Facility Location Games with Thresholds

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ABSTRACT

In classic facility location games, a facility is to be placed based on the reported locations from agents. Each agent wants to minimize the cost (distance) between her location and the facility. In real life, the cost of an agent may not strictly increase with the distance. In this paper, we introduce two types of thresholds to the agent's cost. For the model with lower thresholds, the agent's cost is 0 if the distance is within the threshold, otherwise it increases linearly until the value 1. Similarly, for the model with upper thresholds, the cost is 1 if the distance is beyond the threshold, otherwise it is a linear function with the value from 0 to 1. We aim to prevent the agent from misreporting her location while optimizing social objectives in both models. For the first model, we design a strategyproof mechanism optimal for the social cost objective and a strategyproof mechanism with an approximation ratio of 3 for the maximum cost objective. For the second model, we use the median mechanism for the social cost with a threshold-based approximation ratio and design a new mechanism for the maximum cost with tight bounds. We also show lower bounds for both models. Finally, we derive results for the scenario where each agent has both thresholds.

KEYWORDS

Facility Location; Mechanism Design; Threshold

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1 INTRODUCTION

Facility location games consider the scenario of locating facilities with the reported locations from agents, aiming to optimize some social objectives while guaranteeing that misreporting by any agent cannot bring a better outcome for her. Procaccia and Tennenholtz [16] first studied approximate mechanism design in facility location games. In their setting, agents are on a real line and the cost of an agent is defined as the distance between her location and the facility location. Recently, facility location games have been studied extensively with many interesting models arising (See Chan et al. [4] for a survey).

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Motivated by real life, the cost of an agent may not strictly increase with the distance. If we think of the cost as the satisfaction of an agent, there will be satisfaction thresholds for each agent. For instance, an undemanding agent will be satisfied as long as the facility is within a threshold distance, while a demanding agent will be unsatisfied as long as the facility is located outside of a threshold. We call the threshold in the former case lower threshold, and the one in the latter case upper threshold.

For the lower threshold, agents are satisfied if the facility is within a certain distance (for example, 5 minutes walk can be ignored basically). However, a larger lower threshold does not imply a larger upper threshold. For instance, parents are satisfied if the school is built within their lower thresholds, which may differ due to the variety of vehicles to bring their children to school. However, all their upper thresholds are the same since the school only admits students within 5km, and their children are not able to be enrolled if they live 5km away. Namely, different agents may have different lower thresholds but can have the same upper thresholds, which can be interpreted as all agents having only a lower threshold. Another example is building a base station. The experience of chatting and watching YouTube on some devices does not vary within a certain distance from the base station, where the non-affected radii depend on the type of devices. However, the base station has its own signal coverage. If the base station is out of the devices' normal receiving radii, the signal strength will drop, and then the entertainment experience will deteriorate. All devices will lose signal when the distance reaches the maximum serving range of the base station, corresponding to the uniform cost of 1 for all devices.

Compared with the setting where agents only have lower thresholds, it is much more common for agents to have higher thresholds since the cost starts to increase as long as the distance from the facility becomes positive in most previous works. As we have discussed, a demanding agent will not go to the facility if the distance is large (beyond the threshold). Therefore, she will be totally dissatisfied in that case. In addition, different people may have different thresholds. For instance, if we want to build a supermarket on the street, the elderly will care more about distance, while young people will not care much about distance.

A simple way to model the above settings is to divide agents by different categories based on criteria (e.g., age, vehicle, or mobile phone brand) such that agents from different categories have different thresholds. This setting will raise a series of questions: do previous mechanisms still work in the new setting, and how to design new mechanisms if the previous mechanisms do not work?

Motivated by the above observations of realistic scenarios, we study the facility location game with thresholds.

1.1 Our contributions

In this paper, we introduce thresholds to the classic facility location game and investigate two types of thresholds: lower thresholds and upper thresholds. We first study the facility location game with one type of thresholds. Then we extend our results to a setting where each agent has both thresholds. Our results are summarized in Table 1. More specifically,

- For the facility location game with lower thresholds, we show that an optimal mechanism for minimizing the social cost is strategyproof. We design a strategyproof mechanism with an approximation ratio of 3 for minimizing the maximum cost. We also show that the approximation ratio can be better when there are restrictions on the thresholds.
- For the facility location game with upper thresholds, we study two objectives. For the social cost objective, we show that the median mechanism has an approximation ratio of $\max\{\frac{d_{max}}{d_{min}}, 2\}$ if $d_{min} \leq 1/2$; otherwise $\frac{1}{d_{min}}$, where d_{max} and d_{min} are the maximum and minimum upper thresholds among all agents, respectively. For the maximum cost objective, we show that the endpoint mechanism has an approximation ratio of at least $\frac{d_{max}+d_{min}}{d_{min}}$, and design a new strategyproof mechanism with an approximation ratio of 2.
- We combine the results for lower thresholds and upper thresholds to design new strategyproof mechanisms with approximation ratios dependent on thresholds for the general setting where each agent has both thresholds.
- We also give lower bounds for both cost objectives for deterministic strategyproof mechanisms in all models.

