0, 1]$ can be partitioned into a finite number of intervals such that its value density function f_i is constant on each interval. In addition, if there is some constant c_i such that f_i only attains the values 0 or c_i, we say that v_i is *piecewise-uniform.*

Valid allocations. A *contiguous allocation* is a partition of all customers in $[0, 1]$ into exactly n pairwise-disjoint intervals, along with an assignment of each interval to a facility. Formally, in a contiguous allocation $\mathbf{A} = (A_1, \dots, A_n)$, $\bigcup_{i \in N} A_i = [0, 1]$ and each A_i is a single interval assigned to facility $i \in N$, referred to as the *bundle* of i.

Definition 2.1. *We call an allocation* $\mathbf{A} = (A_1, \dots, A_n)$ valid, *if there exists a location profile* $\mathbf{x} = (x_1, \dots, x_n)$ *of facilities on* $[0, 1]$ *such that the facility locations are pairwise distinct (i.e.,* $x_i \neq x_j$*), and every point/customer is assigned to one of the closest facilities (i.e.,* $\forall i \in N, \forall y \in A_i$, $i \in \arg\min_{i' \in N} |y - x_{i'}|$*).*

The constraint that all customers are assigned to their closest facilities is called the *proximity rule* [25]. Therefore, once given a location profile $\mathbf{x} = (x_1, \dots, x_n)$ of facilities, we can uniquely induce a valid allocation $\mathbf{A} = (A_1, \dots, A_n)$ of customers to n facilities, by assigning each customer to one of his closest facilities. Clearly, each facility receives a single interval, and thus any valid allocation $\mathbf{A}$ is also contiguous.

Social welfare. Under allocation $\mathbf{A}$, each facility $i \in N$ obtains a nonnegative *utility* $v_i(A_i)$, equal to the valuation of the interval assigned to him. We consider two kinds of social welfare, *utilitarian social welfare* and *egalitarian social welfare.* Formally, the utilitarian social welfare is defined as $u(\mathbf{A}) = \sum_{i \in N} v_i(A_i)$, and the egalitarian social welfare is defined as $eg(\mathbf{A}) = \min_{i \in N} v_i(A_i)$.

Fairness criteria. Let $(N, \{v_i\}_{i \in N})$ be an instance of FLCD, and $\mathcal{I}_n$ be the set of all possible instances with n facilities. We are interested in finding valid allocations satisfying some fairness criteria for facilities. Two fairness criteria are considered, namely *proportionality* and *envy-freeness.* When proportional (resp., envy-free) valid allocations are not available, we are trying to find an *approximately proportional* (resp., *approximately envy-free*) valid allocation. This paper studies multiplicative approximation guarantees [3].

Definition 2.2. *An allocation* $\mathbf{A}$ *is proportional if* $v_i(A_i) \geq 1/n$ *for any* $i \in [n]$*. For* $\rho \in [0, 1]$*, an allocation* $\mathbf{A}$ *is* ρ*-(multiplicative) proportional if* $v_i(A_i) \geq \rho \cdot 1/n$ *for any* $i \in [n]$.

Definition 2.3. *An allocation* $\mathbf{A}$ *is envy-free if* $v_i(A_i) \geq v_i(A_j)$ *for any* $i, j \in [n]$*. For* $\rho \in [0, 1]$*, an allocation* $\mathbf{A}$ *is* ρ*-(multiplicative) envy-free if* $v_i(A_i) \geq \rho \cdot v_i(A_j)$ *for any* $i, j \in [n]$.

To ease notation, we will use ρ-proportional to represent approximately proportional. Obviously, a 1-proportional allocation is proportional, and the closer ρ is to 1, the stronger is the proportionality guarantee. This property also holds for envy-freeness.

2.2 Preliminary Results

We have noted that a location profile of facilities uniquely determines a valid allocation. However, a valid allocation may correspond to multiple feasible location profiles of facilities. For example, considering a valid allocation $\mathbf{A} = ([0, 0.4], [0.4, 1])$ for two facilities, both location profiles $(0.2, 0.6)$ and $(0.3, 0.5)$ can induce $\mathbf{A}$. We also notice that a contiguous allocation may be not valid. For example, consider a contiguous allocation $([0, 0.2], [0.2, 0.8], [0.8, 1])$ for three facilities and an arbitrary location profile $\mathbf{x} = (x_1, x_2, x_3)$. Clearly, $0 \le x_1 \le 0.2$ and $0.8 \le x_3 \le 1$, while x_2 cannot satisfy the proximity rule in his own bundle. In particular, when $n = 2$, we make a general observation that any contiguous allocation $([0, y], [y, 1])$ must be valid, because we can locate facility 1 at $y - \epsilon$, and facility 2 at $y + \epsilon$ for some small $\epsilon > 0$, satisfying the proximity rule.

Observation 2.4. *For $n = 2$, every contiguous allocation is valid.*

When $n \ge 3$, we show that, given a contiguous allocation $\mathbf{A} = (A_1, \ldots, A_n)$, there is an efficient algorithm that determines whether it is valid and outputs a location profile of facilities if it exists. Assume that each bundle $A_i \in \mathbf{A}$ is non-empty, otherwise we can remove facility i from the allocation and do not locate it. For simplify, we use the following notation. A contiguous allocation is represented by $n-1$ cut positions $0 < c_1 < c_2 < \ldots < c_{n-1} < 1$ and a permutation $\pi : N \longrightarrow N$ that assigns interval $[c_{i-1}, c_i]$ to facility $\pi(i)$ as his bundle, where $c_0 = 0$ and $c_n = 1$. Let $\mathbf{c} = (c_1, \ldots, c_{n-1})$ be the cut position profile and denote the allocation by $(\mathbf{c}, \pi)$.

