Quantified Announcements and Common Knowledge

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ABSTRACT

Public announcement logic (PAL) extends multi-agent epistemic logic with dynamic operators modelling the effects of public communication. Allowing quantification over public announcements lets us reason about the *existence* of an announcement to reach a certain epistemic goal. Two notable examples of logics of quantified announcements are arbitrary public announcement logic (APAL) and group announcement logic (GAL). The notion of common knowledge plays an important role in PAL, and in particular in characterisations of epistemic states that an agent or a group of agents might make come about by performing public announcements. In this paper, we study extensions of APAL and GAL with common knowledge, which has not been done before. We consider both conservative extensions where the semantics of the quantifiers is not changed, as well as extensions where the scope of quantification also includes common knowledge formulas. We compare the expressivity of these extensions relative to each other and other connected logics, and provide sound and complete axiomatisations.

KEYWORDS

Common Knowledge; Group Announcement Logic; Arbitrary Public Announcement Logic; Dynamic Epistemic Logic; Modal Logic

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1 INTRODUCTION

Common knowledge has played an important part in reasoning about knowledge distribution in the multi-agent setting [15]. It has also been used in epistemic planning [24], machine learning [29], game theory [25], and so on. Informally, common knowledge of some fact φ is usually defined as 'everybody knows that φ, everybody knows that everybody knows that φ, and so on'.

A formalism used in [15], *epistemic logic* (EL) *with common knowledge* (ELC), provides a static description of knowledge in a multi-agent system. Logics that are covered by the umbrella term *dynamic epistemic logic* (DEL) [12] study the effects of various epistemic events on the individual and group knowledge of agents. The prime example of such a logic is *public announcement logic* (PAL) [28] that models public communication. A public announcement is an event where all agents publicly and simultaneously receive the same piece of information. The interaction of epistemic events, in particular of public announcements, and common knowledge was studied in [8].

Arbitrary public announcement logic (APAL) [6] and *group announcement logic* (GAL) [1] are extensions of PAL with quantifiers over possible truthful announcements. APAL extends PAL with constructs of the form $\langle ! \rangle \varphi$ that mean 'after *some* public announcement, φ holds'. GAL has quantifiers with a more limited scope, with group announcement operators $\langle G \rangle \varphi$ meaning that '*there exists* a (joint) announcement by agents from (possibly singleton) group G such that φ is true after the announcement'. GAL thus allows us to reason about the *ability* of an agent or a group of agents to achieve their epistemic goal by a joint public announcement.

Common knowledge plays an significant role in PAL, and in particular in characterisations of epistemic states that an agent or a group of agents might make come about by making public announcements. Investigating logics of quantified announcement (or any other quantified epistemic actions) with common knowledge is long overdue, and it was reiterated as an open question in a recent survey [10]. In this paper, we address this problem. First, we study the languages APALC and GALC obtained by extending APAL and GAL, respectively, with common knowledge without changing the semantics of any of the operators. This allows us to gain further insight into the standard APAL and GAL modalities. There is a subtlety here, however, in the scope of quantification. In both APAL and GAL the quantification is restricted to announcements in the purely epistemic language. The reason for this is, in addition to the fact that the quantification does not range over formulas with quantifiers in them to avoid circularity, that EL and PAL are equally expressive [28]. Thus quantifying over EL has the same effect as quantifying over PAL. Adding common knowledge changes the picture, since EL and ELC are not equally expressive. In this paper, in addition to the 'conservative' variants APALC and GALC, we also study variants of APAL and GAL with common knowledge where the quantification ranges over formulas of ELC, called APALC^X and GALC^X (for 'eXtended semantics'), respectively.

In Section 2 we introduce languages of the logics and the corresponding semantics. We investigate some intuitive properties of the interaction between quantified announcements and common knowledge in Section 3. Section 4 is devoted to the study of the relative expressivity of the languages of GALC, GALC^X, APALC, and APALC^X and situating these languages within a broader landscape of EL-based logics. In Section 5 we give sound and complete proof systems for APALC, GALC, APALC^X, and GALC^X. Like existing complete systems for APAL and GAL, these are infinitary. A detailed proof is given for the case of GALC; the other cases follow by relatively simple modifications.

2 LANGUAGES AND SEMANTICS

Let us fix a finite set of agents A.

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Definition 2.1. Given a countable set of propositional variables P, the *language of group announcement logic with common knowledge* $\mathcal{GALC}$ and the *language of arbitrary public announcement logic with common knowledge* $\mathcal{APALC}$ are inductively defined as

$$\mathcal{GALC}(P) \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [G]\varphi$$
$$\mathcal{APALC}(P) \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [!]\varphi$$

where $p \in P$, $a \in A$, and $G \subseteq A$. Whenever P is clear from the context, we omit it. Duals are defined as $\Diamond_a\varphi := \neg\Box_a\neg\varphi$, $\langle\psi\rangle\varphi := \neg[\psi]\neg\varphi$, $\langle!\rangle\varphi := \neg[!]\neg\varphi$, and $\langle G\rangle\varphi := \neg[G]\neg\varphi$.

Formula $\Box_a\varphi$ is read as 'agent a knows φ'; $[\psi]\varphi$ means that 'after the public announcement of ψ, φ will hold'; $\blacksquare_G\varphi$ is read as 'it is common knowledge among agents from group G that φ'; $[G]\varphi$ is read as 'after any public announcement by agents from group G, φ holds'; $[!]\varphi$ is read as 'after any public announcement, φ holds'.

The fragment of $\mathcal{GALC}$ without $[G]\varphi$ is called *public announcement logic with common knowledge* $\mathcal{PALC}$; the latter without $[\varphi]\varphi$ is *epistemic logic with common knowledge* $\mathcal{ELC}$; $\mathcal{PALC}$ and $\mathcal{ELC}$ minus $\blacksquare_G\varphi$ are, correspondingly, *public announcement logic* $\mathcal{PAL}$ and *epistemic logic* $\mathcal{EL}$. Finally, fragments of $\mathcal{GALC}$ and $\mathcal{APALC}$ without $\blacksquare_G\varphi$ are called *group announcement logic* $\mathcal{GAL}$ and *arbitrary public announcement logic* $\mathcal{APAL}$ correspondingly.

'Everyone in group G knows φ' is denoted by $\Box_G\varphi := \bigwedge_{i\in G}\Box_i\varphi$, and $\Box^n_G\varphi$ is defined inductively as $\Box^0_G\varphi := \varphi$ and $\Box^{n+1}_G\varphi := \Box_G\Box^n_G\varphi$ for all natural numbers n.

Definition 2.2. A *model* M is a tuple (S, R, V), where S is a non-empty set of states, $R : A \to 2^{S\times S}$ is an equivalence relation for each agent, and $V : P \to 2^S$ is the valuation function. We will denote model M with a distinguished state s as M_s. Whenever necessary, we refer to the elements of the tuple as S_M, R_M, and V_M.

It is assumed that for group announcements, agents know the formulas they announce. In the following, we write $\mathcal{EL}^G = \{\bigwedge_{i\in G}\Box_i\psi_i \mid$ for all $i \in G, \psi_i \in \mathcal{EL}\}$ (with typical elements ψ_G) to denote the set of all possible announcements by agents from group G.

