On the Complexity of Two-Stage Majoritarian Rules

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ABSTRACT

Sequential voting rules have been extensively used in parliamentary and legislative decision making. After observing that the prevalent successive rule and the amendment rule fail several fundamental axioms, Horan and Sprumont [2022] proposed very recently a two-stage sequential rule which satisfies a variety of desirable properties. This paper examines this rule by investigating the complexity of Agenda Control, Coalition Manipulation, Possible Winner, Necessary Winner, and eight standard election control problems. Our study offers a comprehensive understanding of the complexity landscape of these problems.

KEYWORDS

two-stage majoritarian rule; successive rule; amendment rule; strategic voting; NP-hard; W[2]-hard

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1 INTRODUCTION

Exploring the complexity of strategic voting problems has been being a vibrant topic in computational social choice (see, e.g., [7, 17, 21, 24, 31]). The motivation is that malicious strategic voting may undermine election results, and it is widely believed that complexity could serve as a barrier against strategic actions [3, 4]. In particular, to what extent a voting rule resists strategic voting has been commonly recognized as a crucial factor to evaluate the applicability of the rule. Over the past three decades, the complexity of many different strategic voting problems under numerous voting rules has been established [5, 20]. Needless to say, as long as a new meritorious voting rule in terms of axiomatic properties has emerged, comparing it with existent rules w.r.t. their resistance degree to strategic voting becomes of great importance.

This paper aims to complete the complexity landscape of several strategic voting problems under a sequential voting rule proposed recently by Horan and Sprumont [25]. Taking as input preferences of voters over candidates and an agenda over candidates (a linear order specifying the priorities of candidates being considered during the decision-making process), a sequential rule outputs one candidate as the winner. Sequential rules are exceedingly useful in parliamentary and legislative decision making. So far, the successive rule and the amendment rule are among the most popular sequential rules used in many countries [32]. However, these rules fail several fundamental axioms from a theoretical point of view. This motivates Horan and Sprumont [25] to study a new rule called *two-stage majoritarian rule (TSMR)*, which has been shown to satisfy a variety of desirable axiomatic properties several of which are failed by other sequential rules.

The work of Horan and Sprumont [25] naturally raises the question of whether the newly proposed rule is comparable to the successive rule and the amendment rule in terms of resistance to strategic voting. This paper aims to answer this question. In addition, we also study two winner determination problems in a scenario where only partial information on voters' preferences are available. Our main contributions are as follows.

(1) We study the Agenda Control problem, which models the scenario where an external agent empowered to set the agenda attempts to make a distinguished candidate the winner.
(2) We study the Coalition Manipulation problem in which a set of voters, called manipulators, aim to make a distinguished candidate the winner by coordinating their votes.
(3) We study eight standard election control problems, namely, CCAV, CCDV, CCAC, CCDC, DCAV, DCDV, DCAC, and DCDC, where "CC"/"DC" stands for "constructive control"/"destructive control", the third letter "A"/"D" stands for "adding"/"deleting", and the last letter "V"/"C" stands for "voters"/"candidates". These problems model the scenario where a powerful external agent aims to make a distinguished candidate the winner (constructive) or not the winner (destructive) by adding or deleting a limited number of voters or candidates.
(4) We study the Possible Winner and the Necessary Winner problems under TSMR. These two problems are relevant to a setting where only partial information on the preferences of voters and agenda are known. Possible Winner consists in determining which candidates have positive chances to win at least one completion of the partial input, and Necessary Winner consists in determining which candidates necessarily win regardless of the missing information.
(5) For the above problems, we offer a comprehensive (parameterized) complexity landscape. Particularly, for the eight election control problems, we study both the special case where the given distinguished candidate p is the first one, and the special case where p is the last one in the agenda. We refer to Table 1 for a summary of our concrete results as well as previous results for the successive rule and the amendment rule.

1.1 Related Works

Agenda Control is arguably one of the most sought-after problems in the context of sequential voting rules and has a long history of study (see, e.g., [6, 30]). However, the complexity of Agenda Control was only first studied several years ago [8]. It should be pointed out that the complexity of some analogous problems

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Table 1: Our main results are in bold face. "first", "last", and "$\overline{\text{last}}$" mean that the distinguished candidate is respectively the first one, the last one, and not the last one in the agenda. P-results spanning two rows hold for the general case. Besides, m is the number of candidates, n is the number of votes, n_{rg} is the number of registered votes, and k is the solution size.

			CCAV	CCDV	CCAC	CCDC
TSMR		first	**W[2]-h** ($k + n_{\text{rg}}$, Thm. 3)	**W[2]-h** ($k, n - k$, Thms. 5, 6)	**W[2]-h** (k, Thm. 9)	**P** (Thm. 10)
		last	**W[2]-h** ($k + n_{\text{rg}}$, Thm. 4)	**W[2]-h** ($k, n - k$, Thms. 7, 8)	immune (Cor. 1)	
successive	[27, 43]	first	P	P	immune	W[1]-h ($k, m - k$)
		last	W[1]-h ($k + n_{\text{rg}}$)	W[2]-h (k)	W[2]-h (k)	P
amendment		first	W[1]-h ($k + n_{\text{rg}}$)	W[1]-h (k)	P	P
		last	W[2]-h ($k + n_{\text{rg}}$)	W[2]-h (k)		

			DCAV	DCDV	DCAC	DCDC
TSMR		$\overline{\text{last}}$	**W[2]-h** ($k + n_{\text{rg}}$, Thm. 11)	**W[2]-h** ($k, n - k$, Thms. 12, 13)	**P** (Thm. 14)	**P** (Cor. 3)
		last	P [4]	P [4]		
successive	[27, 43]	first	P	P	W[2]-h (k)	immune
		last			P	W[1]-h ($k, m - k$)
amendment		first	P	P	P	immune
		last	W[1]-h (k)	W[2]-h (k)		P

		Agenda Control	Coalition Manipulation	Possible Winner	Necessary Winner
TSMR		**P** (Thm. 1)	**P** (Thm. 2)	**NP-h** (Thms. 16, 17)	**P** (Thm. 15)
successive	[8]	P	P	NP-h	P
amendment		P	P	NP-h	coNP-h

in the context of knockout tournaments has been studied earlier [1, 3, 4, 11, 28, 37, 38].

Coalition Manipulation is a natural generalization of the well-known Manipulation problem [3], and was first studied by Conitzer, Sandholm, and Lang [12]. We refer to [5, 13, 34–36] for detailed results on the complexity of this problem for many traditional rules (i.e., voting rules like Borda, Maximin, etc., which do not need an agenda to determine the winner).

