A Hotelling-Downs Game for Strategic Candidacy with Binary Issues

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ABSTRACT

In a pre-election period, candidates may, in the course of the public political campaign, adopt a strategic behavior by modifying their advertised political views, to obtain a better outcome in the election. This situation can be modeled by a type of strategic candidacy game, close to the Hotelling-Downs framework, which has been investigated in previous works via political views that are positions in a common one-dimensional axis. However, the left-right axis cannot always capture the actual political stances of candidates. Therefore, we propose to model the political views of candidates as opinions over binary issues (e.g., for or against higher taxes, abortion, etc.), implying that the space of possible political views can be represented by a hypercube whose dimension is the number of issues. In this *binary strategic candidacy* game, we introduce the notion of local equilibrium, broader than the Nash equilibrium, which is a stable state with respect to candidates that can change their view on at most a given number of issues. We study the existence of local equilibria in our game and identify natural conditions under which the existence of an equilibrium is guaranteed. To complement our theoretical results, we provide experiments to empirically evaluate the existence of local equilibria and their quality.

KEYWORDS

Strategic candidacy; Hotelling-Downs model; Local equilibrium

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1 INTRODUCTION

Strategic voting [20] is a major topic of interest and has been widely studied in Computational Social Choice [4] and Algorithmic Game Theory [22]. While strategic behavior is typically imputed to voters, candidates can also manipulate in real elections. Strategic candidacy [10] occurs when a candidate may strategize by withdrawing from the election in order to obtain a better outcome. Another perspective by which a candidate can be strategic, is to exhibit an insincere political stance [14, 26]. Instead of presenting themselves truthfully, such candidates adopt a dishonest political position whenever it is beneficial.

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In order to model the political stance taken by candidates, most papers use a one-dimensional axis to describe the left-right axis of the political spectrum, and study the existence of equilibria in this context (see, e.g., [14]). However, such left-right representation of the political spectrum fails to capture the complexity of current political debates. Benoit and Laver [3] claim that "*this drastically oversimplified notion of a 'left-right dimension' refers to potentially separable issues [...] Indeed, it is very common to need more than one dimension to describe key political differences*" (see also [16]).

A more accurate perspective to describe candidates' positions in the political spectrum can be to consider a list of *issues* on which each candidate is either "in favor" or "against" (e.g., for or against higher taxes, euthanasia, abortion, etc.). This modeling of the political spectrum can be represented by a hypercube whose dimension is the number of issues. Thus a candidate can stand on a vertex of the hypercube, and attracts voters who agree with her on all issues, but also the voters for whom she is the "closest" candidate.

Consequently, given a distribution of the voters on the hypercube of issues and the position of her competitors, a candidate may be willing to move strategically from the vertex corresponding to her initial truthful political stance, to another position in the hypercube, in order to obtain a better outcome in the election. This game defines a *binary* variant of strategic candidacy that corresponds to a Hotelling-Downs game [7, 15] on a hypercube structure.

In this model, some moves from a position to another in the hypercube of issues can be unlikely to occur, when these positions are too far apart. Indeed, a candidate would not benefit from expressing very contrary opinions because voters would uncover the strategic and insincere aspect of such move, and would not vote for this dishonest candidate. Thus, it seems realistic to assume that only local moves would be performed. This leads to the definition of a new solution concept, called *t-local equilibrium*, which generalizes the notion of Nash equilibrium, and captures stability w.r.t. moves to positions that differ on at most t issues from the current one.

In this article, we investigate the existence of local and Nash equilibria in binary strategic candidacy games, both theoretically and empirically, and focus on several natural restrictions, either on the distribution of voters or on the structure of candidates' strategy sets. Specifically, we study the impact of restricting to a single-peaked distribution of voters. Such restriction can be interpreted as a homogeneous voting body in which there exists a modal position corresponding to the most frequent political stance; the other positions becoming less and less frequent when moving away from this peak position. Another interesting type of restriction is related to the set of positions in the hypercube a candidate can take. A rationale for this restriction comes from the fact that candidates might not want to deviate too much from their truthful position. Moreover, there could be correlations between issues (or more generally some structure over the set of issues) that imply some forbidden positions (e.g., for abortion and against euthanasia).

2 RELATED WORK

Several attempts to tackle similar problems have been found in the literature, coming from a diversity of areas. The Hotelling-Downs model has existed since its original formulation by Hotelling [15] on the well-known problem of ice-cream vendors positioning themselves strategically on a beach. This idea was later translated to voting theory by Downs [7], adapting the strategic location of vendors to a strategic placement of candidates on a political spectrum. The Hotelling-Downs model (HDM) is one of the most widespread models to interpret scenarios coming both from politics and from economics. A range of variants have been studied over the years, both in the context of *facility location* (the game of companies placing their facilities on a given metric space, trying to attract customers assumed to seek for the closest available seller) [6, 11] and in voting models for positioning of candidates [24].

Sengupta and Sengupta [27] were among the first to make links between the literature of the HDM with that of *strategic candidacy*, an election game where candidates may abstain at will, in order to achieve a result closer to their preference. The original model of *strategic candidacy* was introduced by Dutta et al. [10], being followed along the years by multiple different variants, e.g., mixing strategic voting and strategic candidacy [5], understanding its equilibria [18, 25], or assuming given behaviors for candidates [17, 23].

The first papers (to our knowledge) trying to make the fusion between the Hotelling-Downs model for elections and strategic candidacy are Sabato et al. [26] (with their *real candidacy games*), and Harrenstein et al. [14] (with their *HDM for party nominees*). Quite similar models (although with a different perspective) come from the context of Algorithmic Game Theory, with *Voronoi games*: strategic positioning of players on a metric space, seeking to maximize the amount of points that fall the closest to them. Despite the extensive literature on these games for continuous settings and sequential decisions (see, e.g., [1, 2]), the discrete-setting variant of *Voronoi games on graphs* was relatively recently discussed by Dürr and Thang [9] with the complexity analysis of deciding the existence of a Nash equilibrium. In our binary strategic candidacy game, as it is classical in Voronoi games, the voters split their vote among candidates that are the closest to their truthful position. However, our candidates do not aim to maximize the number of votes they receive (contrary to *Voronoi games*), but want to get a better outcome for the election (like in *strategic candidacy*). The analysis of our game, which is based on a hypercube, has some similarities with that of Voronoi games in transitive graphs [12], in particular on the importance of *antipodal positions* in the graph.

As mentioned by Harrenstein et al. [14], there is really scarce literature on the HDM for elections with multiple participants and restricted strategy sets. In particular, and even though similar games have been studied for general graphs, no evidence was found of an attempt of applying the ideas of Hotelling-Downs specifically to a hypercube over issues, as we do in this article. The main idea of such a model comes from the setting of *Judgment Aggregation (JA)* [19]. In this context, Nehring and Puppe [21] have notably defined *general single-peaked structures*, from which we take inspiration to define single-peaked distributions of voters on the hypercube. The use of the Hamming distance in our study was similarly inspired by this field of research (though other alternatives could have been considered from the vast JA literature, see, e.g., [8]).