1.2 Related work

The agenda of mechanism design for facility location games was first explicitly studied by Procaccia and Tennenholtz [16], but can be traced back to the characterization of mechanisms for single-peaked preferences in [14], and single-plateau preferences in [15]. Since the preference of agents with upper thresholds is a special kind of single-peaked preferences and the preference of agents with lower thresholds is a special kind of single-plateau preferences, the characterizations derived earlier definitely help to reduce the searching space of possible strategyproof mechanisms. However, finding strategyproof mechanisms with good approximation ratios in our setting remains a challenging problem. In the basic setting referred to as the classic facility location game, a facility is to be built according to the reported locations on a line. Each agent aims to minimize her cost which is equal to the distance between herself and the facility. Alon *et al.* [1], Schummer and Vohra [17] extended it to more general networks.

The obnoxious facility game was first proposed by Cheng *et al.* [5], where an obnoxious facility will be located on a closed interval and each agent wants to be as far away from the facility as possible. Mei et al. [13] presented a setting where each agent has two thresholds for an obnoxious facility. In their paper, the two thresholds are identical for all agents denoted by d_1, d_2 and $d_1 < d_2$. The utility of an agent is 0 if her distance from the facility is less than d_1 and it is 1 if the distance is beyond d_2; otherwise, the utility increases linearly from 0 to 1. They revisited a mechanism named *majority mechanism*, locating the facility on an endpoint which is preferred by more agents. Given the very different methodology adopted for the classic facility location games and obnoxious facility games, one could foresee that the methodology we use in facility location games with thresholds is totally different from that used for obnoxious facility games with thresholds. Furthermore, we allow different agents to have different thresholds.

The other model on the threshold for facility location games was proposed by Zhang and Li [22] where the cost of each agent is 0 if the distance is within a threshold; otherwise, it is 1. They also showed an optimal mechanism that is strategyproof. In contrast, we give a more complex and complete model including more general cost functions. The mechanism cannot guarantee both optimality and strategyproofness in most settings..

Works on facility location games with more than one facility can be found in [2, 7, 8, 10, 11, 18, 19, 21, 23]. There are also works on dynamic facility location games [6, 20]. More works on various cost functions can be found in [3, 9, 10, 12, 23]. Most works for facility location games can be found in a recent survey [4].

1.3 Paper Organization

In Section 2, we formulate the facility location game with thresholds. In Section 3, we study the facility location game with lower thresholds. In Section 4, we explore the facility location game with upper thresholds. In Section 5, we extend our work to the facility location game with both thresholds. Due to space limit, some proofs are omitted.

2 PROBLEM STATEMENT

Let $N = \{1, 2, ..., n\}$ be a set of agents on a closed interval $I \in [0, 1]$. Let $D = \{D_1, D_2, \dots\}$ be a set of categories of the agents where $D_i \in [0, 1]^2$. Each agent is described by her profile $r_i = \langle x_i, d_i \rangle$, where $x_i \in I$ is her *location* and $d_i = (d_{i1}, d_{i2}) \in D$ is her *category*, where d_{i1}, d_{i2} can be described as the *lower* and *upper threshold* of agent i. Each agent knows the mechanism and reports her location, which may be different from her true location. Let $\mathbf{r} = (r_1, r_2, \dots, r_n)$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denote the agent profile and the location profile, respectively.

A *mechanism* in this setting is a function f that maps a given agent profile $\mathbf{r}$ to a facility location $y \in I$. We denote the distance between two points by $d(\cdot, \cdot)$. Each agent wants to minimize her cost, which is defined below.

$$c(y, r_i) = \begin{cases} 0 & \text{if } 0 \leq d(y, x_i) < d_{i1} \\ \frac{d(y,x_i)-d_{i1}}{d_{i2}-d_{i1}} & \text{if } d_{i1} < d(y, x_i) \leq d_{i2} \\ 1 & \text{otherwise} \end{cases};$$

Note that when $D = \{(0, 1)\}$, all agents belong to the same category and the cost function is $c(y, r_i) = d(y, x_i)$, which coincide with the classic model proposed by Procaccia and Tennenholtz [16]. When $a = b$ for all $(a, b) \in D$, the cost of each agent is either 0 or 1, which coincides with the threshold model proposed by Zhang and Li [22]. Hence, our setting can be viewed as a generalization of both the classic setting and the threshold setting in the previous work.