The constructive control problems were first studied by Bartholdi III, Tovey, and Trick [4], and their destructive counterparts were initiated by Hemaspaandra, Hemaspaandra, and Rothe [23]. Heretofore the complexity of these problems for many rules has been extensively investigated. We refer to the book chapters [5, 20] for important progress by 2016, and refer to [19, 31, 40–42] for some recent new results.

The complexity of Possible Winner and Necessary Winner for the successive rule and the amendment rule has been studied by Bredereck et al. [8]. These two problems for traditional voting rules were first studied by Konczak and Lang [26], and the complexity of the problems for many rules has been subsequently established [9, 10, 39].

2 PRELIMINARIES

We assume the reader is familiar with basic notions in graph theory and (parameterized) complexity theory [2, 14, 15, 33].

Let $[i]$ be the set of positive integers equal to or smaller than i. For a binary relation R, we sometimes use xRy to denote $(x, y) \in R$.

Unless stated otherwise, for a set S we use $\overrightarrow{S}$ to denote an arbitrary but fixed linear order over S. Once such an $\overrightarrow{S}$ is used, $\overleftarrow{S}$ denotes then the reverse of $\overrightarrow{S}$. For $S' \subseteq S$, $\overrightarrow{S}[S']$ denotes $\overrightarrow{S}$ restricted to S', and $\overrightarrow{S} \setminus S'$ denotes $\overrightarrow{S}[S \setminus S']$.

2.1 Graphs

An undirected graph is a tuple $G = (N, A)$, where N is a set of vertices and A is a set of edges. An edge between two vertices v and v' is denoted by $\{v, v'\}$. We use $\Gamma_G(v)$ to denote the set of neighbors of v in G, i.e., $\Gamma_G(v) = \{v' \in N : \{v, v'\} \in A\}$. A digraph is a tuple $G = (N, A)$ where N is a set of vertices and A is a set of arcs. Each arc from a vertex a to a vertex b is denoted by (a, b). The set of inneighbors of a vertex a in G is $\Gamma^-_G(a) = \{b \in N : (b, a) \in A\}$, and the set of outneighbors of a in G is $\Gamma^+_G(a) = \{b \in N : (a, b) \in A\}$. For $S \subseteq N$, let $\Gamma^+_G(S) = \bigcup_{a \in S} \Gamma^+_G(a) \setminus S$. When it is clear from the context which graph G is discussed, we drop G from the notions. For a graph G (be it directed or undirected) and a subset S of vertices, the subgraph of G induced by S is denoted by $G[S]$.

2.2 Elections and Voting Rules

An election is a tuple (C, V) of a set of candidates C and a multiset of votes V, where every vote is a linear order over C. For $c, c' \in C$, we say that c is ranked before c' in a vote $\succ$ if $c \succ c'$. We say that c is ranked immediately before c' if $c \succ c'$ and no other candidates are ranked between them. A vote $\succ$ specifies a preference such that for $a, b \in C$, a is preferred to b by the vote if a is ranked before b. We sometimes write a preference in the format of a sequence of candidates from the most preferred one to the least preferred one. For instance, if we say that a vote has the preference $a\ b\ c$, we mean that a is ranked before b, and b ranked before c in the vote.

An agenda $\rhd$ is a linear order over C. For $c \in C$, we call candidates before c in $\rhd$ the predecessors of c, and call those after c the successors of c. A sequential rule τ maps each election (C, V) and each agenda $\rhd$ to a single candidate $\tau(C, V, \rhd) \in C$, the winner.

For $c, c' \in C$, we use $n_V(c, c')$ to denote the number of votes in V ranking c before c'. We say c beats (resp. ties) c' w.r.t. V if

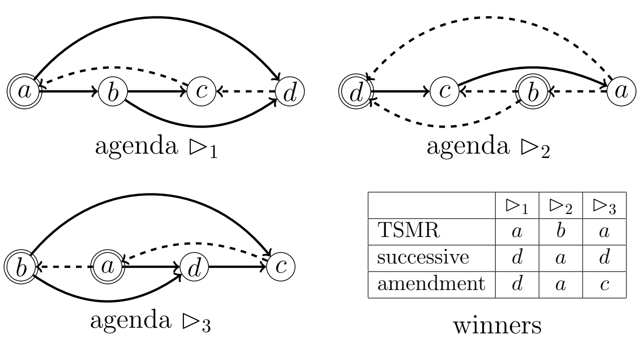
$n_V(c, c') > n_V(c', c)$ (resp. $n_V(c, c') = n_V(c', c)$). A candidate is a weak Condorcet winner of (C, V) if it is not beaten by anyone else. A candidate is the Condorcet winner of (C, V) if it beats all the other candidates. The majority graph of an election $\mathcal{E} = (C, V)$, denoted G_E, is a digraph with the vertex set C so that there is an arc from $c \in C$ to $c' \in C$ if and only if $n_V(c, c') > n_V(c', c)$.

- **Two-stage majoritarian rule (TSMR)** Let $G_E^{\rhd}$ be the subdigraph of G_E with only forward arcs w.r.t. $\rhd$. Precisely, $G_E^{\rhd}$ takes C as the vertex set so that there is an arc from c to c' in $G_E^{\rhd}$ if and only if $c \rhd c'$ and there is an arc from c to c' in G_E. Let $C' \subseteq C$ be the set of candidates without inneighbors in $G_E^{\rhd}$. The TSMR winner is the right-most candidate in C', i.e., the $c \in C'$ such that $c' \rhd c$ for all $c' \in C' \setminus \{c\}$.

We also give formal definitions of the successive rule and the amendment rule as they are closely related to our discussions.

- **Successive** For a candidate $c \in C$ and a subset $C' \subseteq C \setminus \{c\}$, we say c beats C' if there is a strict majority of votes each of which ranks c before all candidates in C'. The successive winner is the first one in the agenda who beats the set of all her successors.
- **Amendment** This procedure takes $|C|$ rounds, where each round determines a temporary winner. Precisely, the winner of the first round is the first candidate in the agenda. The winner of round i where $i \geq 2$ is determined as follows. Let c be the winner of round $i - 1$, and let c' be the i-th candidate in the agenda. The winner of round i is c if c beats c', and is c' otherwise. The amendment winner is the winner of the last round.

Example 1. Let $C = \{a, b, c, d\}$, and let V be a set of three votes respectively with the preferences $\underline{b\ d\ c\ a}$, $\underline{c\ a\ b\ d}$, and $\underline{a\ d\ b\ c}$. The majority graph of (C, V), three different agendas, and the winners under different rules and agendas are shown in Figure 1.