3 THE MODEL

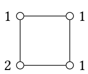
For an integer $k \in \mathbb{N}$, we define $[k] := \{1, \dots, k\}$. We are given a set of voters $N = [n]$, and a set of candidates $C = \{c_1, \dots, c_m\}$. We assume that the population (voters and candidates) is interested in a fixed number $K \in \mathbb{N}$ of relevant binary issues. All possible opinions on these binary issues are given by the set $\mathcal{H} = \{0, 1\}^K$. A position $p \in \mathcal{H}$ representing a global opinion over all issues is a K-vector $p = (p_1, p_2, \dots, p_K)$ where $p_j \in \{0, 1\}$ for all $j \in [K]$. The distance between two positions p and p' in $\mathcal{H}$ is defined as the Hamming distance between the two corresponding vectors, i.e., $dist(p, p') = |\{j \in [K] : p_j \neq p'_j\}|$. The possible positions can be represented on a hypercube graph $G^{\mathcal{H}} := (\mathcal{H}, E)$ where $\{p, p'\} \in E$ iff $dist(p, p') = 1$ for every positions $p, p' \in \mathcal{H}$. The antipodal position $\hat{p}$ of position $p \in \mathcal{H}$ is the position where all opinions are reversed compared to p, i.e., $\hat{p}_i = 1 - p_i$ for every $i \in [K]$.

Each voter $v \in N$ and each candidate $c \in C$ has a position on the hypercube, $p_v \in \mathcal{H}$ and $p_c \in \mathcal{H}$, respectively, representing her truthful opinion about all binary issues. The voters are assumed to focus on the announced opinions of the candidates on the binary issues in order to form their preferences over the candidates. More precisely, the voters prefer the candidates whose announced opinions are closer to theirs. The preferences of each voter $v \in N$ over positions in the hypercube are represented by a weak order $\succsim_v$ over $\mathcal{H}$ such that $p \succsim_v p'$ iff $dist(p, p_v) \leq dist(p', p_v)$ (the strict and symmetric parts of $\succsim_v$ are denoted by $\succ_v$ and $\sim_v$, respectively). The voters can derive, from their fixed preferences over the positions in the hypercube, their preferences over the candidates. The preferences of each voter $v \in N$ over the candidates, w.r.t. a profile of positions $\mathbf{s} = (s_1, \dots, s_m) \in \mathcal{H}^m$ where s_i is the announced position of candidate $c_i \in C$, can be represented by a weak order $\succsim_v^{\mathbf{s}}$ over C defined as follows: $c_i \succsim_v^{\mathbf{s}} c_j$ iff $s_i \succsim_v s_j$, for every $i, j \in [m]$.

The candidates run for an election whose winner is determined by a voting rule $\mathcal{F} : \succsim^{\mathbf{s}} \to C$ that takes as input the preferences of the voters according to a state $\mathbf{s} \in \mathcal{H}^m$ of announced positions of the candidates, or equivalently, the truthful positions of all voters as well as the description of $\mathbf{s}$, and returns a winning candidate in C. We assume $\mathcal{F}$ is resolute therefore, if needed, we use a deterministic tie-breaking rule that is a linear order $\rhd$ over C such that $c_1 \rhd c_2 \rhd \dots c_m$. We focus on a voting rule which is a variant of plurality, where each voter has one point that she divides among the candidates she ranks in the top indifference class of her preference ranking, and $\mathcal{F}$ returns the candidate with the highest score. The score of each candidate $c \in C$ w.r.t. voting rule $\mathcal{F}$ on preference profile $\succsim^{\mathbf{s}}$ is given by a scoring function $sc_{\mathcal{F}}^{\succsim^{\mathbf{s}}} : C \to \mathbb{R}$ (when the context is clear the parameters may be omitted) and $\mathcal{F}(\succsim^{\mathbf{s}}) \in \arg\max_{c \in C} sc_{\mathcal{F}}^{\succsim^{\mathbf{s}}}(c)$.

Example 3.1. Consider an instance with two issues, two candidates c_1 and c_2, and five voters whose truthful positions are

$p_1 = p_2 = (0,0)$, $p_3 = (1,0)$, $p_4 = (0,1)$, and $p_5 = (1,1)$. The two candidates are such that $p_{c_1} = (1,0)$ and $p_{c_2} = (0,1)$ and they announce such truthful positions. The voters can be described as weights related to positions in the hypercube as represented below (left), and their preferences over positions and over candidates can be derived as done below (right). Thus, we have $sc_{\mathcal{F}}(c_1) = sc_{\mathcal{F}}(c_2) = 2.5$ and c_1 wins by the tie-breaking rule.



$$\begin{array}{rccccccccccc}
1,2: & (0,0) & \succ & (1,0) & \sim & (0,1) & \succ & (1,1) & \Rightarrow & c_1 & \sim & c_2 \\
3: & (1,0) & \succ & (0,0) & \sim & (1,1) & \succ & (0,1) & \Rightarrow & c_1 & \succ & c_2 \\
4: & (0,1) & \succ & (0,0) & \sim & (1,1) & \succ & (1,0) & \Rightarrow & c_2 & \succ & c_1 \\
5: & (1,1) & \succ & (1,0) & \sim & (0,1) & \succ & (0,0) & \Rightarrow & c_1 & \sim & c_2
\end{array}$$

3.1 Binary Strategic Candidacy (BSC) Game

The candidates may announce opinions on the issues that do not exactly fit their truthful opinion, in order to alter the outcome of the election towards one they consider *better*. Therefore, analogously to the voters, each candidate $c \in C$ also expresses preferences over the candidates, that are represented by a weak order $\succsim_c$ over C. Basically, since they run for the election, all candidates prefer to be elected than that another candidate is elected, i.e., for every candidate $c \in C$, $\succsim_c$ is such that $c \succ_c c'$ for every $c' \in C \setminus \{c\}$. Note that the candidates may not be willing to announce any possible position in the hypercube (they may not want to lie too much compared to their truthful position). The subset of possible announced positions for candidate $c_i \in C$ is given by $\mathcal{H}_i \subseteq \mathcal{H}$ where $p_{c_i} \in \mathcal{H}_i$.

How the candidates can strategize by advertising political views can be modelled by a strategic game: the *Binary Strategic Candidacy (BSC) game*. In this game, the set of players corresponds to the set of candidates, the set of strategies of each candidate $c_i \in C$ is given by $\mathcal{H}_i$, and a state $\mathbf{s}$ is a profile of announced positions $\mathbf{s} = (s_1, \ldots, s_m)$ where $s_i \in \mathcal{H}_i$ for each candidate $c_i \in C$. A state $\mathbf{s}$ is only evaluated via its winner $\mathcal{F}(\succsim^{\mathbf{s}})$. By abuse of notation, we may directly write $\mathcal{F}(\mathbf{s})$ to denote the winner of the election at state $\mathbf{s}$ according to the fixed preferences of the voters over the positions. A candidate c_i has a better response from state $\mathbf{s}$ if there exists a position $s'_i \in \mathcal{H}_i$ such that $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$. We can thus redefine the well-known solution concept of *Nash equilibrium* for the BSC game.

Definition 3.2 (Nash equilibrium). A state $\mathbf{s} \in \prod_{i=1}^m \mathcal{H}_i$ is a Nash equilibrium if there is no strategy $s'_i \in \mathcal{H}_i$ for a candidate $c_i \in C$ such that $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$.