Table 1: A summary of our results. d_{max} (d_{min}) means the maximum (minimum) thresholds among all agents. Δd_{max} (Δd_{min}) means the maximum (minimum) difference between the lower and the upper threshold among all agents. †: we also give better upper bounds for some cases. *: we use profiles where all agents have the same thresholds d to show the lower bound. **: we use profiles where all agents have the same difference between the lower and the upper threshold Δd to show the lower bound.

	Lower Threshold	Upper Threshold	Both
Social Cost	UB: 1	UB: $\begin{cases} \max\{2, \frac{d_{max}}{d_{min}}\} & \text{if } d_{min} \le \frac{1}{2} \\ \frac{1}{d_{min}} & \text{otherwise} \end{cases}$ LB*: $\begin{cases} \frac{3}{2} & 0 < d < \frac{2}{5} \\ \frac{1}{d} - 1 & \frac{2}{5} \le d < \frac{1}{2} \\ 2 - \frac{1}{2d} & \frac{1}{2} \le d < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2d} & \frac{2}{3} \le d < 1 \end{cases}$	UB: $\begin{cases} \max\{2, \frac{\Delta d_{max}}{\Delta d_{min}}\} & \text{if } \Delta d_{min} \le \frac{1}{2} \\ \frac{1}{\Delta d_{min}} & \text{otherwise} \end{cases}$ LB**: $\begin{cases} \frac{3}{2} & 0 < \Delta d < \frac{2}{5} \\ \frac{1}{\Delta d} - 1 & \frac{2}{5} \le \Delta d < \frac{1}{2} \\ 2 - \frac{1}{2\Delta d} & \frac{1}{2} \le \Delta d < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2\Delta d} & \frac{2}{3} \le \Delta d < 1 \end{cases}$
Maximum Cost	UB: 3† LB: 2	UB: 2 LB: 2	UB: $1 + \frac{\Delta d_{max}}{\Delta d_{min}}$ LB: 2

We aim to design *strategyproof* mechanisms. A mechanism f is *strategyproof* if an agent cannot decrease her cost by reporting a false location, regardless of the strategies of the other agents. Formally, for any agent $i \in N$, any agent profile $\mathbf{r} = (r_1, r_2, \dots, r_n)$, where $r_i = \langle x_i, d_i \rangle$, and any location $x_i' \in I$, let $r_i' = \langle x_i', d_i \rangle$. We have $c(f(\mathbf{r}), r_i) \le c(f(r_i', \mathbf{r}_{-i}), r_i)$, where $\mathbf{r}_{-i} = (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n)$ is the profile of all agents in $N \backslash \{i\}$.

In this paper, we focus on the objectives of minimizing the social cost (sc), and minimizing the maximum cost (mc):

$$sc(y, \mathbf{r}) = \sum_{i=1}^{n} c(y, r_i), \ mc(y, \mathbf{r}) = \max_{i \in N}\{c(y, r_i)\}.$$

We measure the performance of a mechanism f by the approximation ratio. Let y^* denote the optimal facility location, which achieves the optimal objective value. A mechanism f achieves ρ-approximation ratio, taking the social cost objective for example, if given any agent profile $\mathbf{r}$, we have $sc(f(\mathbf{r}), \mathbf{r}) \le \rho \times sc(y^*, \mathbf{r})$. For a special case where $sc(f(\mathbf{r}), \mathbf{r}) = 0$ for any agent profile $\mathbf{r}$, we say f is 1-approximation.

In Section 3, we consider the case where the agents only have lower thresholds (i.e., all agents have upper thresholds of 1, $D_i \in \{(a, 1) | a \in [0, 1]\}$). In Section 4, we consider the case where the agents only have upper thresholds (i.e., all agents have lower thresholds of 0, $D_i \in \{(0, b) | b \in [0, 1]\}$). To simplify the description, we use $D_i \in [0, 1]$ to denote the element in the category set in those two sections.

3 AGENTS WITH LOWER THRESHOLDS

In this section, we study the case where all agents only have lower thresholds.

3.1 Social Cost

Consider an optimization version of this setting where all locations are public. We can observe that each agent i has no cost in the interval $I_i = [x_i - d_i, x_i + d_i] \cap [0, 1], \ i = 1, 2, \dots, n$. Let y be the facility location. Let $L(y)$ be the set of the agents with $x_i + d_i \le y$ and $R(y)$ be the set of the agents with $x_i - d_i > y$, respectively.