	$\rhd_1$	$\rhd_2$	$\rhd_3$
TSMR	a	b	a
successive	d	a	d
amendment	d	a	c

Figure 1: An illustration of TSMR, the successive rule, and the amendment rule. For TSMR, arcs not in $G_E^{\rhd}$ (backward arcs w.r.t. $\rhd_i$) are drawn as dashed lines.

By the definitions of the sequential rules, it is easy to verify that the first and the last candidates in an agenda are somehow related to (weak) Condorcet winner, as summarized below.

Observation 1. *For an election (C, V) and an agenda $\rhd$ over C, the following hold.*

(1) *The first candidate in $\rhd$ is the amendment winner if and only if it is the Condorcet winner of (C, V).*
(2) *The last candidate in $\rhd$ is the TSMR winner of (C, V) if and only if it is a weak Condorcet winner of (C, V).*
(3) *If the first candidate in $\rhd$ is the successive winner of (C, V), then it is also the Condorcet winner of (C, V).*
(4) *If the first candidate in $\rhd$ is the Condorcet winner of (C, V), then it is also the TSMR winner of (C, V).*
(5) *If the last candidate in $\rhd$ is a weak Condorcet winner of (C, V), it is also the successive winner and the amendment winner of (C, V).*
(6) *The converses of (3)–(5) do not necessarily hold.*

2.3 Problem Formulations

For a sequential voting rule τ, we study the following problems.

Agenda Control
I/P: An election (C, V) and a distinguished candidate $p \in C$.
QN: Is there an agenda $\rhd$ over C so that p is the winner of $(C, V, \rhd)$ w.r.t. τ, i.e., $p = \tau(C, V, \rhd)$?

Coalition Manipulation
I/P: An election (C, V), a distinguished candidate $p \in C$, an agenda $\rhd$ over C, and a positive integer k.
QN: Is there a multiset V' of k votes over C s.t. $p = \tau(C, V \cup V', \rhd)$?

For a partial order R over a set X, a linear extension of R is a linear order R' over X so that $(x, y) \in R$ implies $(x, y) \in R'$ for all $x, y \in X$. A partial election is a tuple (C, V) where V is a multiset of partial orders over C. An election (C, V') is a completion of a partial election (C, V) if elements of V' one-to-one correspond to elements of V so that every $v' \in V'$ is a linear extension of the partial order in V corresponding to v'. A partial agenda over C is a partial order over C.

Possible Winner
I/P: A partial election (C, V), a distinguished candidate $p \in C$, and a partial agenda $\rhd$ over C.
QN: Is there a completion (C, V') of (C, V) and a linear extension $\rhd'$ of $\rhd$ so that $p = \tau(C, V', \rhd')$?

Necessary Winner
I/P: A partial election (C, V), a distinguished candidate $p \in C$, and a partial agenda $\rhd$ over C.
QN: Is p the τ winner of every completion of $(C, V, \rhd)$, i.e., $p = \tau(C, V', \rhd')$ for all completions (C, V') of (C, V) and all linear extensions $\rhd'$ of $\rhd$?

We also study eight standard control problems which are special cases of the following problems.

Constructive Multimode Control
I/P: An election $(C \cup D, V \cup W)$ with a set C of registered candidates, a set D of unregistered candidates, a multiset V of registered votes, a multiset W of unregistered votes, a distinguished candidate $p \in C$, an agenda $\rhd$ over $C \cup D$, and four integers k_{AV}, k_{DV}, k_{AC}, and k_{DC}.
QN: Are there $V' \subseteq V$, $W' \subseteq W$, $C' \subseteq C \setminus \{p\}$, and $D' \subseteq D$ such that $\lvert V'\rvert \leq k_{\text{DV}}$, $\lvert W'\rvert \leq k_{\text{AV}}$, $\lvert C'\rvert \leq k_{\text{DC}}$, $\lvert D'\rvert \leq k_{\text{AC}}$, and p wins $((C \setminus C') \cup D', (V \setminus V') \cup W', \rhd')$ w.r.t. τ, where $\rhd'$ is $\rhd$ restricted to $(C \setminus C') \cup D'$?

In Destructive Multimode Control, we have the same input as Constructive Multimode Control, but are asked whether there are V', W', C', and D' as in the above definition so that p is not the τ winner of $((C \setminus C') \cup D', (V \setminus V') \cup W', \rhd')$.

The specifications of the eight standard control problems are summarized in Table 2.

Table 2: Special cases of Constructive/Destructive Multimode Control. Here, X is either CC standing for constructive control or DC standing for destructive control.

problems	restrictions
XAV	$k_{\text{DV}} = k_{\text{AC}} = k_{\text{DC}} = 0$, $D = \emptyset$
XAC	$k_{\text{AV}} = k_{\text{DV}} = k_{\text{DC}} = 0$, $W = \emptyset$
XDV	$k_{\text{AV}} = k_{\text{AC}} = k_{\text{DC}} = 0$, $D = W = \emptyset$
XDC	$k_{\text{AV}} = k_{\text{DV}} = k_{\text{AC}} = 0$, $D = W = \emptyset$

For simplicity, when we study a problem in Table 2, we use k to denote the integer in the input not required to be 0, and omit components in the input requested to be 0 or $\emptyset$. For example, an instance of CCAV is written as $((C, V \cup W), p, \rhd, k)$, where k represents k_{AV}.

Our hardness results are based on the following problem.

Red-Blue Dominating Set (RBDS)	
I/P:	A bipartite graph G with bipartition (R, B) where vertices in R and B are referred to as red vertices and blue vertices respectively, and a positive integer $\kappa \leq \|B\|$.
QN:	Is there a subset $B' \subseteq B$ of cardinality κ that dominates R, i.e., $\|B'\| = \kappa$ and every vertex in R has at least one neighbor from B' in the graph G?

RBDS is NP-complete [22], and it is W[2]-complete w.r.t. κ [16].

2.4 Remarks

Most previous studies make the assumption that there are no ties in elections (see, e.g., [8, 25]). Our results are presented without this assumption, but all of them still hold when the no-tie assumption is made. This is clear for polynomial-time solvability results. Regarding hardness results for voter control problems, some of our reductions can be slightly adapted for showing the same hardness if the no-tie assumption is adopted, and others directly apply to the case where the no-tie assumption is made. We note that in these problems the no-tie assumption means that after the addition or the deletion of votes there are no ties.

All our reductions take polynomial time, and all computationally hard problems proved in the paper are clearly in NP (Necessary Winner is in coNP). Therefore, a problem shown to be W[2]-hard in the paper is also NP-complete.