Proposition 2.5. *Given a contiguous allocation, we can determine whether it is valid in polynomial time. If it is valid, we can also find a feasible location profile of facilities in polynomial time.*

Proof. Let $(\mathbf{c}, \pi)$ be a contiguous allocation, and $\epsilon > 0$ be an arbitrary constant satisfying $\epsilon < \min_{i \in [n]} |c_i - c_{i-1}|$. We use variable x_i to indicate the location of facility $i \in N$, and show that $(\mathbf{c}, \pi)$ is valid if and only if the following linear program has a feasible solution.

$$\begin{array}{lll} \min & 0 & \\ \text{s.t.} & x_{\pi(i)} + x_{\pi(i+1)} = 2c_i, & \text{for } i \in [n-1] \\ & x_{\pi(i)} + \epsilon/2 \le x_{\pi(i+1)}, & \text{for } i \in [n-1] \\ & 0 \le x_{\pi(i)} \le 1. & \text{for } i \in [n] \end{array} \tag{1}$$

The first $n-1$ constraints guarantee the proximity rule, and the second $n-1$ constraints guarantee that the facility locations are pairwise distinct. Clearly, a feasible solution of the above LP (1) is a location profile of facilities, which can induce the allocation $(\mathbf{c}, \pi)$. On the other hand, if LP (1) has no feasible solution, then there is no feasible location profile for $(\mathbf{c}, \pi)$. □

Proposition 2.5 enables us to focus on finding valid allocations, as a corresponding location profile of facilities can be computed efficiently by solving LP (1).

As a preliminary result, we show that a contiguous allocation can induce a valid allocation so that every facility has at least half utility as before. This proposition is a key tool for finding approximately proportional and approximately envy-free valid allocations.

Proposition 2.6. *Given contiguous allocation $\mathbf{A}$, we can obtain a valid allocation $\mathbf{A}'$ by losing at most half utility for each facility, that is, $v_i(A'_i) \ge \frac{1}{2} v_i(A_i)$, for any $i \in N$.*

Proof. Given contiguous allocation $\mathbf{A} = (A_1, \ldots, A_n)$, for any facility $i \in N$, let l_i and r_i be the leftmost and rightmost points in his bundle, i.e., $A_i = [l_i, r_i]$. Denote by $m_i = \frac{l_i + r_i}{2}$ the midpoint of interval $[l_i, r_i]$. Now we construct a location profile $\mathbf{x} = (x_1, \ldots, x_n)$ of facilities in two steps:

Step 1: for each facility $i \in N$, if the interval $[l_i, m_i]$ has greater valuation than interval $[m_i, r_i]$, then initialize as $x_i = l_i$, otherwise $x_i = r_i$.

Step 2: Note that the profile $(x_1, \ldots, x_n)$ may have the same locations. Assume that $x_1 \le \cdots \le x_n$, renaming if necessary. Now we adjust the locations of the facilities so that they are pairwise distinct. Let $0 < \epsilon < \min_{i \in [n]} |A_i|/2$. If there are facilities $i, i+1$ such that $x_i = x_{i+1}$, then let $x_i := x_i - \epsilon$ and $x_{i+1} := x_{i+1} + \epsilon$.

Based on the facilities' location profile $\mathbf{x} = (x_1, \ldots, x_n)$, we obtain an allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$ by assigning every point in $[0, 1]$ to the closest facility. By the definition, allocation $\mathbf{A}'$ is valid. By the above construction of location profile $\mathbf{x}$, we have $v_i(A'_i) \ge \frac{1}{2} \cdot v_i(A_i)$ for any $i \in N$, because every facility i must receive the more valuable one between $[l_i, m_i]$ and $[m_i, r_i]$. □

3 FAIRNESS

In this section, we study two fairness criteria for valid allocations, namely proportionality and envy-freeness, along with their approximations.

3.1 Proportionality

In this subsection, we study the proportionality of valid allocations, and provide impossibility and possibility results for the existence of approximately proportional valid allocations.

For the contiguous cake cutting, Dubins and Spanier [18] proves that there always exists a proportional contiguous allocation for any instance, by presenting a moving-knife algorithm. By Observation 2.4, it is straightforward to have

Observation 3.1. *For any 2-facility instance of the FLCD, there exists a proportional valid allocation.*

However, in the FLCD with more than 2 facilities, the existence of proportional valid allocations is not guaranteed. Further, we show a stronger impossibility result.