A Nash equilibrium is immune to unilateral deviations of candidates to another position that would strictly improve the outcome of the election with respect to their preferences. The considered deviations for a candidate c can be of any type within $\mathcal{H}_c$. However, it may not be realistic for a candidate to pass from one announced position to a radically different one: the voters may not trust her. We thus relax the solution concept of Nash equilibrium by considering stability w.r.t. reasonable deviations that are not too far away from the current position of the candidate. This solution concept is the *t-local equilibrium*, given a maximum distance $t \in [K]$.

Definition 3.3 (t-local equilibrium). A state $\mathbf{s} \in \prod_{i=1}^m \mathcal{H}_i$ is a t-local equilibrium if there is no strategy $s'_i \in \mathcal{H}_i$ for a candidate $c_i \in C$ such that $dist(s'_i, s_i) \leq t$ and $\mathcal{F}((s'_i, s_{-i})) \succ_{c_i} \mathcal{F}(\mathbf{s})$.

A Nash equilibrium is equivalent to a K-local equilibrium. Also, a t-local equilibrium is a t'-local equilibrium for every $1 \leq t' \leq t \leq K$. Therefore, in a given BSC game, if a Nash equilibrium exists then a t-local equilibrium exists for every $t \in [K]$, and if a t-local equilibrium does not exist then no Nash equilibrium can exist.

3.2 Restrictions on the BSC Game

Distribution of voters. Each voter $v \in N$ is characterized by her truthful opinion $p_v \in \mathcal{H}$. This means that we can alternatively formulate the set of voters as a *distribution of voters over* $\mathcal{H}$, i.e., a function $f_N : \mathcal{H} \to \mathbb{N}$ such that $\sum_{p \in \mathcal{H}} f_N(p) = n$, counting how many voters have each position $p \in \mathcal{H}$ as their truthful opinion. Let $[x, z] := \{y \in \mathcal{H} : \exists$ a shortest path in $G^{\mathcal{H}}$ between x and z passing through y$\}$ denote all positions *between* x *and* z. A distribution f_N is said to be *single-peaked* if there exists a peak position $p^* \in \mathcal{H}$ such that for every positions $x, y \in \mathcal{H}$, $y \in [x, p^*]$ implies $f_N(x) \leq f_N(y)$. This definition encodes the idea of having a most popular opinion p^* such that, when walking away from it, we find only positions that are at most as popular. A particular case of single-peaked distribution is the *uniform distribution*, in which $f_N : \mathcal{H} \to \mathbb{N}$ is constant. By abuse of notation, for $S \subseteq \mathcal{H}$, we denote by $f_N(S)$ the number of voters whose truthful position is in S, i.e., $f_N(S) := \sum_{p \in S} f_N(p)$.

Candidates' preferences. Beyond the fact that the preferences of the candidates are such that each candidate strictly prefers herself to any other candidate, they can be of several types. We will particularly focus in the article on the following types:

- *fixed*: the candidates' preferences are not affected by the position chosen by the other candidates;[1]
- *narcissistic*: the candidates do not care about the winner if they are not elected, i.e., for every candidate $c \in C$, $\succsim_c$ is such that $c' \sim_c c''$ for every $c', c'' \in C \setminus \{c\}$.

Note that the two types of candidates' preferences coincide when there are only two candidates, and that narcissistic preferences are a specific type of fixed preferences. It follows that a t-local equilibrium under fixed candidates' preferences is also a t-local equilibrium under narcissistic candidates' preferences.

Candidates' strategies. It would seem unnatural if the only possible positions that a candidate may announce were, e.g., antipodal positions. Therefore, a realistic assumption on the set of strategies of a candidate is its connectedness in the hypercube. Another natural restriction would be to assume that the set of strategies of candidate $c_i \in C$ is a *ball of a given radius* b, meaning that all positions at distance at most b from her truthful position are positions that she accepts to announce (a candidate accepts to lie on at most b issues no matter which they are), i.e., $\mathcal{H}_i := \{p \in \mathcal{H} : dist(p, p_{c_i}) \leq b\}$.[2]

Case of $m = 2$ *candidates.* One can exploit the geometric structure of the hypercube, which provides particular insights for the case of two candidates. When we deal with two candidates, the hypercube $\mathcal{H}$ can be easily partitioned into *sets of influence* associated with each candidate and a *set of indifferent positions*. For an index $i \in \{1, 2\}$, let c_{-i} denote candidate c_{3-i}. Given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of influence of candidate c_i for

[1]Note that candidates' preferences determined by the distances between their truthful and their rivals' truthful positions, are a particular case of *fixed preferences*.

[2]Note that candidates' strategies that are balls induce a symmetric neighborhood around the truthful position, which implicitly assumes independence of the issues.

$i \in \{1,2\}$ is denoted by $P_i^{\mathbf{s}}$ and represents the set of positions which are the truthful positions of the voters who strictly prefer c_i to c_{-i}, i.e., $P_i^{\mathbf{s}} := \{p \in \mathcal{H} : dist(p, s_i) < dist(p, s_{-i})\}$. Given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of indifferent positions is defined by $I^{\mathbf{s}} := \{p \in \mathcal{H} : dist(p, s_i) = dist(p, s_{-i})\}$. It follows that, given a strategy profile $\mathbf{s} = (s_1, s_2)$, the set of all possible positions can be partitioned as follows: $\mathcal{H} = P_1^{\mathbf{s}} \cup P_2^{\mathbf{s}} \cup I^{\mathbf{s}}$. This means that for every voter $v \in N$, it holds that $p_v \in P_i^{\mathbf{s}} \Leftrightarrow c_i \succ_v^{\mathbf{s}} c_{-i}$ and $p_v \in I^{\mathbf{s}} \Leftrightarrow c_i \sim_v^{\mathbf{s}} c_{-i}$. Note that the voters whose truthful position is in $I^{\mathbf{s}}$ do not matter for the computation of the scores of the two candidates, since their vote is equally divided between the two candidates. Therefore, the winner w.r.t. $\mathcal{F}$ in state $\mathbf{s}$ only depends on the number of voters whose truthful positions are in $P_1^{\mathbf{s}}$ and in $P_2^{\mathbf{s}}$, i.e., $\mathcal{F}(\mathbf{s}) \in \arg\max_{c_i \in C} f_N(P_i^{\mathbf{s}})$. Hence, understanding the structure of the sets of influence is key for the analysis of the game.

First observe that we can focus on the parts of the strategy positions that are different between the two candidates. Given $\mathbf{s} = (s_1, s_2)$, let $X_=^{\mathbf{s}}$ and $X_{\neq}^{\mathbf{s}}$ denote the sets of issues on which positions s_1 and s_2 agree and disagree, respectively, i.e., $X_=^{\mathbf{s}} := \{j \in [K] : (s_1)_j = (s_2)_j\}$ and $X_{\neq}^{\mathbf{s}} := \{j \in [K] : (s_1)_j \neq (s_2)_j\}$. By definition, we have $[K] = X_=^{s} \cup X_{\neq}^{s}$ and $|X_{\neq}^{\mathbf{s}}| = dist(s_1, s_2)$. Let $dist_{\neq}^{\mathbf{s}}(.)$ denote the distance calculated only on the issues of $X_{\neq}^{\mathbf{s}}$. The sets of influence can be defined only based on $dist_{\neq}^{\mathbf{s}}(.)$.