Due to space limitations, several proofs have to be omitted. Theorems whose proofs are omitted are marked by ★, and their proofs are available in a full version of the paper posted on arXiv.org [44].

3 STRATEGIC PROBLEMS

In this section, we study the complexity of many strategic voting problems for TSMR.

3.1 Agenda Control and Manipulation

We first present a P-algorithm for Agenda Control.

Theorem 1. *Agenda Control for* TSMR *is in* P.

Proof. Let $I = ((C, V), p)$ be an instance of Agenda Control. Let G be the majority graph of (C, V). We construct an agenda $\rhd$ as follows. Let $A = C \setminus (\Gamma^-_G(p) \cup \{p\})$ be the set of candidates which do not beat p w.r.t. V. We fill all candidates from $A \cup \{p\}$ in the first $|A \cup \{p\}|$ positions in the agenda $\rhd$ so that p is after all candidates from A (the relative orders of candidates from A are set arbitrarily). Then, we fill candidates from $\Gamma^-_G(p)$ into the agenda iteratively as follows. First, let $S = A$. In each iteration we compute the set $S' = \Gamma^+_G(S)$, and fill candidates from S' in the subsequent $|S'|$ positions in the agenda $\rhd$. Then, we update $S := S \cup S'$. The iterations terminate when S' defined above turned out to be empty.

After the iterations terminate, if all candidates in C are in the agenda $\rhd$, p is the TSMR winner of (C, V) w.r.t. $\rhd$. Thus, in this case, we conclude that I is a Yes-instance. If, however, there are still some candidates in C not filled in the agenda, we conclude that I is a No-instance. The reason is as follows. By the above iterations, in this case it holds that (1) none of $C \setminus (S \cup \{p\})$ is beaten by anyone from $S \cup \{p\}$, and (2) everyone in $C \setminus S$ beats p. Condition (2) entails that every candidate from $C \setminus (S \cup \{p\})$ must be after p in the agenda. However, as long as this is the case, Condition (1) warrants the winning of someone from $C \setminus (S \cup \{p\})$. □

For Coalition Manipulation, we have again a P-algorithm.

Theorem 2. *Coalition Manipulation for* TSMR *is in* P.

Proof. Let $I = ((C, V), p, \rhd, k)$ be an instance of Coalition Manipulation. Let B be the set of predecessors of p, and let B' be the set of successors of p in the agenda $\rhd$. Let V' be the multiset of k votes with the same preference $p\ \overrightarrow{B}\ \overrightarrow{B'}$, where $\overrightarrow{B}$ and $\overrightarrow{B'}$ are respectively the linear orders over B and B' consistent with $\rhd$, i.e., $\overrightarrow{B} = \rhd[B]$ and $\overrightarrow{B'} = \rhd[B']$. We conclude that I is a Yes-instance if and only if p is the TSMR winner of $(C, V \cup V', \rhd)$.

The algorithm clearly runs in polynomial time. It remains to prove its correctness. To this end, we assume that I is a Yes-instance, and to complete the proof it suffices to show that I has a feasible solution V' so that every vote in V' has the same preference $p\ \overrightarrow{B}\ \overrightarrow{B'}$. Observe first that I has a feasible solution where p is ranked in the first place in all votes. Let U be a feasible solution of I where p is in the top in all votes in U. If U equals V' defined above, we are done. Otherwise, we show below how to transform U into V' without destroying the feasibility of the solution. If there exists one vote $\succ \in U$, and two candidates $b \in B$ and $b' \in B'$ so that b' is ranked immediately before b in $\succ$, we do the following. Let $\succ'$ be the vote obtained from $\succ$ by swapping b and b', and let $U' = U \setminus \{\succ\} \cup \{\succ'\}$. It is easy to verify that every candidate who is beaten by at least one of her predecessors w.r.t. $V \cup U$ is also beaten by at least one of her predecessors w.r.t. $V \cup U'$, and every candidate which does not beat p w.r.t. $V \cup U$ still does not beat p w.r.t. $V \cup U'$. Therefore, p still wins after the swapping of b and b'. After the swapping operations are exhaustively applied, we obtain a feasible solution $\widetilde{U}$ of I so that p is ranked in the top, and all candidates in B are ranked before all candidates in B' in every vote of $\widetilde{U}$. If $\widetilde{U} = V'$, we are done. Otherwise, there exists at least one vote $\succ \in \widetilde{U}$ such that one of the following conditions holds:

- $\exists a, b \in B$ s.t. a is ranked immediately before b in $\succ$ and $b \rhd a$;
- $\exists a', b' \in B'$ s.t. a' is ranked immediately before b' in $\succ$ and $b' \rhd a'$.

Then, analogous to the above discussion, we can swap a and b (resp. a' and b') in $\succ$ without changing the winning status of p. After the swapping operations are exhaustively used, we obtain V'. □

3.2 Constructive Controls

In this section, we study constructive control problems for TSMR. We first present results for control by adding/deleting votes. We show that these problems are W[2]-hard w.r.t. several meaningful parameters, for both the special case where the distinguished candidate is the first one in the agenda and the speical case where the distinguished candidate is the last one in the agenda.

Theorem 3. *CCAV for TSMR is W[2]-hard w.r.t. the number of added votes plus the number of registered votes. Moreover, this holds even when the distinguished candidate is the first one in the agenda.*

Proof. We prove the theorem via a reduction from RBDS. Let $(G = (R \cup B, A), \kappa)$ be an instance of RBDS. We construct an instance of CCAV for TSMR as follows. We create for each vertex in G a candidate denoted by the same symbol for simplicity. In addition, we create a candidate p. Let $C = B \cup R \cup \{p\}$. The agenda is $\rhd = (p, \overrightarrow{B}, \overrightarrow{R})$. We create the following registered votes:

- κ votes with the preference $\overleftarrow{B}\ \overleftarrow{R}\ p$; and
- one vote with the preference $\overleftarrow{R}\ p\ \overleftarrow{B}$.

Let V be the multiset of the above $\kappa+1$ registered votes. We create $|B|$ unregistered votes corresponding to B. Concretely, for each $b \in B$, we create one vote $\succ_b$ with the preference

$$p\ \left(\overleftarrow{R} \setminus \Gamma_G(b)\right)\ b\ \left(\overleftarrow{R}[\Gamma_G(b)]\right)\ \left(\overleftarrow{B} \setminus \{b\}\right).$$

Let W be the set of the above $|B|$ unregistered votes. Finally, we set $k = \kappa$. The instance of CCAV for TSMR is $((C, V \cup W), p, \rhd, k)$. In the following we show the correctness of the reduction.