Observation. *If $y_1 < y_2$, then $L(y_1) \subseteq L(y_2)$ and $R(y_2) \subseteq R(y_1)$.*

From the observation, we can see that the function $l(y) = \sum_{i \in L(y)} \frac{1}{1-d_i}$ is non-decreasing on y. Analogously, the function $r(y) = \sum_{i \in R(y)} \frac{1}{1-d_i}$ is non-increasing on y. Hence, we can conclude that the following sets are continuous and bounded.

$$Y_1 = \Big\{ y \Big| \sum_{i \in L(y)} \frac{1}{1 - d_i} < \sum_{i \in R(y)} \frac{1}{1 - d_i} \Big\},$$

$$Y_2 = \Big\{ y \Big| \sum_{i \in L(y)} \frac{1}{1 - d_i} > \sum_{i \in R(y)} \frac{1}{1 - d_i} \Big\}.$$

Moreover, we can see that $Y_1 \cap Y_2 = \emptyset$. Let $y_l = \sup Y_1$ and $y_r = \inf Y_2$. Then we characterize the optimal facility location.

Proposition 1. *Given any agent profile $\mathbf{r} = (r_1, r_2, \dots, r_n)$, where $r_i = \langle x_i, d_i \rangle$, y^* is the optimal facility location if and only if $y^* \in [y_l, y_r]$.*

Proof. We will first show that the social costs of all the points in $[y_l, y_r]$ are equal if $y_l \ne y_r$. By the definition of y_l and y_r, any point $y \in (y_l, y_r)$ satisfies

$$\sum_{i \in L(y)} \frac{1}{1 - d_i} = \sum_{i \in R(y)} \frac{1}{1 - d_i}. \tag{1}$$

Consider any two points $y_1, y_2 \in (y_l, y_r)$ and $y_1 < y_2$. From the discussion before Proposition 1, we can see that $L(y_1) = L(y_2)$ and $R(y_1) = R(y_2)$. Consequently, we have

$$\begin{aligned} sc(y_1, \mathbf{r}) &= \sum_{i \in L(y_1)} \frac{y_1 - (x_i + d_i)}{1 - d_i} + \sum_{i \in R(y_1)} \frac{(x_i - d_i) - y_1}{1 - d_i} \\ &= \sum_{i \in L(y_1)} \frac{y_1}{1 - d_i} - \sum_{i \in R(y_1)} \frac{y_1}{1 - d_i} - \sum_{i \in L(y_1)} \frac{x_i + d_i}{1 - d_i} + \sum_{i \in R(y_1)} \frac{x_i - d_i}{1 - d_i} \\ &= - \sum_{i \in L(y_1)} \frac{x_i + d_i}{1 - d_i} + \sum_{i \in R(y_1)} \frac{x_i - d_i}{1 - d_i} \\ &= - \sum_{i \in L(y_2)} \frac{x_i + d_i}{1 - d_i} + \sum_{i \in R(y_2)} \frac{x_i - d_i}{1 - d_i} + \sum_{i \in L(y_2)} \frac{y_2}{1 - d_i} - \sum_{i \in R(y_2)} \frac{y_2}{1 - d_i} \\ &= \sum_{i \in L(y_2)} \frac{y_2 - (x_i + d_i)}{1 - d_i} + \sum_{i \in R(y_2)} \frac{(x_i - d_i) - y_2}{1 - d_i} = sc(y_2, \mathbf{r}), \end{aligned}$$

where the second and the fourth equalities are by Equation (1).

We have already shown the social costs of all the points in (y_l, y_r) are equal. Besides, the social cost function is continuous, which then implies that the social costs of all the points in $[y_l, y_r]$ are equal.

Finally, we need to show that $sc(y_1, \mathbf{r}) > sc(y_2, \mathbf{r})$ for any $y_1 < y_2 \leq y_l$.