Observation 3.1. *For every state* $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1,2\}$, *and position* $p \in \mathcal{H}$, *we have* $p \in P_i^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) < dist_{\neq}^{\mathbf{s}}(p, s_{-i})$, *and* $p \in I^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) = dist_{\neq}^{\mathbf{s}}(p, s_{-i})$.

Secondly, we can observe that the sets of influence can be defined w.r.t. the distance between the strategy positions of the two candidates. Given $\mathbf{s} = (s_1, s_2)$ and $r^{\mathbf{s}} := dist(s_1, s_2)$, let $d_{r^{\mathbf{s}}}$ denote the *critical distance* up to which any given candidate has ensured influence, i.e., $d_{r^{\mathbf{s}}} := \lceil \frac{r^{\mathbf{s}}}{2} \rceil - 1$.

Observation 3.2. *For every state* $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1,2\}$, *and position* $p \in \mathcal{H}$, *we have* $p \in P_i^{\mathbf{s}} \Leftrightarrow dist_{\neq}^{\mathbf{s}}(p, s_i) \leq d_{r^{\mathbf{s}}}$ *and* $p \in I^{\mathbf{s}} \Leftrightarrow r^{\mathbf{s}}$ *is even and* $dist_{\neq}^{\mathbf{s}}(p, s_i) = \frac{r^{\mathbf{s}}}{2}$.

Thus, $r^{\mathbf{s}}$ is even iff $I^{\mathbf{s}} \neq \emptyset$. An interesting further remark is that when we change one of the two strategy positions of a state on exactly one issue, then no position can directly pass from an influence set to another, it must intermediately pass by the indifference set, as stated in the next lemma. We denote by $\tilde{P}_i^{\mathbf{s}}$ the set of positions in $P_i^{\mathbf{s}}$ that are at the limit of the set of influence of candidate c_i in $\mathbf{s}$, i.e., $\tilde{P}_i^{\mathbf{s}} := \{p \in P_i^{\mathbf{s}} : dist(p, s_i) = dist(p, s_{-i}) - 1\}$. For a given subset $P \subseteq \mathcal{H}$, let $P_{|x=e}$ denote the subset of positions from P whose value on issue x is equal to $e \in \{0,1\}$, i.e., $P_{|x=e} := \{p \in P : p_x = e\}$.

Lemma 3.4. *If candidate* c_i *for* $i \in \{1,2\}$ *performs a 1-local deviation from state* $\mathbf{s} = (s_i, s_{-i})$ *to state* $\mathbf{s}' = (s_i', s_{-i})$ *where position strategies* s_i *and* s_i' *differ on issue* $x \in [K]$, *then:*

- *if* $r^{\mathbf{s}}$ *is odd, then* $r^{\mathbf{s}'}$ *is even and* $P_i^{\mathbf{s}'} = P_i^{\mathbf{s}} \backslash (\tilde{P}_i^{\mathbf{s}})_{|x=1-(s_i')_x}$, $P_{-i}^{\mathbf{s}'} = P_{-i}^{\mathbf{s}} \setminus (\tilde{P}_{-i}^{\mathbf{s}})_{|x=(s_i')_x}$ *and* $I^{\mathbf{s}'} = (\tilde{P}_i^{\mathbf{s}})_{|x=1-(s_i')_x} \cup (\tilde{P}_{-i}^{\mathbf{s}})_{|x=(s_i')_x}$,
- *otherwise (i.e.,* $r^{\mathbf{s}}$ *is even), then* $r^{\mathbf{s}'}$ *is odd and* $P_i^{\mathbf{s}'} = P_i^{\mathbf{s}} \cup I^{\mathbf{s}}_{|x=(s_i')_x}$, $P_{-i}^{\mathbf{s}'} = P_{-i}^{\mathbf{s}} \cup I^{\mathbf{s}}_{|x=1-(s_i')_x}$, *and* $I^{\mathbf{s}'} = \emptyset$.

Finally, one can observe that the set of influence of a candidate is composed of the antipodal positions of the positions in the set of influence of the other candidate, i.e., for every state $\mathbf{s} \in \mathcal{H}_1 \times \mathcal{H}_2$, $i \in \{1,2\}$, and position $p \in \mathcal{H}$, we have $p \in P_i^{\mathbf{s}}$ iff $\hat{p} \in P_{-i}^{\mathbf{s}}$, and $p \in I^{\mathbf{s}}$ iff $\hat{p} \in I^{\mathbf{s}}$. Thus, the sets of influence always have the same size for both candidates. Hence, under a uniform distribution of voters, both candidates get the same score in all states, ensuring the existence of Nash equilibria.

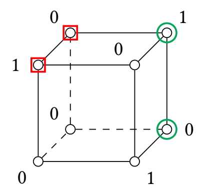
Proposition 3.5. *Every state of a BSC game is a Nash equilibrium when* $m = 2$ *under a uniform distribution of voters.*

4 EXISTENCE OF A LOCAL EQUILIBRIUM

First, a Nash equilibrium may not exist in the game, since even a 1-local equilibrium may not exist under rather strong restrictions.

Proposition 4.1. *A 1-local equilibrium may not exist in a BSC game even when* $m = 2$, *and* $K = 3$.

Proof. Consider a BSC game where $m = 2$, $n = 3$ and $K = 3$. The sets of candidates' strategies are $\mathcal{H}_1 := \{(0,1,0), (0,1,1)\}$ and $\mathcal{H}_2 := \{(1,0,1), (1,1,1)\}$. The distribution of voters on the hypercube as well as the candidates' strategies are represented below on the left (red squares for $\mathcal{H}_1$ and green circles for $\mathcal{H}_2$). The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. From each of these states, there is a 1-local deviation, denoted by an arrow towards a best response for the candidate mentioned next to the arrow.



		$s_2 \in \mathcal{H}_2$		
		$(1,0,1)$		$(1,1,1)$
$s_1 \in \mathcal{H}_1$	$(0,1,0)$	$(1,2)$	$\xleftarrow{c_2}$	$(1.5,1.5)$
		$c_1 \downarrow$		$\uparrow c_1$
	$(0,1,1)$	$(1.5,1.5)$	$\xrightarrow{c_2}$	$(1,2)$

□

However, a 2-local equilibrium always exists for two candidates and an odd number of voters, by considering the outcome $p^m \in \mathcal{H}$ of the *majority rule* from Judgment Aggregation [19] over a voter preference profile given by the truthful positions of all voters, i.e., $(p^m)_j = \arg\max_{e \in \{0,1\}} f_N(\mathcal{H}_{|j=e})$ for all $j \in \mathcal{H}$, in other words p^m captures the *majoritarian view* on each issue.

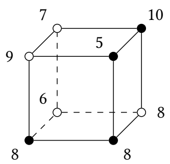
Theorem 4.2. *There always exists a 2-local equilibrium in a BSC game when* $m = 2$, n *is odd, and* $p^m \in \mathcal{H}_1 \cap \mathcal{H}_2$.