Theorem 3.2. *For the FLCD with at least 3 facilities, the existence of a $(\frac{3}{4} + \epsilon)$-proportional valid allocation is not guaranteed for any $\epsilon > 0$, even if there are $n = 3$ facilities and the valuation functions are piecewise-uniform and identical.*

Proof. Consider an instance of 3 facilities, where the facilities have identical valuation density function f satisfying $f(x) = 5$ for $x \in [0, 0.1] \bigcup [0.9, 1]$, and $f(x) = 0$ for $x \in (0.1, 0.9)$. Suppose that there exists a $(\frac{3}{4}+\epsilon)$–proportional valid allocation $\mathbf{A} = (A_1, A_2, A_3)$, which admits a feasible location profile $\mathbf{x} = (x_1, x_2, x_3)$ of facilities satisfying the proximity rule. Assume without loss of generality that $x_1 < x_2 < x_3$ and $x_1, x_2 \in [0, 0.5]$. By the proximity rule, the interval $[0.9, 1]$ must be assigned to the rightmost facility 3, which implies that 3 obtains a utility at least $\frac{1}{2}$, and the total utility of 1 and 2 is at most $\frac{1}{2}$. However, the $(\frac{3}{4} + \epsilon)$-proportionality implies that each facility $i \in N$ has a utility $v_i(A_i) \ge \frac{1}{3} \cdot (\frac{3}{4} + \epsilon) > \frac{1}{4}$, and the total utility of facilities 1 and 2 is larger than $2 \cdot \frac{1}{4} \ge 1/2$, which is a contradiction. □

For the possibility result, we propose Algorithm 1 which returns a 1/2-proportional valid allocation in polynomial time. In Line 1-8, a contiguous proportional allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is obtained by using the discrete moving-knife algorithm [18]. In Line 9-18, we construct a location profile $\mathbf{x} = (x_1, \dots, x_n)$ of facilities, using the same way as in the proof of Proposition 2.6. In Line 19-23, location profile $\mathbf{x}$ induces a valid allocation $\mathbf{A}' = (A'_1, \dots, A'_n)$, which guarantees that the utility of each facility under $\mathbf{A}'$ is at least half of that under $\mathbf{A}$.

Theorem 3.3. *For any instance with n facilities, Algorithm 1 returns a $\frac{1}{2}$-proportional valid allocation in polynomial time.*

Proof. To bound algorithm's time complexity, note that there are in total $n(n-1)$ markings, and each iteration costs constant time. Hence, Algorithm 1 runs in polynomial time. Because the returned allocation $\mathbf{A}'$ is induced by a location profile $\mathbf{x}$, it must be valid. Note that in Line 1-8 we obtain a proportional contiguous allocation $\mathbf{A}$. Given $\mathbf{A}$, in the remaining of this algorithm we use Proposition 2.6 and obtain a valid allocation $\mathbf{A}'$ such that $v_i(A'_i) \geq \frac{1}{2} v_i(A_i) \geq \frac{1}{2n}$ for any facility $i \in N$. Therefore, $\mathbf{A}'$ is $\frac{1}{2}$-proportional. □

Algorithm 1

Require: An instance $(N, \{v_i\}_{i \in N})$.
Ensure: $1/2$-proportional valid allocation $\mathbf{A}'$.
1: Initialize $A_i = A'_i = \emptyset$ for $i \in N$, and $N' = N$.
2: Each facility $i \in N$ makes $n-1$ markings $a_{i,1}, \dots, a_{i,n-1}$, such that $v_i(0, a_{i,1}) = \frac{1}{n}$, and $v_i(a_{i,j}, a_{i,j+1}) = \frac{1}{n}$ for $j \in [n-2]$.
3: Let $t_1 = \min_{i \in N} a_{i,1}$, and set $A_{t_1} \leftarrow [0, a_{t_1,1}]$.
4: **for** $k = 2, \dots, n-1$ **do**
5: $N' \leftarrow N' \setminus \{t_{k-1}\}$, and $t_k = \arg\min_{i \in N'} a_{i,k}$.
6: $A_{t_k} \leftarrow [a_{t_{k-1},k-1}, a_{t_k,k}]$.
7: **end for**
8: For the unique facility $t_n \in N'$, $A_{t_n} \leftarrow [a_{t_{n-1},n-1}, 1]$.
9: Define $\epsilon = \min_{i \in N} |A_i|/4$, $a_{t_0,0} = 0$, $a_{t_n,n} = 1$.
10: **for** $k = 1, \dots, n$ **do**
11: **if** $v_{t_k}(a_{t_{k-1},k-1} + \frac{|A_{t_k}|}{2}, a_{t_k,k}) \geq \frac{1}{2n}$ **then**
12: $x_{t_k} \leftarrow a_{t_k,k}$
13: **else if** $k = 1$ **or** $x_{t_{k-1}} \neq a_{t_{k-1},k-1}$ **then**
14: $x_{t_k} \leftarrow a_{t_{k-1},k-1}$
15: **else**
16: $x_{t_k} \leftarrow a_{t_{k-1},k-1} + \epsilon$, and $x_{t_{k-1}} \leftarrow a_{t_{k-1},k-1} - \epsilon$
17: **end if**
18: **end for**
19: Let $A'_{t_1} \leftarrow [0, \frac{x_{t_1}+x_{t_2}}{2}]$, and $A'_{t_n} \leftarrow [\frac{x_{t_{n-1}}+x_{t_n}}{2}, 1]$.
20: **for** $k = 2, \dots, n-1$ **do**
21: Let $A'_{t_k} \leftarrow [\frac{x_{t_{k-1}}+x_{t_k}}{2}, \frac{x_{t_k}+x_{t_{k+1}}}{2}]$.
22: **end for**
23: **return** $\mathbf{A}' = (A'_1, \dots, A'_n)$

3.2 Envy-Freeness

In this subsection, we study the envy-freeness of valid allocations, and provide lower and upper bounds on the approximation guarantees for envy-freeness.