The social cost of y_1 is

$$\begin{aligned} sc(y_1, \mathbf{r}) &= \sum_{i \in L(y_1)} \frac{y_1 - x_i - d_i}{1 - d_i} + \sum_{i \in R(y_1)} \frac{x_i - d_i - y_1}{1 - d_i} \\ &= \sum_{i \in L(y_1)} \frac{y_2 - (x_i + d_i)}{1 - d_i} - \sum_{i \in L(y_1)} \frac{y_2 - y_1}{1 - d_i} + \sum_{i \in R(y_2)} \frac{x_i - d_i - y_2}{1 - d_i} \\ &+ \sum_{i \in R(y_2)} \frac{y_2 - y_1}{1 - d_i} + \sum_{i \in R(y_1) - R(y_2)} \frac{x_i - d_i - y_1}{1 - d_i} \\ &\geq \sum_{i \in L(y_2)} \frac{y_2 - x_i - d_i}{1 - d_i} + \sum_{i \in R(y_2)} \frac{x_i - d_i - y_2}{1 - d_i} \\ &+ (y_2 - y_1)\Big(\sum_{i \in R(y_2)} \frac{1}{1 - d_i} - \sum_{i \in L(y_1)} \frac{1}{1 - d_i}\Big). \end{aligned}$$

Note that

$$\sum_{i \in L(y_1)} \frac{1}{1 - d_i} \leq \sum_{i \in L(y_2)} \frac{1}{1 - d_i} < \sum_{i \in R(y_2)} \frac{1}{1 - d_i}.$$

Therefore, $sc(y_1, \mathbf{r}) > sc(y_2, \mathbf{r})$. This completes the proof. □

Observe that $\sum_{i \in L(y)} \frac{1}{1 - d_i}$ and $\sum_{i \in R(y)} \frac{1}{1 - d_i}$ can only change at $x_j \pm d_j$, for $j = 1, 2, \ldots, n$. Hence, we have the following mechanism outputting the optimal solution.

Mechanism 1. *Given any agent profile* $\mathbf{r} = (r_1, \ldots, r_n)$*, where* $r_i = \langle x_i, d_i \rangle$*, let* $Y = \{x_i \pm d_i | i = 1, 2, \ldots, n\}$*. Let* $y_i, i = 1, 2, \ldots, 2n$ *denote the elements in* Y*. Without loss of generality, we assume that* $y_1 \leq y_2 \leq \cdots \leq y_{2n}$*. Output the smallest* y_k *such that*

$$\sum_{i \in L(y_k)} \frac{1}{1 - d_i} \geq \sum_{i \in R(y_k)} \frac{1}{1 - d_i}.$$

Now we show the strategyproofness of Mechanism 1.

Theorem 1. *Mechanism 1 is strategyproof and optimal for minimizing the social cost.*

Hence, when the agents only have lower thresholds, the social cost can in fact be minimized using a strategyproof mechanism. Next, we consider minimizing the maximum cost.

3.2 Maximum Cost

Different from the social cost, here the situation becomes more complicated. We first show the approximation ratio of Mechanism 1 in this setting.

Proposition 2. *The approximation ratio of Mechanism 1 is* n *for minimizing the maximum cost.*

Intuitively, the approximation ratio n is quite large since the existing lower bound is 2 for the classic setting [16], which is the special case of our setting. The diversity of thresholds among agents is a big challenge to narrow the gap, since we can easily get a 2-approximation strategyproof mechanism if the thresholds for all the agents are the same , i.e., $D = \{d\}, d \in [0, 1]$. Without loss of generality, we assume that $x_1 \leq x_2 \leq \ldots \leq x_n$. The mechanism outputs $\min\{1, (x_1 + d)\}$. The procedure of checking the strategyproofness is quite standard. The approximation ratio follows from an observation that the optimal solution equals $\max\{\frac{x_n - x_1 - 2d}{2(1-d)}, 0\}$ while the mechanism's solution is $\max\{\frac{x_n - x_1 - 2d}{1-d}, 0\}$.

Proposition 3. *There exists a strategyproof mechanism with* 2*-approximation ratio for minimizing the maximum cost when* $D = \{d\}, d \in [0, 1]$*.*

However, we cannot apply the previously designed mechanism in the general setting, even if there are two categories of thresholds. Consider an instance where $r_1 = \langle 0, 0 \rangle$ and $r_2 = \langle 1, d \rangle$, the mechanism puts the facility at 0 and achieves the maximum cost of 1, while the optimal location is somewhere between 0 and $1 - d$. When d approaches 1, the optimal objective value approaches 0, and therefore the approximation ratio approaches infinity. Despite the unbounded approximation ratio, it does not mean putting the facility at $x_1 + d_1$ is meaningless. One can observe that if we narrow the value range of the thresholds, i.e, restrict the maximum value of the thresholds, a constant approximation ratio can be guaranteed. For instance, if every agent's threshold is less than $1/2$, namely, $\forall i \in N$, $d_i < 1/2$, putting the facility at $y = \min\{\min_i\{x_i + d_i\}, 1\}$ achieves a constant approximation ratio. Before we show the approximation ratio, we introduce a lemma first.

