On the Average-Case Complexity of Predicting Round-Robin Tournaments

Extended Abstract

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ABSTRACT

Round-robin tournaments are, besides single-elimination tournaments, by far the most prominent and widely used tournament format in sports and other competitions. We study the average-case complexity of two problems related to the prediction of round-robin tournaments, namely first the problem of calculating the championship probability of a team and second the well-known sports elimination problem where one has to decide whether a team still has the possibility to become champion. We show that, under certain assumptions, these problems are solvable in expected polynomial time for a distribution which, for the algorithm used, seems to dominate the distribution of real instances in terms of complexity, despite their computational worst-case hardness.

KEYWORDS

average-case complexity; round-robin tournaments; predicting; probabilistic tournaments

ACM Reference Format:
Dorothea Baumeister and Tobias Hogrebe. 2022. On the Average-Case Complexity of Predicting Round-Robin Tournaments: Extended Abstract. In *Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), Online, May 9–13, 2022*, IFAAMAS, 3 pages.

1 INTRODUCTION

One of, if not the, most popular type of tournaments in sports and other competitions is the round-robin tournament. In a round-robin tournament, each team plays a fixed number of matches against each other team, for which the teams receive points according to the match outcomes, and the team with the highest score at the end becomes the champion. The key question that excites fans, media, sponsors, and gamblers as well as bookmakers before and during a tournament is: who will emerge as the champion in the end? This great interest has led many to go to great lengths to come up with predictions or at least certain observations limiting the set of possible champions.

Thus, naturally, these problems were also studied from a theoretical point of view. The problem of prediction was studied as early as 1929 by Zermelo [13] for round-robin chess tournaments. His work later led to the development of the Elo rating system, named after his inventor Arpad Elo, in the middle of the last century which is still used today in chess and other sports and which we will later use to derive and justify the distribution considered here. Apart from sports, this problem has also been studied historically in the design of experiments in which participants are successively presented with each pair of so-called treatments and are asked to decide which one they prefer (see, e.g., Trawinski and David [12]).

We deal with the computational aspects of predicting the outcomes of such tournaments. This study was initiated in 1966 by Schwartz [11], who showed that for certain scoring systems, one can decide in polynomial time whether a given team still has a chance to become champion assuming we are at a certain point in the course of a tournament. This problem is commonly referred to as the (sports) elimination problem. Bernholt et al. [3] first showed that for certain scoring systems, including the FIFA 3-point rule, the elimination problem for round-robin tournaments is actually NP-complete even if each team has at most three matches left to play, whereby probably no polynomial-time algorithm exists. Subsequently, the complexity of the problem was extensively studied e.g., by Kern and Paulusma [7] and Cechlárová et al. [4].

The problem of actually predicting the outcome of a tournament in terms of calculating the championship probabilities of a given team given the probability of the outcomes for each remaining match, is referred to as the evaluation problem, which has received significantly less attention regarding its computational complexity. To the best of our knowledge, the evaluation problem was studied in terms of its complexity almost exclusively, by Mattei et al. [9] with respect to different tournament formats, with a follow-up paper by Saarinen et al. [10] showing that the prediction problem is #P-hard for round-robin tournaments in which the winner of a match receives exactly one point. For round-robin tournaments, the hardness result was sharpened by Baumeister and Hogrebe [1]. They show that the hardness is also preserved for the case that only a fixed number of at least three matchdays remains to be played. However, they also showed in experiments, that this hardness is not reflected on real data and artificial data and thus, as often in the case of worst-case analyses, seems to be based on rather unrealistic and highly constructed instances.

Based on this observation, we investigate the average-case complexity of the evaluation problem, and also of the elimination problem, and show that the FPT algorithm presented by Baumeister and Hogrebe [1] indeed has an expected polynomial running time under certain assumptions, regardless of the #P-hardness, or in the case of elimination the NP-hardness, of the problem. To the best of our knowledge, this is the first average-case complexity result in the field of sports prediction problems, which up to now has focused exclusively on worst-case analysis.

There is a long history of average-case analysis of worst-case computational hard problems. For example, Dyer and Frieze [6]

Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), P. Faliszewski, V. Mascardi, C. Pelachaud, M.E. Taylor (eds.), May 9–13, 2022, Online.

studied the average-case complexity of a number of classical decision problems such as k-colorability and partition, Beier and Vöcking [2] studied the Knapsack problem, Krivelevich and Vilenchik [8] studied the 3-SAT problem, and Coja-Oghlan [5] studied the independent set problem. However, a general problem in average-case analysis is to find a distribution that approximates the distribution of real-world instances or at least dominates it from a complexity perspective, which is why we will justify the distribution considered here in detail.