($\Rightarrow$) Suppose that there exists $B' \subseteq B$ such that $|B'| = \kappa$ and B' dominates R. Let $W' = \{\succ_b : b \in B'\}$ be the set of the κ unregistered votes corresponding to B'. We show below that p becomes the TSMR winner of the election $\mathcal{E} = (C, V \cup W')$. Obviously, $|V \cup W'| = 2\kappa+1$. As one of the registered votes ranks p before B, and all the κ votes in W' rank p before B too, there are $\kappa+1$ votes in $V \cup W'$ ranking p before B. So, none of B is the TSMR winner of $\mathcal{E}$. Let us consider a candidate $r \in R$. Note that there are κ registered votes which rank B before R. As B' dominates R, there is at least one $b \in B'$ so that $r \in \Gamma_G(b)$. By the definition of $\succ_b$, b is ranked before r in $\succ_b$. Therefore, there are in total $\kappa + 1$ votes in $V \cup W'$ which rank b before r, precluding the winning of r. As this holds for all $r \in R$, and candidates from B are before candidates from R in $\rhd$, none of R is the TSMR winner of $\mathcal{E}$ either. This leaves only the possibility that p is the winner.

($\Leftarrow$) Suppose that there exists a subset $W' \subseteq W$ of at most κ votes so that p is the TSMR winner of $(C, V \cup W')$. Observe that W' must contain exactly κ votes, since otherwise someone in B precludes p from winning. Observe that all candidates in R beat p w.r.t. $V \cup W'$ no matter which votes are contained in W'. Furthermore, everyone in R beats all her predecessors in R w.r.t. $V \cup W'$. So, if p wins $(C, V \cup W')$ it must be that every $r \in R$ is beaten by someone in B. This implies that for every $r \in R$, there is at least one vote in W' which ranks some $b \in B$ before r. By the construction of the unregistered votes, this vote must be $\succ_b$ such that b dominates r. It follows that $\{b \in B : \succ_b \in W'\}$ dominates R. This implies that the RBDS instance is a Yes-instance. □

Now we consider the case where the distinguished candidate is the last one in the agenda. Recall that the last candidate in the agenda is the TSMR winner if and only if it is a weak Condorcet winner (Observation 1). The W[1]-hardness of CCAV for Condorcet winner established by Liu et al. [27] can be adapted for showing the same hardness for weak Condorcet winner[1]. We strengthen the result by establishing a W[2]-hardness result, excluding the possibility of the problem being complete to W[1].

Theorem 4. *CCAV for TSMR is W[2]-hard w.r.t. the number of added votes plus the number of registered votes even when the distinguished candidate is the last one in the agenda.*

Proof. We prove the theorem via a reduction from RBDS. Let (G, κ) be an instance of RBDS, where $G = (R \cup B, A)$ is a bipartite graph. We create an instance of CCAV as follows. The candidate set is $C = R \cup \{p, q\}$. Let $\rhd = (\overrightarrow{R}, q, p)$. We create a multiset V of κ registered votes as follows:

- $\kappa - 1$ votes with the preference $q\ p\ \overrightarrow{R}$; and
- one vote with the preference $q\ \overrightarrow{R}\ p$.

For each $b \in B$, we create one unregistered vote $\succ_b$ with the preference $\left(\overrightarrow{R} \setminus \Gamma_G(b)\right)\ p\ \left(\overrightarrow{R}[\Gamma_G(b)]\right)\ q$. For a given $B' \subseteq B$, let $W(B') = \{\succ_b : b \in B\}$ be the multiset of unregistered votes corresponding to B'. Let $k = \kappa$. The instance of CCAV is $((C, V \cup W(B)), p, \rhd, k)$. It remains to show the correctness of the reduction.

($\Rightarrow$) Assume that there exists $B' \subseteq B$ such that $|B'| = \kappa$ and B' dominates R. Let $\mathcal{E} = (C, V \cup W(B'))$. First, observe that p ties q in $\mathcal{E}$. As B' dominates R, for every $r \in R$ there is at least one $b \in B'$ which dominates r. This implies that in the vote $\succ_b \in W(B')$, p is ranked before r, and hence p is not beaten by r in $\mathcal{E}$. As p is the last one in the agenda, it follows that p wins $\mathcal{E}$.

($\Leftarrow$) Assume that there exists $B' \subseteq B$ such that $|B'| \leq k = \kappa$ and p is the TSMR winner of $\mathcal{E} = (C, V \cup W(B'))$. This means that p is not beaten by anyone else in $\mathcal{E}$. Therefore, $|B'| = k$, since otherwise q beats p. It follows that $|V \cup W(B')| = 2\kappa$. Let $r \in R$. As we have exactly $\kappa - 1$ registered votes ranking p before r in V, there is at least one $b \in B'$ so that p is ranked before r in the vote $\succ_b$. By the definition of $\succ_b$, this implies that b dominates r. It follows that B' dominates R. Thus, the RBDS instance is a Yes-instance. □

Let us move on to constructive control by deleting votes. This problem possesses two natural parameters: the solution size k and its dual parameter $n - k$, where n is the number of votes. We show that the problem is W[2]-hard w.r.t. both parameters, even when the distinguished candidate is the first or the last one in the agenda. These results are encapsulated in the following four theorems.

[1]For this, we mean the problem of determining if we can add a limited number of votes to make a distinguished candidate a weak Condorcet winner.

Theorem 5 (★). CCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of deleted votes even when the distinguished candidate is the first one in the agenda.*

Theorem 6 (★). CCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of votes not deleted even when the distinguished candidate is the first one in the agenda.*

Theorem 7. CCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of deleted votes. This holds even if the distinguished candidate is the last candidate in the agenda.*

Proof. We prove Theorem 7 by a reduction from RBDS. Let (G, κ) be an instance of RBDS, where $G = (R \cup B, A)$ is a bipartite graph. We assume that G does not contain any isolated vertices, $\kappa \geq 4$, and every red vertex is of degree ℓ where $\ell \geq 1$. These assumptions do not change the W[2]-hardness of the problem. Let $C = R \cup \{p, q\}$, and let $\rhd$ be an agenda over C where p is the last one (the relative orders of other candidates are immaterial to the correctness of the reduction). We create $2|B| + 2\ell + \kappa$ votes in V:

- $|B| + 1$ votes with the preference $\overleftarrow{R}\ p\ q$;
- $\ell + \kappa$ votes with the preference $q\ p\ \overleftarrow{R}$;
- $\ell - 1$ votes with the preference $p\ q\ \overleftarrow{R}$; and
- for each blue vertex $b \in B$, one vote $\succ_b$ with the preference

$$q\ \left(\overleftarrow{R}[\Gamma_G(b)]\right)\ p\ \left(\overleftarrow{R} \setminus \Gamma_G(b)\right).$$

For a given $B' \subseteq B$, let $V(B') = \{\succ_b : b \in B'\}$ be the multiset of votes corresponding to B'. Finally, we set $k = \kappa$. The instance of CCDV is $((C, V), p, \rhd, k)$. In the following, we prove the correctness of the reduction.