Sketch of proof. Given the geometric structure of the influence sets for two candidates, any 1-local deviation by c_2 from state $\mathbf{s}^0 = (p^m, p^m)$, where c_1 wins, will result in *cutting* the hypercube in half (along the issue that was changed). However, by definition, p^m is always on the half of the hypercube that has the most voters, therefore c_1 would still win. For 2-local deviations from $\mathbf{s}^0$, we use the fact that the influence sets for any 2-local move will be included in those of the 1-local deviations that lead to them (two possible ways). Given that on both cases the set is a *cut* of the hypercube, and that p^m always has more voters on its half of the cut, by adding both inequalities, we can conclude that c_1 would still win. □

However, under the same conditions, this positive result cannot be extended to 3-local equilibria, as stated below.

Proposition 4.3. *A 3-local equilibrium may not exist in a BSC game even when $m = 2$, $K = 3$, and the sets of candidates' strategies coincide, contain p^m and are connected.*

Proof. Consider a BSC game with $m = 2$, $n = 61$ and $K = 3$. The sets of strategies are $\mathcal{H}_1 = \mathcal{H}_2 = \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$. The distribution of voters on the hypercube is represented below (left), where the set of strategies of both candidates is marked by black vertices. In this game, $p^m = (1,1,1)$. The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. One can observe that, from each of these states, there is a 3-local deviation.



	$s_2 \in \mathcal{H}_2$			
$s_1 \in \mathcal{H}_1$	$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(1,1,1)$
$(0,0,0)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$
$(1,0,0)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$	$(30, \mathbf{31})$
$(1,1,0)$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$	$(30, \mathbf{31})$
$(1,1,1)$	$(30, \mathbf{31})$	$(\mathbf{31}, 30)$	$(\mathbf{31}, 30)$	$(\mathbf{30.5}, 30.5)$

□

Moreover, deciding about the existence of a Nash equilibrium, and even a 2-local equilibrium, is computationally hard.

Theorem 4.4. *Deciding whether there exists a t-local equilibrium is NP-hard, for $t \in \{2, \ldots, K\}$, even under narcissistic preferences.*

Sketch of proof. We perform a reduction from Exact Cover by 3-Sets (X3C), a problem known to be NP-complete [13]. In an instance of X3C, we are given a set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a set $S = \{S_1, S_2, \ldots, S_r\}$ of 3-element subsets of X and we ask whether there exists an exact cover, i.e., a subset $S' \subseteq S$ such that every element of X occurs in exactly one member of S', in other words S' is a partition of X. We construct a BSC game as follows. First we consider $K = 3q + 4$ issues, and we create $(3q + 10)w_p + 23$ voters, given an arbitrary integer w_p such that $w_p > 24$, where the voters are distributed as follows on the positions of the hypercube:

- w_p voters on each position $e^i = (0, \ldots, 0, 1, 0, \ldots, 0)$ such that $e^i_i = 1$ and $e^i_j = 0$ for every $j \in [3q + 4] \setminus \{i\}$, for every $i \in [3q]$;
- $\frac{5}{2}w_p + 11$ voters on position $p_1 := (0, \ldots, 0, 1, 1, 0, 0)$;
- 7 voters on position $p_2 := (0, \ldots, 0, 0, 0, 1, 1)$;
- $\frac{5}{2}w_p + 3$ voters on position $p_3 := (0, \ldots, 0, 0, 0, 1, 0)$;
- 2 voters on position $p_4 := (0, \ldots, 0, 0, 0, 0, 1)$.

We create $q + 2$ candidates and denote the set of candidates by $C := C_S \cup \{c_a, c_b\}$, where the set $C_S := \bigcup_{j=1}^{q} c_j$ regroups the so-called subset-candidates. The sets of strategies are:

- $\mathcal{H}_c := \mathcal{H}_S := \bigcup_{j=1}^{r}\{s^j = (s_1, \ldots, s_{3q}, 0, 0, 0, 0) \in \{0,1\}^K : \forall i \in [3q],\ s_i = 1 \text{ iff } x_i \in S_j\}$ for every $c \in C_S$;
- $\mathcal{H}_{c_a} := \{s_a^1 := (0, \ldots, 0, 1, 0, 0, 1), s_a^2 := (0, \ldots, 0, 1, 1, 0, 0)\}$;
- $\mathcal{H}_{c_b} := \{s_b^1 := (0, \ldots, 0, 0, 0, 1, 1), s_b^2 := (0, \ldots, 0, 1, 0, 1, 0)\}$.

The candidates' truthful positions are arbitrary and their preferences are narcissistic. We report in Table 1 the number of votes that candidates c_a and c_b can get from positions p_1, p_2, p_3, and p_4.

One can prove that there exists a Nash equilibrium in the BSC game iff there exists a subset of S that is a partition of X.

The idea is that only candidates c_a and c_b may have an incentive to deviate and they would do so only if there is a position e^i for $i \in [3q]$ not "covered" by the strategy position of a subset-candidate.

Table 1: Number of votes, from the voters whose truthful position is in $\{p_1, p_2, p_3, p_4\}$, that candidates c_a and c_b get according to all their possible strategies.

		$\mathcal{H}_{c_b}$	
		s_b^1	s_b^2
$\mathcal{H}_{c_a}$	s_a^1	$(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11)$	$(\frac{5}{4}w_p + 11, \frac{15}{4}w_p + 12)$
	s_a^2	$(\frac{5}{2}w_p + 11, \frac{5}{2}w_p + 12)$	$(\frac{5}{2}w_p + 12, \frac{5}{2}w_p + 11)$

A better response for candidate c_a or c_b would trigger a cycle of local deviations, preventing a Nash equilibrium to exist, as it can be deduced from Table 1. Moreover, the only deviations that c_a or c_b can make are towards another strategy position at distance 2 from their previous strategy position. It follows that the complexity result also holds for 2-local equilibria. □

The question is nevertheless open whether hardness still holds for 1-local equilibria or connected candidates' sets of strategies. Remark that there exists a fixed-parameter tractable algorithm w.r.t. the number of issues and candidates for deciding the existence of a t-local equilibrium, since it suffices to check all the possible states of the game (by the game's structure, checking whether a candidate has an improving Nash deviation may already take $O(2^K)$ steps).

Nevertheless, positive results can be found when restrictions are added on the distribution of voters or on candidates' strategies.

4.1 Restrictions on the Distribution of Voters

Restricting to a single-peaked distribution of voters allows to guarantee the existence of a Nash equilibrium for two candidates.

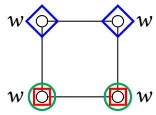
Theorem 4.5. *There always exists a Nash equilibrium in a BSC game under a single-peaked distribution of voters when $m = 2$ and the peak position p^* is included in $\mathcal{H}_1$.*

Sketch of proof. Consider any state $\mathbf{s} = (s_1, s_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ where c_1 chooses the peak position, i.e., $s_1 = p^*$. We will prove that $sc(c_1) \geq sc(c_2)$ must hold and thus, by the tie-breaking rule, c_1 always wins and c_2 cannot change the outcome by deviating to another strategy. If $s_1 = s_2$, then we trivially have $sc(c_1) = sc(c_2)$, and we are done. We thus assume that $s_1 \neq s_2$. One can show that we can construct a perfect matching $\varphi : P_1^{\mathbf{s}} \to P_2^{\mathbf{s}}$ such that each position $p_1 \in P_1^{\mathbf{s}}$ in on a shortest path between $\varphi(p_1) \in P_2^{\mathbf{s}}$ and p^* in $G^{\mathcal{H}}$ implying, by single-peakedness, that $sc(c_1) \geq sc(c_2)$. □

However, this positive result cannot be extended to more than two candidates since even a 1-local equilibrium may not exist.