Definition 2.3. Let $M_s = (S, R, V)$ be a model, $p \in P$, $G \subseteq A$, and $\varphi, \psi \in \mathcal{GALC}(P) \cup \mathcal{APALC}(P)$.

$$\begin{array}{lll}
M_s \models p & \text{iff} & s \in V(p)\\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi\\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi\\
M_s \models \Box_a\varphi & \text{iff} & M_t \models \varphi \text{ for all } t \in S \text{ such that } R(a)(s,t)\\
M_s \models \blacksquare_G\varphi & \text{iff} & \forall n \in \mathbb{N} : M_s \models \Box^n_G\varphi\\
M_s \models [\psi]\varphi & \text{iff} & M_s \models \psi \text{ implies } M^\psi_s \models \varphi\\
M_s \models [!]\varphi & \text{iff} & M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{EL}\\
M_s \models [G]\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for all } \psi_G \in \mathcal{EL}^G
\end{array}$$

where $M^\psi_s = (S^\psi, R^\psi, V^\psi)$ with $S^\psi = \{s \in S \mid M_s \models \psi\}$, $R^\psi(a)$ is the restriction of $R(a)$ to S^ψ for all $a \in A$, and $V^\psi(p) = V(p) \cap S^\psi$ for all $p \in P$.

It is immediate from the semantics that common knowledge of a group consisting of one agent is equivalent to the knowledge of that agent. This fact is characterised by the formula $\blacksquare_{\{a\}}\varphi \leftrightarrow \Box_a\varphi$.

As discussed in the introduction, we now define alternative variants of APAL and GAL extended with common knowledge, where the quantification also ranges over common knowledge. We call these APALC^X and GALC^X, respectively. The languages of the latter are the same as $\mathcal{APALC}$ and $\mathcal{GALC}$ with $[!]\varphi$ and $[G]\varphi$ being substituted by $[!]^X\varphi$ and $[G]^X\varphi$. The critical difference, however, is in the semantics.

Let $\mathcal{ELC}^G = \{\bigwedge_{i\in G}\Box_i\psi_i \mid$ for all $i \in G, \psi_i \in \mathcal{ELC}\}$. Intuitively, $\mathcal{ELC}^G$ is the set of possible group announcements by agents from G that may include common knowledge.

Definition 2.4. The semantics of APALC^X and GALC^X is as in Definition 2.3 with the following modification:

$$\begin{array}{lll}
M_s \models [!]^X\varphi & \text{iff} & M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{ELC}\\
M_s \models [G]^X\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for all } \psi_G \in \mathcal{ELC}^G
\end{array}$$

Note that in a language with *both* types of operators, $[!]^X\varphi \to [!]\varphi$ and $[G]^X\varphi \to [G]\varphi$ would be true in every model.

Definition 2.5. We call formula φ *valid* if and only if for all M_s it holds that $M_s \models \varphi$.

We will also use several notions of bisimulation.

Definition 2.6. Let Q be a set of propositional variables, and $M = (S_M, R_M, V_M)$ and $N = (S_N, R_N, V_N)$ be models. We say that M and N are *Q-bisimilar* if there is a non-empty relation $B \subseteq S_M \times S_N$, called *$Q$-bisimulation* and denoted $M \leftrightarrows_Q N$, such that for all $B(s,t)$, the following conditions are satisfied:

Atoms for all $p \in Q$: $s \in V_M(p)$ if and only if $t \in V_N(p)$,
Forth for all $a \in A$ and $u \in S_M$ such that $R_M(a)(s,u)$, there is a $v \in S_N$ such that $R_N(a)(t,v)$ and $B(u,v)$,
Back for all $a \in A$ and $v \in S_N$ such that $R_N(a)(t,v)$, there is a $u \in S_M$ such that $R_M(a)(s,u)$ and $B(u,v)$.

We say that M_s and N_t are Q-bisimilar and denote this by $M_s \leftrightarrows_Q N_t$ if there is a Q-bisimulation linking states s and t. Also, we omit subscripts Q if $Q = P$.

Theorem 1. Given M_s and N_t, if $M_s \leftrightarrows N_t$, then for all $\varphi \in \mathcal{APALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC} \cup \mathcal{GALC}^X$ we have that $M_s \models \varphi$ if and only if $N_t \models \varphi$.

Proof. The proof is by induction on φ. Propositional, boolean, and epistemic cases are as usual. The case of common knowledge is proven in [12, Theorem 8.35], and the case of public announcements follows from the corresponding result for action models [12, Theorem 6.21]. Finally, the cases of arbitrary and group announcements follow from the fact that public announcements preserve bisimilarity and the induction hypothesis. $\Box$

Note that for the case of Q-bisimulation where $Q \subset P$, Theorem 1 holds only for $\varphi \in \mathcal{PALC}$. The reason this result cannot be extended to a language with quantified announcements is that the quantification is *implicit*, and hence can use propositional variables outside of Q.

If for some M_s and N_t, **Forth** and **Back** can be maintained up to depth $n \in \mathbb{N}$, we say that M_s and N_t are *n-bisimilar* and write $M_s \leftrightarrows^n N_t$. It is a standard result that $M_s \leftrightarrows^n N_t$ implies $M_s \models \varphi$ if and only if $N_t \models \varphi$ for $\varphi \in \mathcal{EL}$ with modal depth less or equal n

(see, e.g, [20]). This does not hold if φ contains either a common knowledge modality or a quantified announcement. In the first case, common knowledge can access a state on an arbitrarily long distance from the origin. In the second case, quantified announcements are not restricted by any modal depth.

3 SHARING COMMON KNOWLEDGE

As one of the main purposes of communication is sharing information, in the context of quantified announcements it is quite natural to ask whether a set of agents can make some fact, which they know, common knowledge among themselves and other agents. We now state a number of results for GALC and APALC, but they do in fact all hold for APALC^X and GALC^X as well.

We start with showing that in general agents cannot make their knowledge common to some other agents neither through their announcements or any announcement at all. In the proof we use the well known Moore sentence (see the extended discussion in the setting of EL in [22]): p is true and agent a does not know this.

Proposition 1. Let $G \neq H$, then there is a φ such that $\Box_G \varphi \rightarrow \langle G \rangle \blacksquare_H \varphi$ and $\Box_G \varphi \rightarrow \langle ! \rangle \blacksquare_H \varphi$ are not valid.

Proof. Immediate for the case when $G = \{b\}$, $H = \{a\}$, and $\varphi := p \land \neg \Box_a p$. □

It is also the case that it is not always possible to share common knowledge of one group with some other group.

Proposition 2. Let $G \neq H$, then there is a φ such that $\blacksquare_G \varphi \rightarrow \langle G \rangle \blacksquare_H \varphi$ and $\blacksquare_G \varphi \rightarrow \langle ! \rangle \blacksquare_H \varphi$ are not valid.

Proof. Follows from Proposition 1 and $\blacksquare_{\{b\}} \varphi \leftrightarrow \Box_b \varphi$. □

We have the next proposition as a corollary with $\psi := p \lor \neg p$.

Proposition 3. There are φ and ψ such that $\blacksquare_G \varphi \land \blacksquare_H \psi \rightarrow \langle G \cup H \rangle \blacksquare_{G \cup H} (\varphi \land \psi)$ and $\blacksquare_G \varphi \land \blacksquare_H \psi \rightarrow \langle ! \rangle \blacksquare_{G \cup H} (\varphi \land \psi)$ are not valid.

Finally, it is not always possible to make group knowledge common even among the members of the group.

Proposition 4. There is a φ such that $\Box_G \varphi \rightarrow \langle G \rangle \blacksquare_G \varphi$ and $\Box_G \varphi \rightarrow \langle ! \rangle \blacksquare_G \varphi$ are not valid.