($\Rightarrow$) Assume there exists $B' \subseteq B$ of cardinality κ so that B' dominates R. Let $\mathcal{E} = (C, V \setminus V(B'))$. Clearly, $|V \setminus V(B')| = 2|B| + 2\ell$. We show below that p is not beaten by anyone else in $\mathcal{E}$, and hence p is the TSMR winner of $\mathcal{E}$. As all votes in $V(B')$ rank q before p, it holds that $n_{V \setminus V(B')}(p, q) = (|B| + 1) + (\ell - 1) = |B| + \ell$, meaning that p ties q in $\mathcal{E}$. As B' dominates R, for every $r \in R$, there exists $b \in B'$ dominating r. By the construction of the votes, r is ranked before p in the vote $\succ_b \in V(B')$. It follows that at most $\kappa - 1$ votes in $V(B')$ rank p before r. By the construction of the votes, there are at least $(\ell + \kappa) + (\ell - 1) + (|B| - \ell) - (\kappa - 1) = |B| + \ell$ votes ranking p before r in $V \setminus V(B')$, meaning that p ties or beats r in $\mathcal{E}$.

($\Leftarrow$) Assume there exists $V' \subseteq V$ such that $|V'| \leq k$ and p is the TSMR winner of $\mathcal{E} = (C, V \setminus V')$ w.r.t. $\rhd$. As p is the last one in the agenda, it holds that p beats or ties everyone else in $\mathcal{E}$. As a consequence, all votes in V' must rank q before p and, moreover, it must be that $|V'| = k = \kappa$, since otherwise p is beaten by q in $\mathcal{E}$. There are two groups of votes ranking q before p: those corresponding to the blue vertices, and those with the preference $q\ p\ \overleftarrow{R}$. We may assume that all votes in V' are from $V(B)$. Indeed, if V' contained some vote with the preference $q\ p\ \overleftarrow{R}$, we can obtain another feasible solution V'' from V' by replacing this vote with any vote in $V(B) \setminus V'$. Let $r \in R$. As $n_V(r, p) = (|B| + 1) + \ell$ and $|V \setminus V'| = 2|B| + 2\ell$, there is at least one vote $\succ_b \in V'$ which ranks r before p. By the reduction, we know that the vertex b corresponding to $\succ_b$ dominates r. It is clear now that $\{b \in B : \succ_b \in V'\}$ dominates R, implying that the RBDS instance is a Yes-instance. □

We point out that Theorem 7 strengthens the W[1]-hardness of CCAV for (weak) Condorcet winner w.r.t. k by Liu et al. [27].

Theorem 8 (★). CCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of votes not deleted. This holds even when the distinguished candidate is the last one in the agenda.*

Let us now explore the complexity landscape of constructive control by adding or deleting candidates. Unlike voter controls, we have only one hardness result as stated in the following theorem.

Theorem 9. CCAC *for* TSMR *is* W[2]-hard *w.r.t. the number of added candidates. This holds even when the distinguished candidate is the first one in the agenda.*

Proof. We prove the theorem via a reduction from RBDS. Let $(G = (R \cup B, A), \kappa)$ be an instance of RBDS. We construct an instance of CCAC for TSMR as follows. For each vertex in G we create one candidate denoted by the same symbol for notational simplicity. In addition, we create a distinguished candidate p. Let $C = R \cup \{p\}$ and let $D = B$. Besides, let $k = \kappa$ and let $\rhd = (p, \overrightarrow{B}, \overrightarrow{R})$. We create a multiset V of votes in a way so that

- every candidate from R beats all her predecessors in $R \cup \{p\}$;
- p beats every candidate from B; and
- for each $r \in R$ and each $b \in B$, if b dominates r in G, then b beats r; otherwise, r beats b.

By the famous McGarvey's theorem [29] such votes can be constructed in polynomial time. The instance of CCAC for TSMR is $((C \cup D, V), p, \rhd, k)$. The correctness of the reduction is easy to see. In particular, if there exists $B' \subseteq B$ of κ vertices dominating R, then after adding the candidates corresponding to B', every $r \in R$ has at least one predecessor from B' who beats her, excluding the winning of r. Candidates in B' cannot win as they are beaten by p. Therefore, after adding these candidates, p becomes the winner. If, however, the RBDS instance is a No-instance, no matter which at most k candidates from B are added, there is at least one candidate in R who beats all her predecessors in the resulting election. In this case we cannot add at most k candidates to make p the winner. □

When the distinguished candidate is the last one in the agenda, we have the following corollary as a consequence of Observation 1 and the immunity of weak Condorcet to CCAC [4].

Corollary 1. *If the distinguished candidate is the last in the agenda,* TSMR *is immune to* CCAC.

For CCDC, a greedy P-algorithm can be easily obtained.

Theorem 10. CCDC *for* TSMR *is in* P.

Proof. Let $I = ((C, V), p, \rhd, k)$ be an instance of CCDC. We first remove all predecessors of p in $\rhd$ who beat p w.r.t. V. Then, we iteratively remove each successor c of p so that c is not beaten by any of her predecessors. We conclude that I is a Yes-instance if and only if at most k candidates are removed in total. □

3.3 Destructive Controls

Now we start the exploration on destructive control problems.

Theorem 11. DCAV *for* TSMR *is* W[2]-hard *w.r.t. the number of added votes plus the number of registered votes. This holds as long as the distinguished candidate is not the last one in the agenda.*

Proof. We prove Theorem 11 via a reduction from RBDS. Let $(G = (R \cup B, A), \kappa)$ be an instance of RBDS. We create an instance of DCAV as follows. Let $C = R \cup \{p, q\}$, and let $\rhd$ be an agenda where q is the last candidate. We create the following registered votes:

- $\kappa - 1$ votes with the preference $p\ \ q\ \ \overrightarrow{R}$;
- two votes with the preference $p\ \ \overrightarrow{R}\ \ q$; and
- one vote with the preference $q\ \ p\ \ \overrightarrow{R}$.