Proposition 4.6. *A 1-local equilibrium may not exist in a BSC game even when $m = 3$, $K = 2$, the candidates' preferences are fixed, and the distribution of voters is uniform.*

Proof. Consider a BSC game with $m = 3$, $K = 2$ and n is a multiple of 2^K. The voters are distributed in $\mathcal{H}$ in such a way that there are $w := \frac{n}{2^K}$ voters on each position $p \in \mathcal{H}$. The sets of strategies are $\mathcal{H}_1 = \mathcal{H}_2 = \{(0,0), (1,0)\}$ and $\mathcal{H}_3 = \{(0,1), (1,1)\}$. The distribution and the strategies are represented below on the left (red squares for $\mathcal{H}_1$, green circles for $\mathcal{H}_2$, and blue diamonds for $\mathcal{H}_3$). The candidates' preferences are fixed and given below (right).



c_1:	c_1	$\succ$	c_3	$\succ$	c_2
c_2:	c_2	$\succ$	c_1	$\succ$	c_3
c_3:	c_3	$\succ$	c_2	$\succ$	c_1

The table below reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. One can observe that, from each of these states, there is a 1-local deviation.

$s_3 = (0,1)$:

$s_1 \in \mathcal{H}_1$	$s_2 \in \mathcal{H}_2$: $(0,0)$		$(1,0)$
$(0,0)$	$(w, w, \mathbf{2w})$	$\xrightarrow{c_2}$	$(w, \frac{3}{2}\mathbf{w}, \frac{3}{2}w)$
	$c_1 \downarrow$		$\downarrow c_1$
$(1,0)$	$(\frac{3}{2}\mathbf{w}, w, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(\frac{7}{6}w, \frac{7}{6}w, \frac{5}{3}\mathbf{w})$

$s_3 = (1,1)$:

$s_1 \in \mathcal{H}_1$	$s_2 \in \mathcal{H}_2$: $(0,0)$		$(1,0)$
$(0,0)$	$(\frac{7}{6}w, \frac{7}{6}w, \frac{5}{3}\mathbf{w})$	$\xrightarrow{c_2}$	$(\frac{3}{2}\mathbf{w}, w, \frac{3}{2}w)$
	$c_1 \uparrow$		$\uparrow c_1$
$(1,0)$	$(w, \frac{3}{2}\mathbf{w}, \frac{3}{2}w)$	$\xleftarrow{c_2}$	$(w, w, \mathbf{2w})$

c_3

c_3

□

4.2 Restrictions on Candidates' Strategies

The counterexample of Proposition 4.1 for the existence of a 1-local equilibrium is specific because the sets of candidates' strategies are disjoint and contain only two strategies. However, for two candidates and sets of strategies that coincide, there always exists a 1-local equilibrium, as stated more generally in the next theorem.

Theorem 4.7. *There always exists a 1-local equilibrium in a BSC game when $m = 2$ and $\mathcal{H}_2 \subseteq \mathcal{H}_1$. Such an equilibrium can be found in polynomial time.*

Sketch of proof. We construct a particular sequence $s = \langle \mathbf{s}^0, \mathbf{s}^1, \ldots, \mathbf{s}^T \rangle$ of improving 1-local deviations that eventually ends in a 1-local equilibrium after at most $2K$ steps. In the sequence, each state $\mathbf{s}^t$, where t is even, is unanimous with $\mathbf{s}^t = (s_1^{t-1}, s_2^{t-1})$ such that $s_1^{t-1} = s_2^{t-1}$ and makes c_1 win, whereas each state $\mathbf{s}^t$, where t is odd, is such that $\mathbf{s}^t = (s_1^{t-1}, s_2^t)$ with $dist(s_1^{t-1}, s_2^t) = 1$ and makes c_2 win. One can prove that, once c_2 has changed her mind on one issue x, during sequence s, she cannot reverse this opinion on issue x. Hence, c_2 cannot make more than K 1-local deviations in sequence s and thus sequence s eventually ends in a state $\mathbf{s}^t$, with even t, such that $\mathbf{s}^t$ is a 1-local equilibrium. □

This positive result contrasts with the case of Nash equilibria, and even 3-local equilibria, where the same conditions are not sufficient to guarantee the existence, as it can be observed in Proposition 4.3. The question is open whether the existence is guaranteed for 2-local equilibria under the same conditions. Beyond the connections between sets of candidates' strategies, another type of restriction that can be considered concerns the structure of these sets.

Theorem 4.8. *There always exists a 1-local equilibrium in a BSC game when $m = 2$ and candidates' strategies are balls of radius one. Such an equilibrium can be found in polynomial time.*

Sketch of proof. Consider the truthful state $\mathbf{s}^0 = (s_1^0, s_2^0)$ where $s_1^0 = p_{c_1}$ and $s_2^0 = p_{c_2}$. Say that c_i wins in $\mathbf{s}^0$ for some $i \in \{1, 2\}$. Suppose there exists a strategy $s_{-i}^1 \in \mathcal{H}_{-i}$ such that c_i wins in state $\mathbf{s}^1 := (s_i^0, s_{-i}^1)$. The only 1-local deviation that c_{-i} could perform from $\mathbf{s}^1$ is towards her truthful strategy s_{-i}^0. However, this deviation is not a better response because it leads to $\mathbf{s}^0$, thus $\mathbf{s}^1$ is a 1-local equilibrium. Hence, suppose that all possible 1-local deviations of c_{-i} from $\mathbf{s}^0$ lead to a state where c_{-i} wins. Denote by $\mathbf{s}^{1x}$ the state resulting from the deviation from $\mathbf{s}^0$ where c_{-i} changes her strategy s_{-i}^0 only on issue x. Consider the state $\mathbf{s}^{2x}$ which is the same as $\mathbf{s}^{1x}$ except that c_i changes her strategy s_i^0 only on issue x.

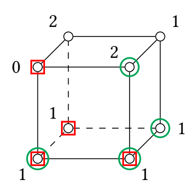
Suppose that $r := dist(s_i^0, s_{-i}^0)$ is even. If there exists an issue $x \in X_{=}^{\mathbf{s}^0}$, then one can prove that $\mathbf{s}^{2x}$ is a 1-local equilibrium. Let us thus assume that all issues are in $X_{\neq}^{\mathbf{s}^0}$. This implies that s_i^0 and s_{-i}^0 are antipodal positions. By Lemma 3.4, among the $n^0 := f_N(I^{\mathbf{s}^0})$ voters whose truthful position is in $I^{\mathbf{s}^0}$, we must have strictly more than $\frac{n^0}{2}$ voters whose truthful position has a value equal to $(s_i^0)_x$ on issue x, for all $x \in [K]$. In the same time, by Observation 3.2, the truthful position of each such voter must have exactly $\frac{K}{2}$ issues with the same value as s_i^0, because they belong to $I^{\mathbf{s}^0}$. By the pigeonhole principle, these two requirements cannot be simultaneously fulfilled, a contradiction.