Proof. Let $G = \{a, b\}$ and $\varphi := \Diamond_a (\Diamond_a p \land \Diamond_b \Box_a \neg p)$, and consider model M_s in Figure 1.

$$
\begin{array}{ccccccc}
 & s & & t & & u \\
M: & p & \text{——}\ a\ \text{——} & \neg p & \text{——}\ b\ \text{——} & \neg p \\
 & s & & t & & s \\
 & p & \text{——}\ a\ \text{——} & \neg p & & p
\end{array}
$$

Figure 1: Model M and some of its submodels.

It is easy to verify that $M_s \models \Box_{\{a,b\}} \varphi$ and at the same time $M_s \not\models \blacksquare_{\{a,b\}} \varphi$ (the rightmost state of the model, u, does not satisfy φ). Now let us consider all updates of M_s depicted in Figure 1. The reader can check that none of the updates satisfy $\blacksquare_{\{a,b\}} \varphi$. Hence, $M_s \not\models \langle G \rangle \blacksquare_G \varphi$ and $M_s \not\models \langle ! \rangle \blacksquare_G \varphi$. □

All the negative results of this section should not come as a surprise. Target formulas in our proof contained modalities expressing that an agent *does not know* something. Achieving an epistemic goal that also requires someone to remain ignorant of some fact is quite tricky in the setting of public communication. Indeed, formulas with negated knowledge modalities are unstable in the sense that providing additional public information may make them false.

However, for many applications in AI and multi-agent systems, having a stable, easily verifiable epistemic goal is enough (see more on this in [11]). Formulas that remain true after public communication are called *positive*, and we show that for positive formulas our intuitions regarding sharing common knowledge are indeed true.

Definition 3.1. The *positive fragment of epistemic logic with common knowledge* $\mathcal{ELC}^+$ is defined by the following BNF:

$$\mathcal{ELC}^+(P) \ \ni\ \varphi^+ ::= p \mid \neg p \mid (\varphi^+ \land \varphi^+) \mid (\varphi^+ \lor \varphi^+) \mid \Box_a \varphi^+ \mid \blacksquare_G \varphi^+$$

where $p \in P$, $a \in A$, and $G \subseteq A$.

The distinctive feature of positive formulas is that they are preserved under submodels, i.e. if φ^+ holds in a model, then φ^+ also holds in all submodels of the model in the same state of evaluation. In particular, this fact implies the following result.

Lemma 1. Let $\varphi^+ \in \mathcal{ELC}^+$, then $[\varphi^+] \blacksquare_G \varphi^+$ is valid for any $G \subseteq A$.

Proof. The proof for the case of common knowledge of the whole set of agents $\blacksquare_A \varphi^+$ can be found in [14], but it can be easily adapted to any $G \subseteq A$. □

Proposition 5. All of the following are valid:

(1) $\Box_G \varphi^+ \rightarrow \langle G \rangle \blacksquare_H \varphi^+$
(2) $\blacksquare_G \varphi^+ \rightarrow \langle G \rangle \blacksquare_H \varphi^+$
(3) $\blacksquare_G \varphi^+ \land \blacksquare_H \psi^+ \rightarrow \langle G \cup H \rangle \blacksquare_{G \cup H} (\varphi^+ \land \psi^+)$
(4) $\Box_G \varphi^+ \rightarrow \langle G \rangle \blacksquare_G \varphi^+$
(5) $\Box_G \varphi^+ \rightarrow \langle ! \rangle \blacksquare_H \varphi^+$
(6) $\blacksquare_G \varphi^+ \rightarrow \langle ! \rangle \blacksquare_H \varphi^+$
(7) $\blacksquare_G \varphi^+ \land \blacksquare_H \psi^+ \rightarrow \langle ! \rangle \blacksquare_{G \cup H} (\varphi^+ \land \psi^+)$
(8) $\Box_G \varphi^+ \rightarrow \langle ! \rangle \blacksquare_G \varphi^+$

Proof. We outline the general idea for proving all of the statements. First, note that formula $\Box_G \varphi^+$ is already in a form of a group announcement by G (also, for the case of common knowledge we have that $\blacksquare_G \varphi^+ \rightarrow \Box_G \varphi^+$). Moreover, $\Box_G \varphi^+$ is positive and holds in the current state of a model. These two facts, in conjunction with Lemma 1, yield $\Box_G \varphi^+ \land [\Box_G \varphi^+] \blacksquare_G \Box_G \varphi^+$. The latter is equivalent to $\langle \Box_G \varphi^+ \rangle \blacksquare_G \Box_G \varphi^+$ due to the validity of $\psi \land [\psi] \varphi \leftrightarrow \langle \psi \rangle \varphi$. Noting that $\blacksquare_G \Box_G \varphi^+ \rightarrow \blacksquare_G \varphi^+$ is valid, we have that $\langle \Box_G \varphi^+ \rangle \blacksquare_G \Box_G \varphi^+$ implies $\langle \Box_G \varphi^+ \rangle \blacksquare_G \varphi^+$. The latter is equivalent to $\langle G \rangle \blacksquare_G \varphi^+$ by the semantics. Finally, $\langle ! \rangle \blacksquare_G \varphi^+$ is implied by $\langle G \rangle \blacksquare_G \varphi^+$. □

Again, all the results above hold for APAL^X and GALC^X as well, substituting the corresponding modalities.

4 EXPRESSIVITY

In the previous section we did not find any explicit distinction between GALC and GALC^X, since all the results were true for both. An interesting question, then, is whether there is any difference in expressive power between GALC and GALC^X, and APALC and APALC^X. In this section we show that they are indeed different, and also situate these languages within a wider context of logics based on EL.

Definition 4.1. Let $\varphi \in \mathcal{L}_1$ and $\psi \in \mathcal{L}_2$. We say that φ and ψ are *equivalent*, if for all M_s: $M_s \models \varphi$ if and only if $M_s \models \psi$.

Definition 4.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. If for every $\varphi \in \mathcal{L}_1$ there is an equivalent $\psi \in \mathcal{L}_2$, we write $\mathcal{L}_1 \leqslant \mathcal{L}_2$ and say that $\mathcal{L}_2$ is *at least as expressive as* $\mathcal{L}_1$. We write $\mathcal{L}_1 < \mathcal{L}_2$ if and only if $\mathcal{L}_1 \leqslant \mathcal{L}_2$ and $\mathcal{L}_2 \not\leqslant \mathcal{L}_1$, and we say that $\mathcal{L}_2$ is *strictly more expressive than* $\mathcal{L}_1$. If $\mathcal{L}_1 \not\leqslant \mathcal{L}_2$ and $\mathcal{L}_2 \not\leqslant \mathcal{L}_1$, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *incomparable.*

It is known from the literature that $\mathcal{EL} < \mathcal{ELC} < \mathcal{PALC}$ [9]. Now we compare these languages to the logics of quantified announcements discussed in this paper..

Theorem 2. $\mathcal{PALC} < \mathcal{GALC}, \mathcal{PALC} < \mathcal{GALC}^X, \mathcal{PALC} < \mathcal{APALC}$, and $\mathcal{PALC} < \mathcal{APALC}^X$.

Proof. The proof is very similar to the one for $\mathcal{PAL} < \mathcal{GAL}$ [1, Theorem 19] and $\mathcal{PAL} < \mathcal{APAL}$ [6, Proposition 3.13] (noting that the models in the proof are $P \setminus \{q\}$-bisimilar), and we do not present it here. □

Theorem 3. Both pairs $\mathcal{ELC}$ and $\mathcal{GAL}$, and $\mathcal{ELC}$ and $\mathcal{APAL}$, are incomparable.