Let V be the multiset of the above $\kappa + 2$ registered votes. The unregistered votes are created according to B: for each $b \in B$, we create one vote $\succ_b$ with the preference $\left(\overrightarrow{R} \setminus \Gamma_G(b)\right)\ \ q\ \ p\ \ \left(\overrightarrow{R}[\Gamma_G(b)]\right)$. For a given $B' \subseteq B$, let $W(B') = \{\succ_b : b \in B'\}$ be the multiset of unregistered votes corresponding to B'. For simplicity, let $W = W(B)$. Let $k = \kappa$. The instance of DCAV is $((C, V \cup W), p, \rhd, k)$. We prove the correctness of the reduction as follows.

($\Rightarrow$) Suppose that there is a $B' \subseteq B$ of κ vertices which dominate R in G. Then, one can check that q beats or ties every other candidate w.r.t. $V \cup W(B')$, implying that q is the winner of $(C, V \cup W(B'))$. Thus, in this case the instance of DCAV is a Yes-instance.

($\Leftarrow$) Suppose that there exists a subset $W' \subseteq W$ of at most k votes so that p is not the TSMR winner of $\mathcal{E} = (C, V \cup W')$. Observe that no matter which at most k votes are contained in W', p beats all candidates in R, implying that the only candidate which is able to preclude p from winning is q. As q is the last candidate in the agenda $\rhd$, q is the winner if and only if q beats or ties everyone else. This implies that W' contains exactly κ votes, since otherwise p beats q in $\mathcal{E}$. Moreover, for each $r \in R$, at least one vote in W' ranks q before r. By the construction of the unregistered votes, an unregistered vote $\succ_b$ ranks q before r if and only if b dominates r in G. This implies that the set of vertices corresponding to W' dominates R, and hence the instance of RBDS is a Yes-instance. □

It is known that DCAV and DCDV for weak Condorcet are polynomial-time solvable [23]. By Observation 1, we have the following corollary.

Corollary 2 ([23]). DCAV *and* DCDV *for* TSMR *are in* P *if the distinguished candidate is in the last position of the agenda.*

However, the complexity of DCDV increases if the distinguished candidate is not the last one in the agenda.

Theorem 12. DCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of deleted votes. This holds as long as the distinguished candidate is not the last one in the agenda.*

Proof. The reduction is the same as the one in the proof of Theorem 7 with only the difference that q is the distinguished candidate. The correctness hinges upon the fact that no matter which at most k votes are deleted, q beats all candidates in R, which leaves p the unique candidate preventing q from winning and, moreover, this holds as long as q is not the last one in the agenda. □

For the dual parameter of the solution size, we have the same result.

Theorem 13 (★). DCDV *for* TSMR *is* W[2]-hard *w.r.t. the number of votes not deleted. This holds as long as the distinguished candidate is not the last one in the agenda.*

For destructive control by modifying the set of candidates, we have polynomial-time solvability results, regardless of the position of the distinguished candidate in the agenda.

Theorem 14. DCAC *for* TSMR *is in* P.

Proof. Let $I = ((C \cup D, V), p, \rhd, k)$ be an instance of DCAC. We assume that $k \geq 1$ and p is the winner of (C, V), since otherwise I can be solved trivially. As p wins (C, V), p is not beaten by any of her predecessors, and each successor $c \in C \setminus \{p\}$ of p is beaten by at least one of c's predecessors. If there exists $c \in D$ which is before p in the agenda and beats p, we conclude that I is a Yes-instance because p does not win $(C \cup \{c\}, V)$. Additionally, if there exists $c \in D$ so that $p \rhd c$, and c is not beaten by any of her predecessors in C, we also conclude that I is a Yes-instance, since p does not win $(C \cup \{c\}, V)$. If neither of the two cases occurs, then no matter which unregistered candidates are added, p remains the winner. Therefore, in this case, we conclude that I is a No-instance. □

The following result is a consequence of Theorem 10.

Corollary 3. DCDC *for* TSMR *is in* P.

4 NECESSARY AND POSSIBLE WINNER

In this section, we study Necessary Winner and Possible Winner. Bredereck et al. [8] showed that except Necessary Winner for the successive rule which is polynomial-time solvable, other cases of the two problems for the successive rule and the amendment rule are computationally hard (coNP-hardness for Necessary Winner and NP-hardness for Possible Winner). We show below that TSMR behaves the same as the successive rule in terms of the complexity of determining necessary and possible winners.

Theorem 15. Necessary Winner *for* TSMR *is in* P.

Proof. Let $I = ((C, V), p, \rhd)$ be an instance of Necessary Winner. We determine if there is a completion of (C, V) and a completion of the agenda $\rhd$ so that p is not the TSMR winner of the completion. Note that p is not the winner if and only if

(1) either some of her predecessors beats her,
(2) or some of her successors, say c, is not beaten by any of the predecessors of c.

We consider first if there is a completion leading to the occurrence of Case 1. For this purpose, let $B = \{c \in C \setminus \{p\} : (p, c) \notin \rhd\}$ be the set of all candidates that can be predecessors of p in some completion of $\rhd$. We consider candidates in B one by one, and for each considered $c \in B$, we greedily complete the preference profile to determine if there exists at least one completion so that c beats p. More precisely, for every partial vote $\succ \in V$ such that $(p, c) \notin \succ$, we complete it so that c is ranked before p. If in the completion of (C, V) obtained this way c beats p, we conclude that I is a No-instance.

If we cannot draw the conclusion that I is a No-instance above, we consider whether it is possible to make the second case happen. To this end, we enumerate all candidates which can be successors of p in some completion of the partial agenda. More precisely, these candidates are those in $B' = \{c \in C \setminus \{p\} : (c, p) \notin \rhd\}$. For each enumerated $c \in B'$, we compute the minimum set A_c of candidates that are necessarily predecessors of c under the restriction that p is before c in the agenda, and then we greedily complete the preference profile to check if it can be completed so that c is not beaten by anyone in A_c. To be more precise, for each enumerated $c \in B'$, we compute $A_c = \{c' \in C : (c', c) \in \rhd\}$, and for each partial vote $\succ \in V$, we complete $\succ$ so that c is ranked as high as possible, i.e., we complete $\succ$ so that c is ranked below all candidates in $\{c' \in C : (c', c) \in \succ\}$ and is above all the other candidates. If in the completion c is not beaten by anyone from $A_c \cup \{p\}$, we conclude that I is a No-instance. If none of the above enumerations provides us with a "No"-answer, we conclude that I is a Yes-instance. □

Unlike the above problems, we show that Possible Winner for TSMR is NP-hard.