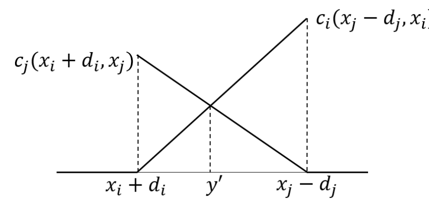


Figure 1: Cost functions of agent i and agent j in the model with lower thresholds.

Lemma 1. *As Figure 1 shows, given two agent profiles* r_i *and* r_j *where* $x_i + d_i < x_j - d_j$*. Let* $y' \in (x_i + d_i, x_j - d_j)$ *be a location that achieves* $c(y', r_i) = c(y', r_j)$*. We have* $\frac{c(x_i + d_i, r_j)}{c(y', r_j)} = 1 + \frac{1 - d_i}{1 - d_j}$ *and* $\frac{c(x_j - d_j, r_i)}{c(y', r_i)} = 1 + \frac{1 - d_j}{1 - d_i}$*.*

Proposition 4. *There exists a strategyproof mechanism with* $1 + \frac{1 - d_{min}}{1 - d_{max}}$*-approximation ratio for minimizing the maximum cost when* $D_i \in [0, \frac{1}{2})$ *for all* $D_i \in D$*, where* $d_{max} = \max_{i \in D}\{D_i\}$ *and* $d_{min} = \min_{D_i \in D}\{D_i\}$*.*

Proof. Recall that we put the facility at $\min\{\min_i\{x_i + d_i\}, 1\}$.

If $\min_{i \in N}\{x_i + d_i\} > 1$, the facility location $y = 1 \in [x_i, x_i + d_i)$ for all agent i, implying that the costs of all agents are 0. Hence, they have no incentive to misreport their locations.

If $\min_{i \in N}\{x_i + d_i\} \leq 1$, without loss of generality we assume that $mc(y, \mathbf{r}) = c(y, r_j)$ where y is the facility location. For agent i who is on the left of y, because $y = \min_{i \in N}\{x_i + d_i\}$, we have $x_i \leq y \leq x_i + d_i$ and her cost is 0. Hence, she has no incentive to misreport her location. For agent i who is on the right of y. If $x_i - d_i \leq y$, her cost is 0 and she has no incentive to misreport her location. If $x_i - d_i > y$, she cannot change the facility location unless she moves to the left side of y and makes $x_i + d_i < y$, which will move the facility farther away from her. Hence, she has no incentive to misreport her location.

Therefore, strategyproofness has been proved. Then we show the approximation ratio.

If $\min_{i\in N}\{x_i + d_i\} > 1$, because all agents' costs are 0, the approximation ratio is 1.

If $\min_{i\in N}\{x_i + d_i\} \le 1$, without loss of generality we assume that $mc(y,\mathbf{r}) = c(y, r_j)$ and $l = \arg\min_i\{x_i + d_i\}$. Then we have the facility location $y = x_l + d_l$. For agent j, if $x_j - d_j \le y$, from the proof of strategyproofness we know that her cost is 0. Then we have $mc(y,\mathbf{r}) = c(y, r_j) = 0$ and the approximation ratio is 1. If $x_j - d_j > y$. Let y^* be the optimal facility location. The cost of agent l is increasing in $[y, x_j - d_j]$ and the cost of agent j is decreasing in $[y, x_j - d_j]$. By Lemma 1 we know that there exists a $y' \in (y, x_j - d_j)$ with $c(y', r_l) = c(y', r_j)$, and we have

$$\frac{c(y,r_j)}{c(y',r_j)} = 1 + \frac{1-d_l}{1-d_j}$$

If $y^* > y'$, we have $mc(y^*,\mathbf{r}) > c(y^*, r_l) > c(y', r_l) = c(y', r_j)$. If $y^* \le y'$, we have $mc(y^*,\mathbf{r}) \ge c(y^*, r_j) \ge c(y', r_j)$. Finally, we have the approximation ratio

$$\rho = \frac{mc(y,\mathbf{r})}{mc(y^*,\mathbf{r})} \le \frac{c(y,r_j)}{c(y',r_j)} = 1 + \frac{1-d_l}{1-d_j} \le 1 + \frac{1-d_{min}}{1-d_{max}}.$$

□

When $d_{min} = 0$ and d_{max} approaches $\frac{1}{2}$, the approximation ratio approaches 3. Since we can get a constant approximation ratio when $D_i \in [0, \frac{1}{2})$, one may wonder what if $D_i \in (\frac{1}{2}, 1]$. If every agent's threshold is at least $1/2$, we can put the facility at $1/2$ and get the maximum cost of 0 since the distance between an agent and the facility is at most $1/2$.