Proof. In one direction, the proof is a straightforward modification of those for $\mathcal{GAL} \not\leqslant \mathcal{EL}$ [1, Theorem 19] and $\mathcal{APAL} \not\leqslant \mathcal{EL}$ [6, Proposition 3.13] .

To see that $\mathcal{ELC} \not\leqslant \mathcal{GAL}$, consider $\blacksquare_{\{a,b\}}\neg p \in \mathcal{ELC}$ and assume that there is an equivalent $\psi \in \mathcal{GAL}$. As ψ is finite, it must have some finite number of symbols n.

Now, let us consider models M and N depicted in Figure 2. Lengths of the models are $n+1$. It is easy to see that $M_s \not\models \blacksquare_{\{a,b\}}\neg p$

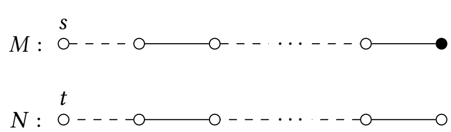

Figure 2: Models M and N. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines. Propositional variable p is true in the black state.

and $N_t \models \blacksquare_{\{a,b\}}\neg p$

To show that $M_s \models \psi$ if and only if $N_t \models \psi$, we use the induction on the size of ψ. Since the models are n-bisimilar, no $\mathcal{EL}$ formula of modal depth n can distinguish M_s and N_t.

Case $\psi := [\chi]\tau$ and for some $m \in \mathbb{N}$, u and v, M_u and N_v are $(n-m)$-bisimilar. There are two possible cases. First, update of M with χ preserves the path to the black state. Then, however, τ has a modal depth of at most $(n-m)-1$, while M_u^χ and N_v^χ are $(n-m)-1$-bisimilar. Second, update with χ may not preserve the path to the black state. In this case the two models become bisimilar, and thus cannot be distinguished by any τ.

Cases $\psi := [G]\chi$ and $\psi := \langle ! \rangle \chi$ are like the previous one. □

We have the following two theorems as corollaries, noting that $\blacksquare_{\{a,b\}}\neg p \in \mathcal{PALC}, \mathcal{GALC}, \mathcal{APALC}$.

Theorem 4. Both pairs $\mathcal{PALC}$ and $\mathcal{GAL}$, and $\mathcal{PALC}$ and $\mathcal{APAL}$, are incomparable.

Theorem 5. $\mathcal{GAL} < \mathcal{GALC}$, $\mathcal{GAL} < \mathcal{GALC}^X$, $\mathcal{APAL} < \mathcal{APALC}$, and $\mathcal{APAL} < \mathcal{APALC}^X$.

Now we turn to the question of the relative expressivity of $\mathcal{GALC}$ and $\mathcal{GALC}^X$ (and of $\mathcal{APALC}$ and $\mathcal{APALC}^X$). We show in Theorem 6 that there are some properties of models that can be captured by the extended versions of the logics, and cannot be captured by the conservative versions. The main intuition of the proof is that having a finite formula φ we can always assume that there are some propositional variables that are not in φ. At the same time, as the quantification is implicit, we still quantify over formulas that contain those variables.

Theorem 6. $\mathcal{GALC}^X \not\leqslant \mathcal{GALC}$ and $\mathcal{APALC}^X \not\leqslant \mathcal{APALC}$.

Proof. Let $\varphi := p \wedge \Diamond_b(\neg p \wedge \Box_a \Diamond_b p) \wedge \Diamond_b(\Diamond_a \Box_b \neg p \wedge \Box_a(\neg \Diamond_b p \to \Box_b \Diamond_a \Diamond_b p))$, and $\langle \{c\} \rangle^X \varphi \in \mathcal{GALC}^X$. Assume that there is a $\psi \in \mathcal{GALC}$ that is equivalent to $\langle \{c\} \rangle^X \varphi$. Since ψ has a finite number of symbols, there must be a $q \in P$ such that q does not occur in ψ.

Now consider models M and N in Figure 3. In both of the models, there are chains starting from s and t correspondingly of length $n+2$ for each $n \in \mathbb{N}$. Chains end with numbered states. In model N there is also an infinite *vertical* chain starting from state u. Propositional variable p is true in s and t, and q is true in numbered states at the ends of finite chains.

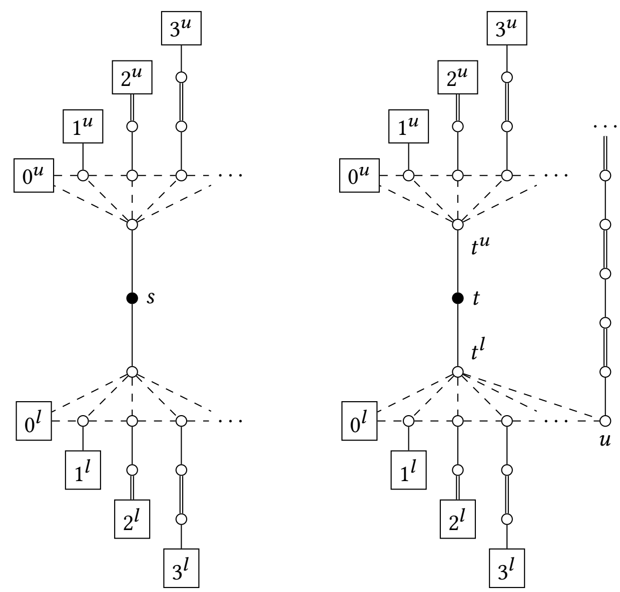

Figure 3: Models M (left) and N (right). Relation for agent a is depicted by dashed lines, relation b is shown by solid lines, and c's relations are double lines. Propositional variable p is true in black states and q is true in numbered states.

Let us examine formula $\langle \{c\} \rangle^X \varphi$. In order to see that $N_t \models \langle \{c\} \rangle^X \varphi$, consider the following c-announcement: $\psi_c := \Box_c(\neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p) \wedge q \to \Box_a \neg \blacksquare_{\{b,c\}} \neg q)$. Since the quantification over announcements of c is implicit, we can use announcements with q. Also note that this announcement belongs to $\mathcal{ELC}^G$. In model N, formula $\blacksquare_{\{b,c\}}\neg q$ is true only in states t, t^u, t^l, and all states of the infinite vertical chain including u.

We argue that the result of updating N with the c-announcement is presented in Figure 4. It is easy to check that $O_t \models \varphi$.

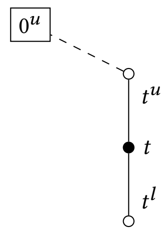

Figure 4: Submodel O of model N.

Pick any non-zero numbered state, i.e. let $n^* \in \{n^* \mid n \in \mathbb{N} \setminus \{0\}$ and $* \in \{u, l\}\}$. We have that $N_{n^*} \not\models \neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p)$ as p is true only in the black state and thus cannot be reached by b, and there is always either a b- or c-arrow to a neighbour circle node with $\neg q$. Hence, $N_{n^*} \not\models \psi_c$. Now consider state 0^l: it holds that $N_{0^l} \not\models q \to \Box_a \neg \blacksquare_{\{b,c\}} \neg q$ since there is an a-arrow to state u and $N_u \models \blacksquare_{\{b,c\}} \neg q$. On the other hand, there are no a-arrows from 0^u to states where $\blacksquare_{\{b,c\}} \neg q$ holds, since each reachable finite chain ends with a q-state. It is left to check that $N_{0^u} \models \neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p)$, and indeed $N_{0^u} \models \Box_{\{b,c\}} q$, and hence $N_{0^u} \models \psi_c$.