Suppose now that r is odd. If there exists an issue $x \in X_{\neq}^{\mathbf{s}^0}$, then one can prove that $\mathbf{s}^{2x}$ is a 1-local equilibrium. Let us thus assume that all issues are in $X_{=}^{\mathbf{s}^0}$. It follows that the sets of strategies of both candidates coincide, thus we can use the proof of Theorem 4.7 to construct a 1-local equilibrium. □

The previous positive result for the existence of t-local equilibria when $t = 1$ cannot be extended to larger t, as stated below.

Proposition 4.9. *A 2-local equilibrium may not exist in a BSC game, even when $m = 2$, $K = 3$, and both candidates' strategies are balls of radius one.*

Proof. Consider a BSC game with $m = 2$, $n = 9$ and $K = 3$. The sets of strategies are $\mathcal{H}_1 := \{(0,0,0), (0,1,0), (0,0,1), (1,0,0)\}$ and $\mathcal{H}_2 := \{(1,0,0), (0,0,0), (1,1,0), (1,0,1)\}$, which are balls of radius one around truthful positions $p_{c_1} = (0,0,0)$ and $p_{c_2} = (1,0,0)$, respectively. The distribution of voters and the sets of candidates' strategies (red squares for $\mathcal{H}_1$ and green circles for $\mathcal{H}_2$) are represented below (left). The table below (right) reports all possible states of the game; the number of votes that each candidate gets is given for each state, and it is written in bold to represent the winner. From each of these states, there is a 2-local deviation.

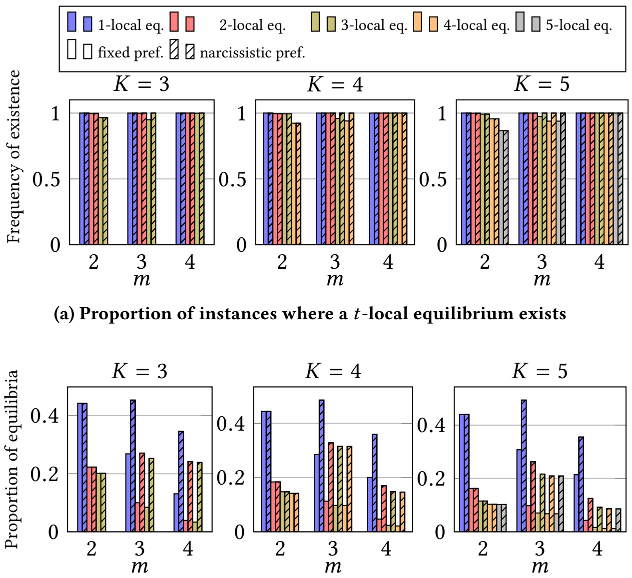


$s_1 \in \mathcal{H}_1$	$s_2 \in \mathcal{H}_2$: $(1,0,0)$	$(0,0,0)$	$(1,0,1)$	$(1,1,0)$
$(0,0,0)$	$(4, \mathbf{5})$	$(\mathbf{4.5}, 4.5)$	$(4, \mathbf{5})$	$(4, \mathbf{5})$
$(1,0,0)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$	$(4, \mathbf{5})$
$(0,1,0)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$
$(0,0,1)$	$(\mathbf{4.5}, 4.5)$	$(\mathbf{5}, 4)$	$(4, \mathbf{5})$	$(\mathbf{5}, 4)$

□

5 EMPIRICAL STUDY OF LOCAL EQUILIBRIA

We also perform an experimental study on synthetic data in order to investigate the behavior of local equilibria in practice. In particular, we will perform two types of analysis: on the equilibria themselves and on the dynamics of local deviations. In general, we generate 1,000 instances of BSC games with 5,000 voters whose truthful position is selected via a uniform distribution over the hypercube of issues (the number of voters does not impact the experiments, if it is large enough, since it only affects the scale of the "weights" associated with each position). The candidates' strategies are "random balls": for each candidate $c \in C$, are generated using a



(a) Proportion of instances where a t-local equilibrium exists

(b) Average proportion of states that are t-local equilibria

Figure 1: Existence experiments for t-local equilibria; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 4, 5\}$ issues, 5,000 voters, all $t \in [K]$, under fixed or narcissistic candidates' preferences.

uniform distribution both distance b and position p_c, and then her set of strategies is defined as a ball of radius b around p_c. Due to computational burden, the choice for the rest of the parameters of the BSC game depends on the type of experiments we perform.

5.1 Existence of Local Equilibria

We first analyze how often local equilibria exist and the proportion of states that are local equilibria. We generate BSC games for $K \in \{3, 4, 5\}$ issues and $m \in \{2, 3, 4\}$ candidates. The candidates' preferences are either fixed and uniformly generated, or narcissistic. For each set of parameters, Figure 1 (a) presents the proportion of the games, over the 1,000 generated games, that admit a t-local equilibrium, for each $t \in [K]$.

The most noteworthy observation from Figure 1, which contrasts with our theoretical results exhibiting several negative results, is that a t-local equilibrium almost always exists for every $t \in [K]$. Indeed, for all sets of experiments under consideration, the frequency of existence is around 95% and is also very often close to 100%. In accordance with the theoretical connection between t-local equilibria, stating that a t-local equilibrium is also a t'-local equilibrium for $t' \leq t$, we observe that the frequency of existence of 1-local equilibria is greater than the frequency of existence of 2-local equilibria, and so on. In particular, for all our choices of parameters, a 1-local equilibrium always exists in the generated games.

We know that an equilibrium under fixed candidates' preferences is also stable under narcissistic preferences. This fact is clearly visible in Figure 1 (a) since the frequency of existence is always greater for narcissistic preferences. Interestingly, in our experiments, all games under narcissistic preferences admit a t-local equilibrium.

Now, we investigate how many states are equilibria. More precisely, by generating all possible states, we verify whether each one is a t-local equilibrium for each $t \in [K]$ and then we compute the average proportion of states that are t-local equilibria over all the 1,000 generated games. The results are presented in Figure 1 (b).

Like for the question of existence, we can recover the connections between t-local equilibria w.r.t. distance t, i.e., there are more 1-local equilibria than 2-local equilibria, and so on. The proportion of t-local equilibria is rather close for all $t \in \{2, \dots, K\}$ however, interestingly, there is a large gap with the proportion of 1-local equilibria which is approximately twice as high as the proportion of 2-local equilibria. This particular behavior of 1-local equilibria is already notable in our theoretical results since our counterexamples for the existence of a Nash equilibrium are typically already counterexamples for the existence of 2-local equilibria.

Note that the number of t-local equilibria under narcissistic preferences is around twice that number under fixed preferences, for all sets of experiments (except for $m = 2$ where they coincide). While the proportion of t-local equilibria tends to decrease when the number of candidates increases under fixed candidates' preferences, this tendency is not visible under narcissistic candidates' preferences. This can be explained by the fact that candidates may have less freedom to strategize when the hypercube is divided among several candidates' sets of influence: it can be more difficult for a candidate to find enough space for a deviation that would make her win.