Proposition 5. *There exists a strategyproof mechanism which can minimize the maximum cost when $D_i \in [\frac{1}{2}, 1]$ for all $D_i \in D$.*

Now, we are ready to deal with the following problem, how to design a strategyproof mechanism with a constant approximation ratio when there are two categories of thresholds, i.e, $D = \{d_{min}, d_{max}\}$, $d_{min} < \frac{1}{2}$ and $d_{max} \ge \frac{1}{2}$. Inspired by Propositions 4 and 5, we aim to design a mechanism that make the agents with thresholds d_{max} have costs 0. Hence, only agents with thresholds d_{min} can have the positive costs. Our new mechanism is putting the facility at $y = \min\{x_{lmax} + d_{lmax}, x_r - d_r\}$ where $lmax = \arg\min_{i\in N; d_i = d_{max}}\{x_i + d_i\}$ and $r = \arg\max_{i\in N}\{x_i - d_i\}$.

Lemma 2. *The previously designed mechanism will make the costs of all agents with a threshold of d_{max} be 0, when $D = \{d_{min}, d_{max}\}$, $d_{min} < \frac{1}{2}$ and $d_{max} \ge \frac{1}{2}$.*

Then we show its strategyproofness and approximation ratio.

Proposition 6. *There exists a strategyproof mechanism with approximation ratio 2 for minimizing the maximum cost when $D = \{d_{min}, d_{max}\}$, $d_{min} < \frac{1}{2}$ and $d_{max} \ge \frac{1}{2}$.*

Hence, if the agent threshold is either d_{min} or d_{max}, the tight bounds can be achieved. However, directly applying the previously defined mechanism to the arbitrary D setting cannot achieve a constant approximation ratio. One example is $\mathbf{r} = \{\langle 0, d_{max}\rangle, \langle 0, d_{max} - \epsilon\rangle, \langle 1, 1 - d_{max}\rangle\}$, for which we put the facility at d_{max} and get the approximation ratio of $1 + \frac{d_{max}-\epsilon}{1-d_{max}}$. The intuitive explanation is that the mechanism can only make all agents with the largest thresholds have cost 0, which is effective for the setting $D = \{d_{min}, d_{max}\}$. While for the general setting, there will be agents with the second largest thresholds (like the second agent in the above example) and so on, who have a big effect on the approximation ratio. Hence, is there a mechanism that combines the advantages of the first three mechanisms and, most importantly, makes all agents with thresholds of at least $\frac{1}{2}$ have cost 0? Finally, we have the following mechanism.

Mechanism 2. *If $\forall i$, $d_i < \frac{1}{2}$, put the facility at $\min\{\min_i\{x_i + d_i\}, 1\}$. Otherwise, put the facility at $y = \min\{x_{lm} + d_{lm}, x_r - d_r\}$ where $lm = \arg\min_{i\in N; d_i \ge \frac{1}{2}}\{x_i + d_i\}$ and $r = \arg\max_{i\in N}\{x_i - d_i\}$.*

Lemma 3. *The output by Mechanism 2 will make the cost of all agents with a threshold of at least $\frac{1}{2}$ be 0.*

Theorem 2. *Mechanism 2 is strategyproof and has an approximation ratio of 3 for minimizing the maximum cost without restrictions of thresholds.*

Proof. If all agents have a threshold less than $\frac{1}{2}$, we can use the proof of Proposition 4 to show its strategyproofness and the approximation ratio.

For the remaining cases, by Lemma 3 we know that all the agents with a threshold of at least $\frac{1}{2}$ have no incentive to misreport their locations and their costs are 0. Then we focus on agents with a threshold of less than $\frac{1}{2}$ below.

If $x_{lm} + d_{lm} \le x_r - d_r$, for agents on the left of $x_{lm} + d_{lm}$ with positive costs, they cannot move the facility by misreporting. For agents on the right of $x_{lm} + d_{lm}$, they can only move the facility by making $x_r - d_r < x_{lm} + d_{lm}$, which makes the facility move to the left, farther away from them. Therefore, they have no incentive to misreport their locations. Then we show the approximation ratio. We assume that $mc(y,\mathbf{r})$ is achieved by agent p with $d_p < \frac{1}{2}$. Then we have $x_p + d_p < y$ or $x_p - d_p > y$, otherwise $c(y, r_p) = 0$ and the approximation ratio is 1.