Pick any circle state apart from t^u and t^l. To see that $N_\circ \not\models \neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p)$, notice that $N_\circ \models \neg p$, $N_\circ \not\models \Box_{\{b,c\}} q$ (q is false in the current state) and $N_\circ \not\models \Diamond_b p$ (as p is true only in the black state, which is not reachable via b from any white circle state apart from t^u and t^l). So, $N_\circ \not\models \psi_c$. In both t^u and t^l, $\Diamond_b p$ is true and hence the whole formula is true. Finally, we have $N_\bullet \models \psi_c$ vacuously, since $N_\bullet \not\models \neg p$ and $N_\bullet \not\models q$.

To argue that $M_s \not\models \langle \{c\} \rangle^X \varphi$, we note that the upper and lower halves of model M (relative to state s) are bisimilar. Hence, by Theorem 1, there is no formula of $\mathcal{ELC}$ that can be announced by c so that the update is asymmetric similar to O_t: if we preserve a state in one half, we need to preserve the corresponding state in the other half. This implies that all updates of M_s with announcements by c do not yield a model isomorphic to O_t.

To show that no $\mathcal{GALC}$ formula ψ can distinguish M_s and N_t, we can use formula games studied in [16]. Since the definitions related to formula games are quite lengthy and due to the lack of space, we present a sketch of a proof. First, ψ can be equivalently rewritten into a negation-normal form (NNF), where negations appear only in front of propositional variables. Also, recall that q does not appear in ψ. Then, we play a game between two players: the existential player ($\exists$-player) and the universal player ($\forall$-player). The players take turns according to the form of the current subformula of ψ in NNF: the universal player takes 'universal' turns (conjunctions and boxes), and the existential player takes 'existential' turns (propositional variables, disjunctions, and diamonds). The existential player wins if the current state of a model reached in a game satisfies propositional variables specified by ψ, otherwise the universal player wins. Hence, $M_s \models \psi$ if and only if the $\exists$-player has a winning strategy in the game for ψ over M_s.

We play two games simultaneously: one over M_s, and another one over N_t. Now, without loss of generality, assume that $M_s \not\models \psi$ and $N_t \models \psi$. This means that the $\forall$-player has a winning strategy in M_s, and the $\exists$-player has a winning strategy in N_t. The intuition is that at the end of the game we will end up in a pair of states that satisfy the same propositional variables, and thus arrive at a contradiction. For this, we need to maintain an invariant that after k-steps of the game, we are in $(n + 1 - k)$-bisimilar states, where n is the modal depth of ψ.

The case of propositional variables is immediate, and for boolean cases players perform analogous actions in both models, e.g. choose the same conjunct for both models in the case of a conjunction.

For epistemic cases, including common knowledge[1], if a player chooses a successor in M, due to the structure of the models, the player can choose the same successor in N (by 'the same' we mean the state in the same position in N in Figure 3). The similar situation is when a player chooses a successor in N first. The only exception is that it either may be possible to make a move from the current state to u, which is missing in M, or the current state in N already lies on the infinite vertical chain, which is also missing in M.

In the first case, i.e. when one of the players makes a move to u in N for the first time, by the construction of the models, they can make a move to the first state of r^l-chain in M, where r equals to the modal depth of the remaining subformula plus one. In such a way it is guaranteed that the finite chain r^l is long enough to be $r - 1$-bisimiliar to u. In the second case, i.e. when the current state is already on the infinite chain, we note the ordinal number *pos* of the current position on the infinite chain. If $pos > r$, or a move by a player will take us to some pos' such that $pos' > r$, then in M we stay at the current state if it is the last state on r^l-chain, or we move to the last state on r^l-chain otherwise. Intuitively, we play the same moves over the infinite chain and the corresponding finite chain until we reach deeper states on the infinite chain that cannot be matched by the same move on the finite one. However, since the length of the finite chain is r and the remaining modal depth is $r - 1$, and due to the fact that both chains are identically constructed, it is enough to stay in the last state of r^l-chain.

Suppose a player makes a public announcement move, i.e. they choose subsets of the states in the two models. Observe that the models are constructed in such a way that if for some chain r^u or r^l its depth is 'trimmed' in the update to some m, $m < r$, then all chains in the update of the model are trimmed to m. Moreover, trimming the infinite chain to depth m, makes the updated N bisimilar to M trimmed to depth m. If an update does not include the black state, then, choosing the corresponding subset in the other model, both models become $P \setminus \{q\}$-bisimilar to a single-white-state model.

Finally, group announcements are treated in a similar fashion to public announcements. The only twist is that now a formula $\psi_G \in \mathcal{EL}^G(P)$ chosen by a player can include q. First, notice that for each state v in M there is a corresponding state w in N at

[1]Even though games in [16] did not include moves corresponding to common knowledge, it is quite straightforward to expand the definition of a game to include them. For the box version, the $\forall$-player chooses a successor reachable via a sequence of relations marked by agents from the corresponding group. Similarly for the diamond version and the $\exists$-player.

the same position such that $M_v \leftrightarrows^n N_w$ for all $n \in \mathbb{N}$ (a formal argument would be similar to [12, Lemma 8.14]). The only special case to consider is states on the infinite chain in N. Without loss of generality, let us pick u. This means that in model M we are already on a finite chain of sufficient depth. If a player chooses an announcement without q, she chooses the same announcement in the other model. Now, assume that q is in ψ_G. Depending on the resulting updated model, we can construct a χ_G that will have the same effect. If ψ_G is such that a model is trimmed to some finite depth k, then we can use a formula of modal depth $k+2$ for the same effect. For example, to trim a model to depth 2, we may use $\Diamond_c \Diamond_b \Diamond_a \Diamond_b p$ in χ_G (see [16] for more intuitions on this). We can treat similarly the updates when ψ_G 'cuts out' several chains up to some chain r. For example, to cut out chains up to depth 2, we may use $\neg(\Box_{\{b,c\}} q \vee (\Diamond_b \Box_c q \wedge \Diamond_b \Diamond_a \Diamond_b p))$ in χ_G. Combining these two approaches we can obtain a χ_G for all other intermediate updates.

As a result of these two simultaneous games over M_s and N_t we end up in states where the $\exists$-player (resp. the $\forall$-player) has a winning strategy. This contradicts the assumption that the $\forall$-player (resp. the $\exists$-player) has a winning strategy in the other model, or, equivalently, it contradicts the fact that $M_s \not\models \psi$ iff $N_t \models \psi$.

The proof of $\mathcal{APALC}^X \not\leqslant \mathcal{APALC}$ is quite similar. Let $\langle ! \rangle^X \varphi \in \mathcal{APALC}^X$. We have that $M_s \not\models \langle ! \rangle^X \varphi$ exactly for the same reason as $M_s \not\models \langle \{c\} \rangle^X \varphi$: upper and lower halves of M are bisimilar. Above we showed that $N_t \models \langle \psi_c \rangle^X \varphi$. As $\psi_c \in \mathcal{EL}^X$, $N_t \models \langle \psi_c \rangle^X \varphi$ implies that $N_t \models \langle ! \rangle \varphi$. In order to see that no $\psi \in \mathcal{APALC}$ can distinguish M_s and N_t we can apply the same formula game reasoning as for the case of $\mathcal{GALC}$ substituting $\mathcal{EL}^G$ with $\mathcal{EL}$. □

We leave the other direction of Theorem 6 as an open question, and conjecture that both pairs $\mathcal{GALC}$ and $\mathcal{GALC}^X$, and $\mathcal{APALC}$ and $\mathcal{APALC}^X$ are incomparable. We also leave as an open problem the relative expressivity of $\mathcal{GALC}$ and $\mathcal{APALC}$. Taking into account that $\mathcal{APAL}$ and $\mathcal{GAL}$ are incomparable [16, Theorem 5.6], we conjecture that $\mathcal{GALC}$ and $\mathcal{APALC}$ are incomparable as well. The expressivity map of $\mathcal{GALC}$, $\mathcal{APALC}$, and other connected logics is shown in Figure 5.