2 PRELIMINARIES

A *single-round round-robin tournament* $\mathcal{T} = (T, M)$ consists of a set of teams $T = \{t_1, \dots, t_n\}$ with even $n \geq 2$ and a set of matches M which contains each pairing of distinct teams exactly once. A round-robin tournament with $K \geq 1$ rounds is a concatenation of K single-round round-robin tournaments over the same set of teams as defined previously. Depending on the sport and the competition, the teams receive points based on the outcome of the matches. We denote the *set of possible outcomes*, often referred to as a scoring system, by $O = \{(\alpha_1, \beta_1), \dots, (\alpha_\ell, \beta_\ell)\}$ with $\ell \geq 2$ and $\alpha_s, \beta_s \in \mathbb{N}_0$ for $1 \leq s \leq \ell$ is given, with at least one $(\alpha_s, \beta_s) \in O$ with $\alpha_s \neq \beta_s$. We call a set of possible outcomes *symmetric*, if it holds that $(\beta_s, \alpha_s) \in O$ for all $(\alpha_s, \beta_s) \in O$. At the end of the tournament, the champion(s) is (are) the team(s) with the highest score. Practical examples include football with $O = \{(3, 0), (1, 1), (0, 3)\}$, also called the FIFA 3-point rule, baseball with $O = \{(1, 0), (0, 1)\}$, basketball with $O = \{(2, 0), (1, 1), (0, 2)\}$, volleyball with $O = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$, and in the context of decision making, the Copeland rule with $O = \{(1, 0), (0, 1)\}$.

The *evaluation problem* for round-robin tournaments with respect to a given set of possible outcomes O is defined as follows.

O-RRT-Evaluation	
Given:	A round-robin tournament $\mathcal{T} = (T, M)$ with $M = M_p \uplus M_o$, the outcomes for all matches in M_p, the outcome probabilities for all matches in M_o, and a distinguished team $p \in T$.
Question:	What is the probability for p to become the unique champion of the tournament?

We refer to the matches in M_p as the played matches, and to the matches in M_o as the open or remaining matches. In the sports elimination problem we only ask if a team can still become champion under the assumption that any outcome for any open match is possible. Note that the evaluation problem is a generalization of the elimination problem whereby worst-case and average-case efficiency results carry over to the elimination problem.

For our study, we consider a uniform distribution over the outcomes (α_1, β_1) and $(\alpha_\ell, \beta_\ell)$ for the played matches for symmetric $O = \{(\alpha_1, \beta_1), \dots, (\alpha_\ell, \beta_\ell)\}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell$, which we assume to be a distribution dominating the real-world distribution of instances with respect to the complexity of the algorithm by Baumeister and Hogrebe [1]. For many problems, such kind of uniform distributions are assumed to be highly unrealistic. In particular, average-case polynomial-time results for NP-hard problems assuming them are not considered very meaningful, since it is generally not assumed that they dominate, or are at least somewhat similar to, more realistic distributions in terms of expected running time. However, this is not the case for the problem and algorithm considered here, which according to Baumeister and Hogrebe [1] is particularly hard for this distribution. However, their experiments showed that even for this distribution the runtime of their algorithm increases relatively slowly w.r.t. the number of teams in the long run, after an initial rapid exponential increase and decrease. In particular, the experiments suggest that instances generated according to this distribution tend to be harder than real-world instances, whereby we assume the distribution to be a well founded proxy for studying the complexity of the problem regarding the distribution of real-world instances.

3 RESULTS

Our key result is that the FPT algorithm presented by Baumeister and Hogrebe [1] solves the evaluation problem in expected polynomial time[1] for single-round round-robin tournaments with outcomes $O = \{(1, 0), (0, 1)\}$ and at most r open matches per team for fixed r, even if we assume that we are given the probabilities of the outcomes over the remaining matches and the distinguished team by the, complexity-wise, adversary and assume that each already played match was decided uniformly at random. Note, that the decision to let the adversary select the distinguished team is in line with the experiments of Baumeister and Hogrebe [1] in which the maximum running time across all teams was considered. This result is in contrast to the worst-case result of Baumeister and Hogrebe [1] who showed the #P-hardness of the problem in this case. Moreover, this result can be easily extended to the case with an arbitrary number of rounds and an arbitrary symmetric set of outcomes as considered in the experiments.

Theorem 3.1. *Given fixed integers $r \geq 0$, $K \geq 1$, and a symmetric $O = \{(\alpha_1, \beta_1), \dots, (\alpha_\ell, \beta_\ell)\}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\ell$, O-RRT-Evaluation for a round-robin tournament with K rounds, and at most r open matches per team can be solved in expected polynomial time assuming a uniform distribution over the outcomes of the played matches w.r.t. (α_1, β_1) and $(\alpha_\ell, \beta_\ell)$.*

As mentioned earlier, this result also carries over to the sports elimination problem which, according to Bernholt et al. [3], is NP-complete in the considered case for the FIFA 3-point-rule and at most $r = 3$ open matches per team.

In terms of future work, the most striking approach would be to derive complexity-wise proxy distributions for further computational problems in sports due to the excellent availability of data and to evaluate the average-case complexity of these problems. For the problems presented here, it would be interesting to examine how a variable number of remaining matchdays, or a number of matchdays bounded by a function of the number of teams n, affects the expected running time and, consequently, the development of average-case hardness results.

ACKNOWLEDGMENTS

This work is supported by the DFG-grant BA6270/1-1.

[1]An algorithm A has *expected polynomial time* on some sequence of distributions $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{N}}$ if $\mathrm{E}_{\mathcal{D}_n}[T_A] = \sum_{|I|=n} \mathrm{Pr}_{\mathcal{D}_n}[I] \cdot T_A(I) \in O(n^k)$ for some fixed $k \in \mathbb{N}$, where $T_A(I)$ denotes the running time of A on instance I.

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