For the contiguous cake cutting, an envy-free contiguous allocation can be found by the cut-and-choose algorithm [30]. Then by Observation 2.4, it is easy to see that there exists an envy-free valid allocation when there are only 2 facilities. However, when $n \geq 3$, the existence of envy-free valid allocations is not guaranteed. Further, we show a stronger impossibility result, using the instance constructed in the proof of Theorem 3.2.

Theorem 3.4. *For the FLCD with at least* 3 *facilities, the existence of a $(\frac{1}{2} + \epsilon)$-envy-free valid allocation is not guaranteed for any $\epsilon > 0$, even if there are $n = 3$ facilities and the valuation functions are piecewise-uniform and identical.*

Proof. Consider the 3-facility instance constructed in the proof of Theorem 3.2. Suppose that there exists a $(\frac{1}{2} + \epsilon)$-envy-free valid allocation $\mathbf{A}$, which admits a feasible location profile $\mathbf{x}$ of facilities satisfying the proximity rule. Using the argument in Theorem 3.2, the total utility of facility 1 and 2 is at most $\frac{1}{2}$. However, the $(\frac{1}{2} + \epsilon)$-envy-freeness implies that facility 1 has a utility $v_1(A_1) \geq (\frac{1}{2} + \epsilon) \cdot v_1(A_3) \geq \frac{1}{4} + \frac{\epsilon}{2}$, and similarly $v_2(A_2) \geq \frac{1}{4} + \frac{\epsilon}{2}$. It follows that $v_1(A_1) + v_2(A_2) > \frac{1}{2}$, which is a contradiction. □

For the contiguous cake cutting, Arunachaleswaran *et al.* [3] propose an algorithm ALG that returns a $\frac{1}{3+o(1)}$-envy-free contiguous allocation in polynomial time[1]. We modify it in Algorithm 2 to construct a valid allocation by Proposition 2.6, which admits a feasible location profile of facilities.

Algorithm 2

Require: An instance $(N, \{v_i\}_{i \in N})$.
Ensure: $\frac{1}{12+o(1)}$-envy-free valid allocation $\mathbf{A}'$.
1: Initialize $A'_i = \emptyset$ for $i \in N$.
2: Run ALG in [3] to output a $\frac{1}{3+o(1)}$-envy-free contiguous allocation $\mathbf{A} = (A_1, \dots, A_n)$.
3: Arrange the bundles in an order that $A_{t_1} < A_{t_2} < \cdots < A_{t_n}$ from left to right.
4: Denote the bundles by $A_1 = [0, a_{t_1,1}]$, $A_{t_n} = [a_{t_{n-1},n-1}, 1]$ and $A_{t_k} = [a_{t_{k-1},k-1}, a_{t_k,k}]$ for $k = 2, \dots, n-1$.
5: Run Line 9-22 of Algorithm 1.
6: **return** $\mathbf{A}' = (A'_1, \dots, A'_n)$

Theorem 3.5. *For any n-facility instance of FLCD, Algorithm 2 returns a $\frac{1}{12+o(1)}$-envy-free valid allocation in polynomial time.*

Proof. The running time of ALG in [3] and Algorithm 1 are both polynomial. So Algorithm 2 runs in polynomial time. Let $\mathbf{A} = (A_1, \dots, A_n)$ and $\mathbf{A}' = (A'_1, \dots, A'_n)$ be the allocations returned by ALG in [3] and Algorithm 2, respectively. Then $v_i(A_i) \geq \frac{v_i(A_j)}{3+o(1)}$ for any $i, j \in N, i \neq j$. Noting that Line 3-5 of Algorithm 2 is an application of Proposition 2.6, we have $v_i(A'_i) \geq v_i(A_i)/2$ for any $i \in N$. For allocation $\mathbf{A}$, let A_{i^*} be the bundle with highest valuation to facility i, i.e., $v_i(A_{i^*}) = \max_{j \in N} v_i(A_j)$. By the construction of

[1]Though in their Theorem 2 they just claim that ALG works efficiently for any instance with "piecewise-constant valuations", this result still holds when discarding the restriction and adopting the Robertson-Webb model we are using, because this piecewise-constant restriction is only used to explicitly give the valuations as input, whereas in the Robertson-Webb model which supports oracle access to the valuations this is no longer an obstacle.

$\mathbf{A}'$, each bundle A'_j is contained in the union of A_j and a neighbor bundle of A_j, which indicates that $v_i(A'_j) \leq v_i(A_j) + v_i(A_{i^*})$. Then

$$\begin{aligned} v_i(A'_i) \geq \frac{1}{2} v_i(A_i) &= \frac{1}{4}(v_i(A_i) + v_i(A_i)) \\ &\geq \frac{1}{4} \cdot \frac{1}{3 + o(1)} \cdot (v_i(A_j) + v_i(A_{i^*})) \\ &\geq \frac{1}{12 + o(1)} v_i(A'_j), \end{aligned}$$

which implies that allocation $\mathbf{A}'$ is $\frac{1}{12+o(1)}$-envy-free. □

4 EFFICIENCY

In this section, we study the problem of finding valid allocations from the perspective of efficiency, that is, maximizing the utilitarian or egalitarian social welfare. We first present NP-hardness results for both types of welfare, and then show that there is a $4+o(1)$ approximation for maximizing the utilitarian welfare. Here, the facilities' valuation functions are not necessarily have to be normalized over the interval $[0, 1]$.