5 PROOF SYSTEM

In this section we start with the presentation of a proof system of GALC and a detailed completeness proof for it. We then discuss how both are modified to get corresponding results for GALC^X, APALC, and APALC^X.

Let us first introduce an auxiliary notion.

Definition 5.1. Let $\varphi \in \mathcal{GALC}$, $a \in A$, $G \subseteq A$, and $\sharp \notin P$. The set of *necessity forms* [19] is defined recursively below:

$$\eta(\sharp) ::= \sharp \mid \varphi \rightarrow \eta(\sharp) \mid \Box_a \eta(\sharp) \mid [\varphi]\eta(\sharp)$$

We will denote the result of replacing of $\sharp$ with φ in a necessity form $\eta(\sharp)$ as $\eta(\varphi)$.

Definition 5.2. The proof system of GALC is the following extension of the proof system of GAL [1]:

$(A0)$ Theorems of propositional logic

$(A1)$ $\Box_a(\varphi \rightarrow \psi) \rightarrow (\Box_a \varphi \rightarrow \Box_a \psi)$

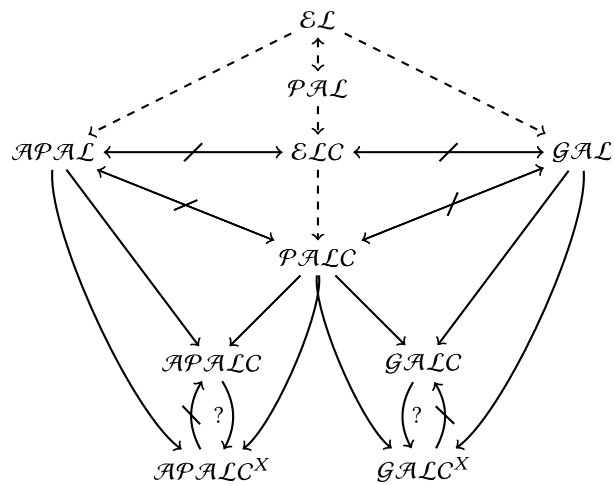

Figure 5: Overview of the expressivity results. An arrow from $\mathcal{L}_1$ to $\mathcal{L}_2$ means $\mathcal{L}_1 \leqslant \mathcal{L}_2$. If there is no symmetric arrow, then $\mathcal{L}_1 < \mathcal{L}_2$. This relation is transitive, and we omit transitive arrows in the figure. An arrow from $\mathcal{L}_1$ to $\mathcal{L}_2$ is crossed-out, if $\mathcal{L}_1 \not\leqslant \mathcal{L}_2$. Arrows marked with the question mark open problems. Dashed arrows depict results known from literature, and solid arrows show the results proven in this paper.

$(A2)$ $\Box_a \varphi \rightarrow \varphi$

$(A3)$ $\Box_a \varphi \rightarrow \Box_a \Box_a \varphi$

$(A4)$ $\neg \Box_a \varphi \rightarrow \Box_a \neg \Box_a \varphi$

$(A5)$ $[\psi]p \leftrightarrow (\psi \rightarrow p)$

$(A6)$ $[\psi]\neg\varphi \leftrightarrow (\psi \rightarrow \neg[\psi]\varphi)$

$(A7)$ $[\psi](\varphi \wedge \chi) \leftrightarrow ([\psi]\varphi \wedge [\psi]\chi)$

$(A8)$ $[\psi]\Box_a \varphi \leftrightarrow (\psi \rightarrow \Box_a[\psi]\varphi)$

$(A9)$ $\blacksquare_G \varphi \rightarrow \Box_G^n \varphi$ for any $n \in \mathbb{N}$

$(A10)$ $[G]\varphi \rightarrow [\psi_G]\varphi$ for any $\psi_G \in \mathcal{EL}^G$

MP From $\varphi \rightarrow \psi$ and φ, infer ψ

NK From φ, infer $\Box_a \varphi$

NA From φ, infer $[\psi]\varphi$

IC From $\{\eta(\Box_G^n \varphi) \mid n \in \mathbb{N}\}$, infer $\eta(\blacksquare_G \varphi)$

IG From $\{\eta([\psi_G]\varphi) \mid \psi_G \in \mathcal{EL}^G\}$, infer $\eta([G]\varphi)$.

We call GALC the minimal set that contains axioms $A0$–$A10$ and is closed under MP, NK, NA, IC, and IG.

Like existing complete systems of APAL and GAL [6, 13], this proof system of GALC is infinitary as it has inference rules that require an infinite number of premises. Note that one of them is the infinitary rule for common knowledge, which is less standard than the usual fixed point approach (see, for example, [8], and also [21] for an alternative axiomatisation of ELC). In an already infinitary system, this treatment is both more intuitive and technically simpler. The infinitary approach to common knowledge has also been discussed in [3], where the authors consider a corresponding Gentzen-type system.

Lemma 2. *IC* and *IG* are truth preserving.

Proof. The proof is a straightforward induction on necessity forms with the application of the definition of semantics. □

Necessitation rules for common knowledge and group announcements are derivable in GALC.

Lemma 3. Rules 'From φ, infer $\blacksquare_G\varphi$' and 'From φ, infer $[G]\varphi$' are derivable in GALC.

Theorem 7. GALC is sound.

Proof. Due to the soundness of GAL, Lemma 2, and the validity of $(A9)$. □

In order to prove the completeness, we adapt the completeness proof of APAL from [5–7].

Whenever we will use induction on the formula structure of some $\varphi \in \mathcal{GALC}$, we will use the following measure.

Definition 5.3. Let $\varphi \in \mathcal{GALC}$. The *quantifier depth* $\delta_\forall(\varphi)$ of φ is defined inductively as

$$\delta_\forall(p) = 0 \qquad \delta_\forall([\psi]\varphi) = \delta_\forall(\psi) + \delta_\forall(\varphi)$$
$$\delta_\forall(\neg\varphi) = \delta_\forall(\Box_a\varphi) = \delta_\forall(\varphi) \qquad \delta_\forall(\blacksquare_G\varphi) = \delta_\forall(\varphi)$$
$$\delta_\forall(\varphi \wedge \psi) = \max(\delta_\forall(\varphi), \delta_\forall(\psi)) \qquad \delta_\forall([G]\varphi) = \delta_\forall(\varphi) + 1$$

The $\blacksquare$*-depth* $\delta_\blacksquare(\varphi)$ of φ is defined similarly to the quantifier depth $\delta_\forall$ with the following exceptions:

$$\delta_\blacksquare([G]\varphi) = \delta_\blacksquare(\varphi) \qquad \delta_\blacksquare(\blacksquare_G\varphi) = \delta_\blacksquare(\varphi) + 1$$

The *complexity* $c(\varphi)$ of φ is

$$c(p) = 1 \qquad c([\psi]\varphi) = c(\psi) + 3 \cdot c(\varphi)$$
$$c(\neg\varphi) = c(\Box_a\varphi) = c(\varphi) + 1 \qquad c(\blacksquare_G\varphi) = c(\varphi) + 1$$
$$c(\varphi \wedge \psi) = \max(c(\varphi), c(\psi)) + 1 \qquad c([G]\varphi) = c(\varphi) + 1$$

Let $\varphi, \psi \in \mathcal{GALC}$. We have that $\varphi <^\forall_\blacksquare \psi$ if and only if $\delta_\forall(\varphi) < \delta_\forall(\psi)$, or, otherwise, $\delta_\forall(\varphi) = \delta_\forall(\psi)$, and either $\delta_\blacksquare(\varphi) < \delta_\blacksquare(\psi)$, or $\delta_\blacksquare(\varphi) = \delta_\blacksquare(\psi)$ and $c(\varphi) < c(\psi)$.