Theorem 16. Possible Winner *for* TSMR *is* NP-hard, *even if the given agenda is complete and the distinguished candidate is the first one in the agenda.*

Proof. We prove the theorem via a reduction from RBDS. Let (G, κ) be an instance of RBDS where G is a bipartite graph with the partition (R, B). We assume that G does not contain any isolated vertices, and all vertices in R have the same degree ℓ where $\ell \geq 1$. We create an instance of Possible Winner for TSMR as follows. Let $C = R \cup \{p, q\}$ and let $\rhd = (p, q, \overrightarrow{R})$. We create five groups of votes where only the first group contains partial votes:

- for each $b \in B$, one partial vote $\succ_b$ with the following partial preference $\left(\overleftarrow{R}[\Gamma_G(b)]\right)$ p $\left(\overleftarrow{R} \setminus \Gamma_G(b)\right)$ and q $\left(\overleftarrow{R} \setminus \Gamma_G(b)\right)$;
- a multiset V_1 of $|B|$ votes with the preference $\overleftarrow{R}$ q p;
- a multiset V_2 of $2\ell + \kappa$ votes with the preference q $\overleftarrow{R}$ p;
- a multiset V_3 of $\ell + 2\kappa + 1$ votes with the preference $\overleftarrow{R}$ p q;
- a multiset V_4 of $\ell + \kappa$ votes with the preference p q $\overleftarrow{R}$.

Let $V(B) = \{\succ_b : b \in B\}$ be the set of the $|B|$ partial votes in the first group. Let V be the multiset of the above $2|B| + 4\ell + 4\kappa + 1$ votes, and let $V(\overline{B}) = V \setminus V(B)$. The instance of Possible Winner is $((C, V), p, \rhd)$. We prove the correctness of the reduction as follows.

($\Rightarrow$) Suppose that there is a subset $B' \subseteq B$ such that $|B'| = \kappa$ and B' dominates R. We complete each $\succ_b$ where $b \in B$ as follows:

- if $b \in B'$, we complete it as q $\left(\overleftarrow{R}[\Gamma_G(b)]\right)$ p $\left(\overleftarrow{R} \setminus \Gamma_G(b)\right)$,
- otherwise, we complete it as $\left(\overleftarrow{R}[\Gamma_G(b)]\right)$ p q $\left(\overleftarrow{R} \setminus \Gamma_G(b)\right)$.

It is fairly easy to verify that w.r.t. the completion, p beats q, and q beats all candidates in R. Then, by the definition of the agenda, p is the TSMR winner w.r.t. the above completion of (C, V).

($\Leftarrow$) Suppose that there is a completion V' of $V(B)$ so that p wins the completion $\mathcal{E} = (C, V(\overline{B}) \cup V')$ of (C, V). Observe that in all completions of (C, V), everyone in R beats all her predecessors in $R \cup \{p\}$. Then, by the definition of the agenda, and the fact that p wins $\mathcal{E}$, it holds that (1) q beats all candidates in R, and (2) q is beaten by p in $\mathcal{E}$. As $V(\overline{B})$ contains exactly $2\ell + 3\kappa + 1$ votes (those in $V_3 \cup V_4$) ranking p before q, Condition (2) implies that there are at least $|B| - \kappa$ votes in V' ranking p before q. Let B' be the subset of B corresponding to votes in V' ranking p before q, and let $B'' = B \setminus B'$. Clearly, $|B''| \leq \kappa$. We show below that Condition (1) implies that B'' dominates R. For the sake of contradiction, assume that there exists $r \in R$ not dominated by any vertex in B''. In other words, all the ℓ neighbors of r in G are contained in B'. This implies that there are ℓ votes in V' (the ℓ completions of votes corresponding to the ℓ neighbors of r) ranking r before q. Together with the $|B| + \ell + 2\kappa + 1$ votes in $V(\overline{B})$ ranking r before q (those from $V_1 \cup V_3$), we have $|B| + 2\ell + 2\kappa + 1$ votes ranking r before q, implying that r beats q in $\mathcal{E}$. However, this is impossible, since otherwise r beats all her predecessors in $\mathcal{E}$ which contradicts that p wins $\mathcal{E}$. This completes the proof that B'' dominates R. Then, from $|B''| \leq \kappa$, we know that the RBDS instance is a Yes-instance. □

Our reduction in the proof of Theorem 16 is completely different from those used in [8]. In fact, their reductions are from different problems. Moreover, in their reductions for Possible Winner under the successive rule and the amendment rule the distinguished candidate is respectively the penultimate and the third candidates in the agenda. Our reduction can be adapted for showing the NP-hardness of Possible Winner for TSMR when the distinguished candidate is the i-th candidate in the agenda for every constant i, by adding $i - 1$ dummy candidates before p in the agenda, and ranking all of them below all the other candidates in all votes.

Notice that Possible Winner for TSMR becomes polynomial-time solvable if the given agenda is complete and p is the last one in the agenda. This follows from Observation 1 and the polynomial-time solvability of determining if a partial election can be completed so that a candidate becomes a (weak) Condorcet winner [26]. The algorithm in [26] can be also trivially adapted for showing that Possible Winner for the amendment rule becomes polynomial-time solvable if the given agenda is complete and p is in the top-2 positions of the agenda. So, there is a radical complexity shift for the amendment rule as the distinguished candidate moves from the second to the third place in the agenda. Our next result also reveals a seamless complexity shift for TSMR.

Theorem 17 (★). Possible Winner *for* TSMR *is* NP-hard *even when the given agenda is complete and the distinguished candidate being the penultimate candidate in the agenda.*

5 CONCLUSION

We conducted the complexity of many well-motivated voting problems under the recently proposed voting rule TSMR, and obtained fruitful results (see Table 1 for a summary). Many of our hardness results hold even when the distinguished candidate is the first or the last one in the agenda. Our exploration offers a complete picture of the complexity of these problems under TSMR, enabling us to compare TSMR with the successive rule and the amendment rule. Our results indicate that TSMR resists most of the control problems, but is vulnerable to Agenda Control and Coalition Manipulation. In addition, we showed that for TSMR, Necessary Winner is polynomial-time solvable, while Possible Winner is NP-hard. Compared with previous works, our study suggests that TSMR behaves at least as well as the other two important sequential rules regarding their resistance to strategic voting problems, and their complexity of calculating possible and necessary winners. We point out that our exploration is a pure theoretic analysis. Whether many problems are hard to solve in specific practical settings demands further investigation. An important topic for future research is to investigate if restricting the preference domains radically changes the complexity. We refer to [18] for a comprehensive survey on many restricted preference domains.

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