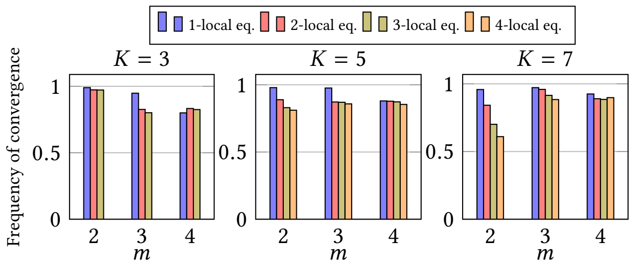
5.2 The Dynamics of Local Deviations

For the experimental study of the dynamics of t-local deviations, we consider successive *rounds* of the game, in which at every given iteration exactly one player is selected (at random) to choose (at random) any t-local best response she might have from the current state. The initial state of the dynamics is the *truthful* state where every candidate $c \in C$ is placed in her truthful position p_c. Whenever such a simulated dynamic *converges*, it is because a t-local equilibrium is reached; we will say the simulated dynamics are *non-convergent* whenever the sequence of visited states *cycles*, i.e., the dynamic returns to an already visited state.

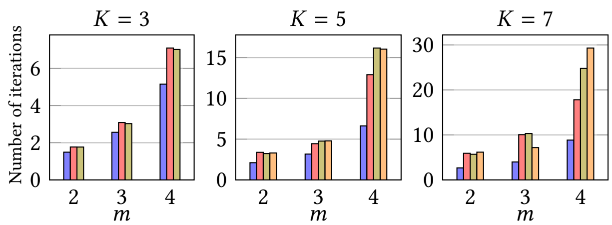
We simulate BSC games for $K \in \{3, 5, 7\}$ issues and $m \in \{2, 3, 4\}$ candidates. All candidates' preferences are fixed and uniformly generated. For each set of parameters, the proportion of games from which the simulated dynamic reached a t-local equilibrium (for $t \in \{1, 2, 3, 4\}$) is represented in Figure 2 (a).

Similarly to our results for the existence, in most cases, t-local equilibria can be reached by randomly following an improving move dynamic from the *truthful* state. Note nevertheless that, for some parameters, around 20% of the time, we do stumble upon *cycles* in the dynamics. It seems like the 1-local dynamic has a higher tendency towards reaching 1-local equilibria, than the rest of dynamics, which would be in line with the fact that 1-local equilibria are more frequent than other t-local equilibria (see Figure 1 (b)).

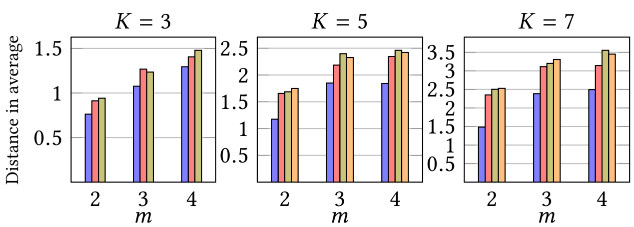
Figure 2 (b) displays, for all the different configurations of our parameter space, the number of *iterations* (or turns) that were required for the dynamic to converge. It is seen, as expected, that for a greater number of candidates, the amount of deviations required to get to a stable state is significantly larger. This is, of course, due to the greater amount of possible candidates who might have an



(a) Average proportion of the dynamics that reached a t-local equilibrium



(b) Average number of iterations before reaching a t-local equilibrium



(c) Average distance between the initial winner's displayed strategy and the winner's displayed strategy at the t-local equilibrium reached by the dynamic

Figure 2: Convergence experiments for the simulated t-local dynamics from the initial *truthful* state; with $m \in \{2, 3, 4\}$ candidates, $K \in \{3, 5, 7\}$ issues, 5,000 voters, and $t \in \{1, 2, 3, 4\}$, under fixed candidates' preferences.

improving deviation. We notice that the number of iterations required to converge increases both with the number of candidates and issues; however this increase is not as significant for the 1-local dynamics as for the other ones (which all behave in a somewhat similar manner). Once again, this may be explained by the fact that 1-local equilibria are significantly more frequent, and thus they might be found *faster* within the dynamic.

Another metric was studied to evaluate the quality of the dynamics: the *distance* between the stance displayed by the initial state's winner and the stance displayed by the winning candidate at the final state of the dynamic. This idea is to identify whether our t-local dynamic brought us to a *radically different winner* from that corresponding to the truthful positions. Figure 2 (c) shows the average distance for each considered combination of parameters.

We clearly see that, given a fixed number of issues, as we increase the number of candidates in the game (and thus, the possibilities of deviating), the average distance gets closer and closer to $\frac{K}{2}$. In some sense, for those cases the *end* position is as good as if it had been chosen uniformly at random (in which case we would see a distance of $\frac{K}{2}$ in expectation). In general, the 1-local dynamic converges to states whose winner is significantly closer to that of the initial state than the dynamics for larger t, whose average distance is nevertheless still below $\frac{K}{2}$. This may also be explained by the proportion of states that are 1-local equilibria: as they are relatively common, a dynamic will likely converge without too drastic deviations from candidates. This result tells us, in a way, that the 1-local dynamic is the most *robust* in terms of how far away the BSC game may take us: it generally converges to a new winner that is not dramatically far away (in *displayed position*) from the original truthful winner. Remark that when there are two candidates, the dynamic in general does not drift too far away from the state of the original winner, possibly due to the high number of equilibria present in the game.

6 CONCLUSION

We have introduced a Hotelling-Downs game to capture the strategic behavior of candidates that may lie about their true opinions in an election. Beyond the classical left-right axis, we have proposed to model political views via binary opinions over issues, leading to work with a very structured environment, i.e., the hypercube. In this context, a natural notion of distance arises, giving birth to the solution concept of local equilibrium. While in general local equilibria may not exist, we have identified several meaningful conditions under which the existence is guaranteed. Moreover, our experimental results balance the apparently negative theoretical results since equilibria almost always exist in practice and can be mostly reached by successive local deviations. All our findings highlight a very interesting behavior for t-local equilibria: it seems that there is a clear frontier for positive results between $t = 1$ and the rest. Since 1-local deviations are the most realistic moves, this suggests that the outcome of an election with strategic candidates may not be disastrous: we would stabilize rather quickly on a equilibrium electing a candidate not that far from the sincere outcome.

Our work opens several interesting and challenging questions. First of all, there are still some gaps in our theoretical results that would be worth investigating. In particular, does an equilibrium always exist under narcissistic preferences, as our experiments suggest? Similarly, it may be of interest to consider voting rules other than plurality when $m \geq 3$. In our specific setting on binary issues, aggregation rules from *Judgment Aggregation* [19] would be particularly relevant, think, e.g., about the classical majority rule which takes the majoritarian outcome on each issue independently. Integrating withdrawal as an additional possible strategy for candidates or assuming that both voters and candidates are strategic (see, e.g., [5]) are also immediate extensions of our model.

A model even closer to that of Harrenstein et al. [14] and to the setting of *Voronoi games on graphs* would be one on which candidates choose to deviate if they are able to *increase the amount of votes that they receive* (without necessarily winning the election). Such lane of study certainly seems like an interesting development to consider. This would nevertheless take us away from the original idea of strategic candidacy where candidates may choose to favor other candidates if they cannot be elected themselves.

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