Lemma 4. Let $\varphi, \psi, \chi \in \mathcal{GALC}$ and $G \subseteq A$. The following inequalities hold:

$$\varphi <^\forall_\blacksquare \neg\varphi \qquad [\psi]\varphi \wedge [\psi]\chi <^\forall_\blacksquare [\psi](\varphi \wedge \chi)$$
$$\varphi <^\forall_\blacksquare \varphi \wedge \psi \qquad [\psi]\Box^n_G\varphi <^\forall_\blacksquare [\psi]\blacksquare_G\varphi$$
$$\varphi <^\forall_\blacksquare \Box_a\varphi \qquad [\psi][\psi_G]\varphi <^\forall_\blacksquare [\psi][G]\varphi$$
$$p <^\forall_\blacksquare [\psi]p \qquad \Box^n_G\varphi <^\forall_\blacksquare \blacksquare_G\varphi$$
$$\psi \rightarrow \neg[\psi]\varphi <^\forall_\blacksquare [\psi]\neg\varphi \qquad [\psi_G]\varphi <^\forall_\blacksquare [G]\varphi$$

Our completeness proof is based on the canonical model construction. We will use *theories* as states in the canonical model.

Definition 5.4. A set x is called a *theory* if it contains all theorems and is closed under *MP*, *IC*, and *IG*. The smallest theory is GALC. Theory x is *consistent* if there is no $\varphi \in \mathcal{GALC}$ such that $\varphi, \neg\varphi \in x$. Theory x is *maximal* if for all $\varphi \in \mathcal{GALC}$ we have that either $\varphi \in x$ or $\neg\varphi \in x$.

Lemma 5. Let $\varphi, \psi \in \mathcal{GALC}$, if x is a theory, then $x + \varphi := \{\chi \mid \varphi \rightarrow \chi \in x\}$, $\Box_a x := \{\chi \mid \Box_a\chi \in x\}$, and $[\psi]x := \{\chi \mid [\psi]\chi \in x\}$ are theories as well. Also, $x + \varphi$ is consistent if and only if $\neg\varphi \notin x$.

Proof. An extension of the proof of Lemma 4.11 in [6], where common knowledge cases are dealt with using $(A9)$ and *IC*. □

Lemma 6. For all theories x and all $\varphi \in \mathcal{GALC}$, it holds that $x \subseteq x + \varphi$.

Proof. Let us for some $\psi \in \mathcal{GALC}$ have that $\psi \in x$. Since x is a theory and thus contains all the instances of propositional tautologies, $\psi \rightarrow (\varphi \rightarrow \psi) \in x$. As x is closed under *MP*, $\varphi \rightarrow \psi \in x$, and, by Lemma 5, $\psi \in x + \varphi$. □

Next, we prove the Lindenbaum lemma.

Lemma 7. If x is a consistent theory, then it can be extended to a maximal consistent theory y such that $x \subseteq y$.

Proof. The proof is a variation of the Lindenbaum Lemma for APAL [6, Lemma 4.12]. We give here a sketch of an extended proof.

Let $\{\varphi_0, \varphi_1, \ldots\}$ be an enumeration of formulas of $\mathcal{GALC}$, and let $y_0 = x$. Assume that for some $n \geq 0$, $x \subseteq y_n$ and y_n is a consistent theory. If $\neg\varphi_n \notin y_n$, then $y_{n+1} = y_n + \varphi_n$. Otherwise, there are three cases to consider.

First, if $\neg\varphi_n \in y_n$ and φ_n is not of either the form $\eta(\blacksquare_G\psi)$ or the form $\eta([G]\psi)$, then $y_{n+1} = y_n$. Second, if $\neg\varphi_n \in y_n$ and φ_n is of the form $\eta(\blacksquare_G\psi)$, then $y_{n+1} = y_n + \neg\eta(\Box^n_G\psi)$, where $\neg\eta(\Box^n_G\psi)$ is the first formula in the enumeration such that $\eta(\Box^n_G\psi) \notin y_n$. Third, if $\neg\varphi_n \in y_n$ and φ_n is of the form $\eta([G]\psi)$, then $y_{n+1} = y_n + \neg\eta([\psi_G]\psi)$, where $\neg\eta([\psi_G]\psi)$ is the first formula in the enumeration such that $\eta([\psi_G]\psi) \notin y_n$.

In all these cases it is clear that y_{n+1} is consistent. Also, using the inductive construction of y_{n+1}, the fact that $x \subseteq y_{n+1}$, it is relatively straightforward to show that $y = \bigcup_{n=0}^\infty y_n$ is a maximal consistent theory such that $x \subseteq y$. □

Now we are ready to define the canonical model, where states are maximal consistent theories.

Definition 5.5. We call model $\mathfrak{M} = (\mathfrak{S}, \mathfrak{R}, \mathfrak{V})$, where $\mathfrak{S} = \{x \mid x \text{ is a maximal consistent theory}\}$, $\mathfrak{R}(a) = \{(x, y) \mid \Box_a x \subseteq y\}$, and $\mathfrak{V}(p) = \{x \mid p \in x\}$, the *canonical model*.

Next, we prove the truth lemma.

Lemma 8. For all maximal consistent theories x and $\varphi \in \mathcal{GALC}$, $\varphi \in x$ if and only if $\mathfrak{M}_x \models \varphi$.

Proof. Proofs for boolean, epistemic, some of public announcement cases are quite similar to those in [7, Lemma 11], and can be shown using the axioms of GALC and Lemma 4. We show here only the cases that include group announcements and common knowledge.

Induction hypothesis (IH): For all maximal consistent theories y and formulas $\psi \in \mathcal{GALC}$, if $\psi <^\forall_\blacksquare \varphi$, then $\psi \in y$ iff $\mathfrak{M}_y \models \psi$.

Case $\varphi = [\chi]\blacksquare_G\psi$. $(\Rightarrow)$: Suppose that $[\chi]\blacksquare_G\psi \in x$. Since x contains all theorems of GALC, we have for all $n \in \mathbb{N}$, $[\chi](\blacksquare_G\psi \rightarrow \Box^n_G\psi) \in x$ and $[\chi](\blacksquare_G\psi \rightarrow \Box^n_G\psi) \rightarrow ([\chi]\blacksquare_G\psi \rightarrow [\chi]\Box^n_G\psi) \in x$ (Proposition 4.46.3 of [12]). Using *MP* twice, we get $[\chi]\Box^n_G\psi \in x$ for all $n \in \mathbb{N}$. By the IH, this is equivalent to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models [\chi]\Box^n_G\psi$.