Theorem 4.1. *The problem of finding a valid allocation maximizing the utilitarian social welfare is NP-hard, even if the valuation functions are piecewise-uniform.*

Proof. We reduce from EXACT-3-COVER (X3C), which is an NP-complete problem [21]. An instance of X3C is given by $I = (X, \mathcal{T})$, where $X = \{x_1, \ldots, x_{3s}\}$ is a set of elements, and $\mathcal{T} = \{T_1, \ldots, T_r\}$ is a family of 3-element subset of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathcal{T}$, i.e., each element in X is covered by exactly one of the s sets. For a set $T \in \mathcal{T}$, order the three elements of T in some canonical way (e.g., alphabetically) and write T^1, T^2, T^3 for the elements in that order.

Consider an instance $I = (X, \mathcal{T})$ of X3C, where the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathcal{T}$. We construct an instance of our problem as follows. There are three subintervals y_T^1, y_T^2, y_T^3 for each set $T \in \mathcal{T}$, and $r-1$ dummy subintervals $D = \{d_1, d_2, \ldots, d_{r-1}\}$. All of these $m = 4r-1$ subintervals have the same length of $\delta < \frac{1}{m^3}$, and are pairwise disjoint. The order of these subintervals in the line segment $[0, 1]$ is

$$y_{T_1}^1 < y_{T_1}^2 < y_{T_1}^3 < d_1 < y_{T_2}^1 < y_{T_2}^2 < y_{T_2}^3 < d_2 < y_{T_3}^1 < \cdots < y_{T_r}^3.$$

Let the left endpoint of interval $y_{T_1}^1$ be 0, and the right endpoint of interval $y_{T_r}^3$ be 1. Define a T-set to be $V_T = \{y_T^1, y_T^2, y_T^3\}$ for each $T \in \mathcal{T}$. We arrange the m subintervals in $[0, 1]$ uniformly such that the distance between every two adjacent subintervals in a T-set is ϵ (for example, the distance between the right endpoint of $y_{T_1}^2$ and the left endpoint of $y_{T_1}^3$ is ϵ), and the distance between every dummy interval and its adjacent interval is 3ϵ. Denote by Y the union of these m subintervals.

There are a total of $n = 2s + 2r - 1$ facilities: $r - s$ identical T-type facilities, one facility F_x for each $x \in X$, and one facility F_d for each dummy subinterval $d \in D$. For each facility $i \in N$, we have $v_i(Y) = 1$, and each facility has uniform valuation over each subinterval. For each subinterval y, the valuations are defined as:

$$v_T(y) = \begin{cases} 1 & \text{if } y \notin D \\ 0 & \text{otherwise} \end{cases}$$

$$v_x(y) = \begin{cases} 3 & \text{if } y = y_T^k \text{ and } x = x_T^k \text{ for some } k, T \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_d(y) = \begin{cases} L & \text{if } y = d \\ 0 & \text{otherwise} \end{cases}$$

where $L > 10mn$ is a large number. That is, every T-type facility values 2 for each non-dummy subinterval, every x-type facility values 3 for each subinterval corresponding element x, and every d-type facility F_{d_i} has a large value L for the corresponding dummy subinterval $d_i \in D$. Clearly, this instance is constructed in polynomial time.

It is easy to see that, in any optimal allocation, every d-type facility F_{d_i} must receive the dummy subinterval $d_i \in D$, and the bundle of every x-type or T-type facility cannot contain any piece in a dummy subinterval, indicating that $v_x(\mathbf{A}) \leq 3$ and $v_T(\mathbf{A}) \leq 3$ by the contiguous constraint. So the optimal social welfare is at most $L(r-1) + 3 \cdot 3s + 3(r-s)$.

We claim that the X3C instance $I = (X, \mathcal{T})$ has a solution iff the optimal social welfare of a valid allocation is $L(r-1)+3 \cdot 3s+3(r-s)$. This gives a reduction.

Suppose there is an exact cover $\mathcal{T}'$. We can construct a valid allocation: every x-type facility receives a corresponding subinterval in the cover $\mathcal{T}'$, every T-type facility receives a T-set not in the cover $\mathcal{T}'$, and every d-type facility F_{d_i} receives the corresponding dummy subinterval $d_i \in D$. This allocation is valid because we can locate each facility on the midpoint of its bundle, satisfying the proximity rule. The social welfare of this allocation is $L(r-1)+3 \cdot 3s+3 \cdot (r-s)$, and thus it is optimal.