The latter is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models \Box_G^n \psi$ for all n. By the semantics of common knowledge we have that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models \blacksquare_G \psi$, and the latter is $\mathfrak{M}_x \models [\chi]\blacksquare_G \psi$ by the semantics of public announcements.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [\chi]\blacksquare_G \psi$. By the semantics, this is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models \blacksquare_G \psi$. By the semantics of common knowledge, the latter is $\forall n \in \mathbb{N} : \mathfrak{M}_x^\chi \models \Box_G^n \psi$. We can 'fold' the public announcement back: $\forall n \in \mathbb{N} : \mathfrak{M}_x \models [\chi]\Box_G^n \psi$. By the IH, $\forall n \in \mathbb{N} : [\chi]\Box_G^n \psi \in x$. Observe that this formula is in a necessity form. Hence, we conclude, by rule *IC*, that $[\chi]\blacksquare_G \psi \in x$.

Case $\varphi = [\chi][G]\tau$. ($\Rightarrow$): Suppose that $[\chi][G]\tau \in x$. Since x contains all theorems of GALC, we have for all $\psi_G \in \mathcal{EL}^G$, $[\chi]([G]\tau \to [\psi_G]\tau) \in x$ and $[\chi]([G]\tau \to [\psi_G]\tau) \to ([\chi][G]\tau \to [\chi][\psi_G]\tau) \in x$ (Proposition 4.46.3 of [12]). Using *MP* twice, we get $[\chi][\psi_G]\tau \in x$ for all $\psi_G \in \mathcal{EL}^G$. By the IH, this is equivalent to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\chi][\psi_G]\tau$. The latter is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models [\psi_G]\tau$ for all $\psi_G \in \mathcal{EL}^G$. By the semantics of group announcements we have that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models [G]\tau$, and the latter is $\mathfrak{M}_x \models [\chi][G]\tau$ by the semantics of public announcements.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [\chi][G]\tau$. By the semantics, this is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}_x^\chi \models [G]\tau$. By the semantics of group announcements, the latter is $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x^\chi \models [\psi_G]\tau$. We can 'fold' the public announcement back: $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\chi][\psi_G]\tau$. By the IH, $\forall \psi_G \in \mathcal{EL}^G : [\chi][\psi_G]\tau \in x$. Observe that this formula is in a necessity form. Hence, we conclude, by rule *IG*, that $[\chi][G]\tau \in x$.

Case $\varphi = \blacksquare_G \psi$. ($\Rightarrow$): Assume that $\blacksquare_G \psi \in x$. By ($A9$), $\forall n \in \mathbb{N} : \Box_G^n \psi \in x$, which is equivalent, by the IH, to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models \Box_G^n \psi$. This is equivalent to $\mathfrak{M}_x \models \blacksquare_G \psi$ by the semantics.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models \blacksquare_G \varphi$. By the semantics, this is equivalent to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models \Box_G^n \varphi$. Furthermore, by the IH, we have $\forall n \in \mathbb{N} : \Box_G^n \varphi \in x$. Since x is closed under *IC*, we have $\blacksquare_G \varphi \in x$.

Case $\varphi = [G]\chi$. ($\Rightarrow$): Assume that $[G]\chi \in x$. By ($A10$), $\forall \psi_G \in \mathcal{EL}^G : [\psi_G]\chi \in x$, which is equivalent, by the IH, to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\psi_G]\chi$. This is equivalent to $\mathfrak{M}_x \models [G]\chi$ by the semantics.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [G]\chi$. By the semantics, this is equivalent to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\psi_G]\varphi$. Furthermore, by the IH, we have $\forall \psi_G \in \mathcal{EL}^G : [\psi_G]\varphi \in x$. Since x is closed under *IG*, we can infer that $[G]\chi \in x$. □

Finally, we can prove the completeness of GALC.

Theorem 8. For all $\varphi \in \mathcal{GALC}$, if φ is valid, then $\varphi \in$ GALC.

Proof. Assume towards a contradiction that φ is valid and $\varphi \notin$ GALC. Since GALC is a consistent theory, it follows that GALC$+\neg\varphi$ is a consistent theory as well. By Lemma 5, there is a maximal consistent theory x such that GALC $+ \neg\varphi \subseteq x$. By Lemma 6, $\neg\varphi \in$ GALC $+ \neg\varphi$, and hence $\neg\varphi \in x$. Since x is a maximal consistent theory, it follows that $\varphi \notin x$. According to Lemma 8, $\varphi \notin x$ is equivalent to $\mathfrak{M}_x \not\models \varphi$, which contradicts φ being valid. □

The proof system of GALC^X is the same as in Definition 5.2 with following differences:

($A10$)$'$ $[G]\varphi \to [\psi_G]\varphi$ for any $\psi_G \in \mathcal{ELC}^G$

IG' From $\{\eta([\psi_G]\varphi) \mid \psi_G \in \mathcal{ELC}^G\}$, infer $\eta([G]\varphi)$.

The completeness proof is exactly as for GALC, with each $[G]$ replaced by $[G]^X$ and $\mathcal{EL}^G$ replaced by $\mathcal{ELC}^G$.

Theorem 9. GALC^X is sound and complete.

The axiomatisation of APALC is the same as the proof system of GALC with the following differences:

($A10$)$'$ $[!]\varphi \to [\psi]\varphi$ for any $\psi_G \in \mathcal{EL}$

IG' From $\{\eta([\psi]\varphi) \mid \psi \in \mathcal{EL}\}$, infer $\eta([!]\varphi)$.

Again, the completeness proof is exactly the same as for GALC, replacing $[G]$ with $[!]$ and each $\mathcal{EL}^G$ with $\mathcal{EL}$.

Theorem 10. APALC is sound and complete.

Finally, the proof system and the completeness of APALC^X can be obtained from those of APALC in the same way as for GALC^X.

Theorem 11. APALC^X is sound and complete.

6 DISCUSSION

We considered common knowledge in the context of quantified announcements. In particular, we studied the extensions of GAL and APAL with the common knowledge modality, both conservative and with the extended semantics. We observed that difference in the semantics matters: with the extended semantics we can express properties we cannot express with the conservative semantics. We conjecture that the same is true the other way around. This echoes the results for GAL extended with distributed knowledge [18]. Moreover, we presented sound and complete axiomatisations of GALC, GALC^X, APALC and APALC^X.

Note that with common knowledge there is a possibility to extend the scope of quantification beyond both EL and ELC to PALC. This is also left for future work. We conjecture that the resulting logics have equal expressive power as APALC^X and GALC^X.

The results for GALC and APALC go hand-in-hand with each other due to the fact that the underlying logics are relatively similar. We omitted from the discussion, however, an interesting cousin of GAL and APAL, *coalition announcement logic* (CAL) [2, 17]. CAL extends PAL with the modality $[\langle G \rangle]\varphi$, meaning 'whatever agents from group G announce, there is a simultaneous counter-announcement by the agents from outside of the group such that φ holds in the resulting model'. Thus, CAL has a game-theoretic flavour to it and is reminiscent of coalition logic [26], alternating-time temporal logic [4], and game logic [27]. Extending CAL with common knowledge seems to be complicated, since finding an axiomatisation of CAL is an open problem. Apart from that, it is worthwhile to investigate the expressivity of CAL with common knowledge.

We also believe that our approach to common knowledge can be extended beyond quantified announcements and be applied to logics with other types of quantified epistemic actions. One of such logics is arbitrary arrow update logic with common knowledge (AAULC) introduced in [23], where the author shows that AAULC is not finitary axiomatisable, but does not provide a proof system.

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