Conversely, suppose the optimal social welfare is $L(r-1) + 9s + 3(r-s)$. Consider an optimal valid allocation $\mathbf{A}$. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the family of sets in which at least one element corresponds to a subinterval assigned to an x-type facility, that is,

$$\mathcal{T}' = \{T \in \mathcal{T} \mid \text{there is a } y_T^k \text{ assigned to some } F_x \text{ with } x = x_T^k\}.$$

Clearly, the total utility of all x-type facilities is at most $3 \cdot 3s = 9s$, and that of all T-type facilities is at most $3(r-s)$. Then, every x-type facility must receive a corresponding subinterval (as otherwise the maximum social welfare is less than $L(r-1) + 9s + 3(r-s)$), and thus it must be $|\mathcal{T}'| \geq s$. If $|\mathcal{T}'| > s$, then at least one T-type facility cannot receive a full T-set, and the total utility of all T-type facilities is less than $3(r-s)$. It indicates that the maximum social welfare is less than $L(r-1) + 9s + 3(r-s)$, a contradiction. Therefore, it must be $|\mathcal{T}'| = s$, which is an exact cover. □

Using a similar analysis, we can prove that maximizing the egalitarian social welfare is also NP-hard.

Theorem 4.2. *The problem of finding a valid allocation maximizing the egalitarian social welfare is NP-hard, even if the valuation functions are piecewise-uniform.*

Proof sketch. Given an arbitrary instance of X3C, construct an instance of the FLCD as in the proof of Theorem 4.1. We can claim that, the X3C instance has a solution if and only if the optimal egalitarian social welfare of a valid allocation is 3. If there is an exact cover, then every x-type facility and T-type facility is able to receive a utility exactly 3 in an optimal allocation. If there is no exact cover, then in an optimal solution there is a T-type facility has a utility less than 3. This establishes the proof. □

Next, we evaluate the performance of an algorithm on the system efficiency in the standard worst-case approximation framework. Formally, given an instance I, let $opt(I)$ be an optimal valid allocation maximizing the utilitarian welfare, and $\mathcal{A}(I)$ be the allocation output by an algorithm $\mathcal{A}$. Say $\mathcal{A}$ is α-approximate for the objective of maximizing the utilitarian welfare if for every instance I,

$$u(opt(I)) \leq \alpha \cdot u(\mathcal{A}(I)).$$

The approximation ratio for the egalitarian welfare is defined analogously.

We note that, for the contiguous cake cutting, Arunachaleswaran *et al.* [3] provide an algorithm with approximation ratio $2 + o(1)$ for maximing the utilitarian welfare. For our problem, using the algorithm in [3] to obtain a preliminary contiguous allocation, we can construct a valid allocation by locating the facilities in a way as in Proposition 2.6.

Theorem 4.3. *For any n-facility instance, there exists a $(4+o(1))$-approximate algorithm that returns a valid allocation in polynomial time under the objective of maximizing the utilitarian welfare.*

Proof. For any instance $I = (N, \{v_i\}_{i\in N})$, let $\mathbf{A} = (A_1, \ldots, A_n)$ be the contiguous allocation output by the algorithm in the proof of Theorem 6 in [3], which guarantees that $(2+o(1))u(\mathbf{A}) \geq u(opt(I))$. Let $\mathbf{A}' = (A'_1, \ldots, A'_n)$ be the valid allocation obtained by applying Proposition 2.6, based on $\mathbf{A}$. Since $v_i(A'_i) \geq \frac{v_i(A_i)}{2}$ for any $i \in N$, we have

$$u(\mathbf{A}') = \sum_{i\in N} v_i(A'_i) \geq \frac{u(\mathbf{A})}{2} \geq \frac{1}{4+o(1)} u(opt(I)),$$

which completes the proof. □

To end this section, we show that our Algorithm 1 has a bad performance guarantee on both types of social welfare. By Theorem 3.3, Algorithm 1 achieves an egalitarian welfare at least $\frac{1}{2n}$, and thus a utilitarian welfare at least $\frac{1}{2}$. Note that for any instance, the egalitarian and utilitarian welfare are at most 1 and n. So the approximation ratio of Algorithm 1 is $2n$ for both types of social welfare. In the following we give an example to show the bad performance on egalitarian welfare.

Example 4.4. Consider an instance with n facilities. The valuation function of facility 1 satisfies $v_1(0, 0.1) = \frac{1}{n}$, $v_1(0.1, 0.9) = 0$ and $v_1(0.9, 0.9 + \frac{1}{10n}) = \frac{n-1}{n}$. For $i = 2, \ldots, n$, the valuation function of facility i satisfies $v_i(0.9 + \frac{i-1}{10n}, 0.9 + \frac{i}{10n}) = 1$. It is not hard to see that, the allocation induced by location profile $\mathbf{x}^* = (x^*_1, \ldots, x^*_n)$ with $x^*_i = 0.9 + \frac{2i-1}{20n}$ for $i \in N$ guarantees that every facility has a utility 1, and thus the optimal egalitarian welfare is 1. However, Algorithm 1 locates facility 1 at point 0.1, and the utility of facility 1 and the egalitarian welfare is merely $\frac{1}{n}$.

5 PRICE OF FAIRNESS

In this section, we measure the efficiency loss under a fair allocation by the price of fairness. We study two kinds of concepts on price of fairness: best price and worst price, which compare the social welfare of an optimal solution to that of the best/worst fair solution.

Given an instance $I = (N, \{v_i\}_{i\in N})$, its *worst utilitarian price of fairness* w.r.t. criterion F (proportionality or envy-freeness) is defined as the ratio of the utilitarian social welfare of the optimal valid allocation over that of the worst valid allocation satisfying criterion F. Formally,

Definition 5.1. *Let I be an instance, X be the set of all valid allocations, and $X_F \subseteq X$ be the set of valid allocations satisfying criterion F. If $X_F \neq \emptyset$, the* worst utilitarian price of fairness *for instance I w.r.t. criterion F is*

$$\bar{P}^u_F(I) = \frac{\sup_{\mathbf{A}\in X} u(\mathbf{A})}{\inf_{\mathbf{A_F}\in X_F} u(\mathbf{A_F})}.$$

The (overall) worst utilitarian price of fairness w.r.t. criterion F is the supremum over all instances. That is,

$$\bar{P}^u_F = \sup_{I\in\mathcal{I}_n} \bar{P}^u_F(I).$$

As is commonly done, the price of fairness is not defined when there is no valid allocation satisfying criterion F for instance I. The *worst egalitarian price of fairness* $\bar{P}^e_F$ is defined analogously.

We are the first to study the worst price of fairness in FLCD, and derive lower and upper bounds on the worst price of fairness w.r.t. proportionality and envy-freeness, respectively.

Theorem 5.2. *For the FLCD, the worst utilitarian price of proportionality $\bar{P}^u_{pr}$ is in the interval $[n - \frac{1}{n}, n]$.*

Proof. *Upper bound.* Consider an arbitrary instance I. For any proportional valid allocation $\mathbf{A}$, the utility of each facility is at least $\frac{1}{n}$, and thus the utilitarian welfare is $u(\mathbf{A}) \geq 1$. As the optimal utilitarian welfare is at most n, it follows that $\bar{P}^u_{pr} \leq n$.

Lower bound. Consider an n-facility instance, where the valuation density function f_i of each facility $i \in N$ is defined as follows. For facility $i = 1, \ldots, n-1$,

$$f_i(x) = \begin{cases} 2n, & \text{if } x \in [\frac{2i-1}{2n}, \frac{2i}{2n}] \\ 0, & \text{otherwise} \end{cases}$$

For facility n,

$$f_n(x) = \begin{cases} 2(n-1), & \text{if } x \in [0, \frac{1}{2n}] \\ 2, & \text{if } x \in [\frac{2n-1}{2n}, 1] \\ 0, & \text{otherwise} \end{cases}$$

First, consider a valid allocation $\mathbf{A}$ induced by location profile $\mathbf{x} = (x_1, \ldots, x_n)$, where each facility $i \in N \setminus \{n\}$ is located at the point $x_i = \frac{i}{n}$, and facility n is located at the point $x_n = 0$. That is, $A_i = [\frac{2i-1}{2n}, \frac{2i+1}{2n}]$, for any $i \in N \setminus \{n-1, n\}$, $A_{n-1} = [\frac{2n-3}{2n}, 1]$, and $A_n = [0, \frac{1}{2n}]$. It indicates that each facility $i \in N \setminus \{n\}$ will obtain a utility 1, and facility n will obtain a utility $1 - \frac{1}{n}$ under the allocation $\mathbf{A}$. Then the utilitarian welfare of $\mathbf{A}$ is $u(\mathbf{A}) = n - \frac{1}{n}$, which means that the optimal utilitarian welfare is at least $n - \frac{1}{n}$.

Next construct a proportional valid allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$ induced by location profile of facilities $\mathbf{x}'$, where each facility $i \in N$ is located at the point $x'_i = \frac{i-1}{n} + \frac{1}{2n^2}$. That is, $A'_1 = [0, \frac{1}{2n} + \frac{1}{2n^2}]$, $A'_n = [\frac{2n-3}{2n} + \frac{1}{2n^2}, 1]$, and $A'_i = [\frac{2i-3}{2n} + \frac{1}{2n^2}, \frac{2i-1}{2n} + \frac{1}{2n^2}]$ for $i = 2, \ldots, n-1$. It follows that every facility has utility $\frac{1}{n}$ under the allocation $\mathbf{A}'$. Thus $\mathbf{A}'$ is proportional, and the utilitarian social welfare is $u(\mathbf{A}') = 1$. Therefore, $\bar{P}^u_{pr} \geq \frac{u(\mathbf{A})}{u(\mathbf{A}')} = n - \frac{1}{n}$. □

Theorem 5.3. *For the FLCD, the worst egalitarian price of proportionality* $\bar{P}^e_{pr} = n$.

Proof. *Upper bound.* For any instance I and any proportional valid allocation $\mathbf{A}$, the egalitarian welfare of $\mathbf{A}$ is at least $\frac{1}{n}$, while the optimal egalitarian welfare is at most 1. This implies that $\bar{P}^e_{pr} \le n$.

Lower bound. Consider an n-facility instance, where the valuation density function f_i of each facility $i \in N$ is defined as

$$f_i(x) = \begin{cases} 2n, & \text{if } x \in [\frac{2i-1}{2n}, \frac{2i}{2n}] \\ 0, & \text{otherwise} \end{cases}$$

First, consider a valid allocation $\mathbf{A} = (A_1, \dots, A_n)$ induced by location profile $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i = \frac{2i-1}{2n}$ for each $i \in N$. That is, $A_i = [\frac{i-1}{n}, \frac{i}{n}]$, and each facility i obtains a utility 1 under $\mathbf{A}$. Then the egalitarian social welfare is $u(\mathbf{A}) = 1$, which is also optimal.

Then construct an proportional valid allocation $\mathbf{A}' = (A'_1, \dots, A'_n)$, induced by location profile $\mathbf{x}'$, where each facility $i \in N$ is located at $x_i = \frac{i-1}{n} + \frac{1}{2n^2}$. Under $\mathbf{A}'$, each facility $i \in N \setminus \{n\}$ obtains a utility $\frac{1}{n}$ and facility n obtains a utility 1. Then $\mathbf{A}'$ is proportional, and the egalitarian social welfare of $\mathbf{A}'$ is $eg(\mathbf{A}') = \frac{1}{n}$. Therefore, $\bar{P}^e_{pr} \ge \frac{eg(\mathbf{A})}{eg(\mathbf{A}')} = n$. □

Theorem 5.4. *For the FLCD, the worst utilitarian price of envy-freeness* $\bar{P}^u_{ef}$ *is in the interval* $[\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2} + 1 - o(1)]$, *and the worst egalitarian price of envy-freeness* $\bar{P}^e_{ef}$ *is in the interval* $[\frac{n}{2}, n]$.

Proof. The bounds for $\bar{P}^u_{ef}$ can be obtained by the proof of Theorem 2.1 in [4]. Next we only show the lower bound of $\bar{P}^e_{ef}$. Consider the instance constructed in the proof of Theorem 2.4 [4], where each facility $i \in N \setminus \{n\}$ has a valuation density function f_i:

$$f_i(x) = \begin{cases} \frac{1/2+\epsilon}{2\epsilon}, & \text{if } x \in [\frac{i}{n} - \epsilon, \frac{i}{n} + \epsilon] \\ \frac{1/2-\epsilon}{2\epsilon}, & \text{if } x \in [1 - \frac{2i+1}{2n} - \epsilon, 1 - \frac{2i+1}{2n} + \epsilon] \\ 0, & \text{otherwise} \end{cases}$$

and facility n has $f_n(x) = 1$ for any $i \in [0, 1]$. Assume that n is odd, then consider a valid allocation induced by location profile $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i = \frac{i}{n}$ for $i = 1, \cdots, \frac{n-1}{2}$, $x_i = 1 - \frac{2i+1}{2n}$ for $i = \frac{n-1}{2} + 1, \cdots, n-1$, and $x_n = \frac{n-1}{2n} + 2\epsilon$, in which, each facility obtains a utility at least $1/2 - \epsilon$. However, in any envy-free valid allocation, facility n has a utility less than $1/n + 2\epsilon$, which implies that $\bar{P}^e_{ef} \ge \frac{1/2-\epsilon}{1/n+2\epsilon} \to \frac{n}{2}$ as ϵ approaches zero. □

We also consider the well-studied measurement: *best price of fairness*, which is defined as the ratio of the social welfare of an optimal valid allocation over that of the best valid allocation satisfying some fairness criterion. Formally, for an instance I, if it admits a valid allocation satisfying criterion F, then its *best utilitarian price of fairness* w.r.t. criterion F is

$$P^u_F(I) = \frac{\sup_{\mathbf{A} \in X} u(\mathbf{A})}{\sup_{\mathbf{A_F} \in X_F} u(\mathbf{A_F})},$$

and the (overall) best utilitarian price of fairness w.r.t. criterion F is

$$P^u_F = \sup_{I \in \mathcal{I}_n} P^u_F(I).$$

The *best egalitarian price of fairness* P^e_F is defined analogously.

Aumann and Dombb [4] derive the upper bounds and lower bounds on the best price of proportionality and envy-freeness for contiguous cake cutting. In their problem, every instance admits proportional and envy-free contiguous allocations, and thus the price of proportionality is no more than the price of envy-freeness. However, this is not true in our problem, because the price of fairness may be not defined for some instances. We also note that partial results in [4] are applicable to our problem, since we can define feasible location profiles of facilities in the instances constructed in their proof to obtain valid allocations. Motivated by these, we present the following results.

Theorem 5.5. *For the FLCD, the best utilitarian price of proportionality* P^u_{pr} *is in the interval* $[\frac{\sqrt{n}}{2}, n-1+\frac{1}{n}]$, *and the best egalitarian price of proportionality* P^e_{pr} *is* 1.

Theorem 5.6. *For the FLCD, the best utilitarian price of envy-freeness* P^u_{ef} *is in the interval* $[\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2} + 1 - o(1)]$, *and the best egalitarian price of envy-freeness* P^e_{pr} *is* $\frac{n}{2}$.

6 CONCLUSION

This paper is devoted to the problem of fairly locating facilities and assigning customers, who are continuously distributed on a line, to the facilities. We consider two fairness criteria of proportionality and envy-freeness, and provide upper and lower bounds on their multiplicative approximation guarantees. Compared with the contiguous cake cutting, the existence of an approximately fair valid allocation that admits a feasible location profile of facilities is much harder to be guaranteed.

Our work opens up a number of new directions for future research. The first one is to narrow the gaps between upper and lower bounds on the approximation guarantees for proportionality and envy-freensss. Second, while the existence of a proportional/envy-free valid allocation is not guaranteed, the hardness of the problem of determining the existence of a proportional/envy-free valid allocation is still unknown. In addition, there are some other fairness criteria (e.g., equitability[8]) and other objectives (e.g., Nash social welfare [3]) deserving to be studied. One can also generalize this model to a graph [8, 26] or two-dimensional space [17], rather than a line segment.

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