If $x_p + d_p < y$, we have $c(x_p + d_p, r_r) > c(y, r_r)$ since y is on the left of $x_r - d_r$. By Lemma 1 we know that there exists a $y' \in (x_p + d_p, x_r - d_r)$ with $c_p(y', x_p) = c_r(y', x_r)$ and $\frac{c(x_p+d_p,r_r)}{c(y',r_r)} = 1 + \frac{1-d_p}{1-d_r}$. Because $x_r - d_r \ge x_{lm} + d_{lm} \ge \frac{1}{2}$, we have $d_r \le \frac{1}{2}$. Then we further have $\frac{1-d_p}{1-d_r} \le 3$. Moreover, if the optimal solution $y^* \le y'$, $mc(y^*,\mathbf{r}) \ge c(y^*, r_r) \ge c(y', r_r)$. If $y^* > y'$, then $mc(y^*,\mathbf{r}) > c_p(y^*, r_p) \ge c(y', r_p)$. Therefore, $mc(y^*,\mathbf{r}) \ge c(y', r_r)$. Then the approximation ratio

$$\rho = \frac{mc(y,\mathbf{r})}{mc(y^*,\mathbf{r})} < \frac{c(x_p + d_p, r_r)}{c(y', r_r)} \le 3.$$

If $x_p - d_p > y$, by Lemma 1 we know that there exists a $y' \in (y, x_p - d_p)$ with $c(y', r_p) = c(y', r_{lm})$ and $\frac{c(y,r_p)}{c(y',r_p)} = 1 + \frac{1-d_{lm}}{1-d_p} < 2$. Then we use the similar analysis as the case $x_p + d_p < y$ to prove $mc(y^*,\mathbf{r}) \ge c(y', r_p)$ and the approximation ratio

$$\rho = \frac{mc(y,\mathbf{r})}{mc(y^*,\mathbf{r})} < \frac{c(y,r_p)}{c(y',r_p)} = 1 + \frac{1-d_{lm}}{1-d_p} < 2$$

since $d_p < \frac{1}{2} \le d_{lm}$.

Table 2: The results of the maximum cost for the case of agents with lower thresholds. All lower bounds come from [16].

Constraints	Upper Bound	Lower Bound
$D = \{d\}$	2 (Pro 3)	2
$D_i \in [0, \frac{1}{2})$	$1 + \frac{1-d_{min}}{1-d_{max}} \leq 3$ (Pro 4)	2
$D_i \in [\frac{1}{2}, 1]$	1 (Pro 5)	
$D = \{d_{min}, d_{max}\}$	2 (Pro 6)	2
null	3 (Thm 2)	2

If $x_{lm}+d_{lm} > x_r - d_r$, for agents on the left of $x_r - d_r$, they cannot move the facility since $d_i < \frac{1}{2}$, and all agents on the right of $x_r - d_r$ have costs 0 by the definition of r. Therefore, they have no incentive to misreport their locations. Then we can use the similar way as the case of $x_{lm} + d_{lm} \leq x_r - d_r$ to show the approximation ratio. Assume that $mc(y, \mathbf{r})$ is achieved by agent p, if $x_p + d_p < x_r - d_r$, we use agents p and r with Lemma 1 to prove the approximation ratio is at most $1 + \frac{1-d_r}{1-d_p} = 3$. Moreover, $x_p - d_p$ cannot be larger than $x_r - d_r$. Therefore, all cases have been discussed. □

Table 2 summarizes the results of minimizing the maximum cost for the case of agents with lower thresholds. Despite there is a small gap between the upper bounds and lower bounds for the general case, we achieve tight bounds for many settings.

4 AGENTS WITH UPPER THRESHOLDS

As we have discussed, if $D = \{(0, 1)\}$, it is the classic facility location game, and therefore is excluded in this paper. We first study the social cost objective.

4.1 Social Cost

Proposition 7. *There exists an optimal facility location on an agent's location.*

Therefore, we have the following optimal mechanism.

Mechanism 3. *Given a profile $\mathbf{r} = (r_1, \ldots, r_n)$, where $r_i = \langle x_i, d_i \rangle$, output x_l, where $l = \arg\min_{i \in N} sc(x_i, \mathbf{r})$. If there are multiple candidate points, output the leftmost one.*

By doing an intricate analysis of carefully chosen profile instances, we can check that Mechanism 3 is not strategyproof. Then we consider the median mechanism.

Mechanism 4. *Put the facility at $y = x_{\lceil \frac{n}{2} \rceil}$*

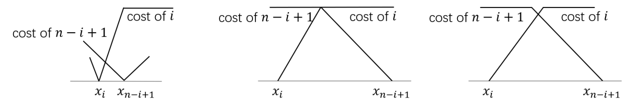
Since it is easy to check that Mechanism 4 is strategyproof, we focus on the approximation ratio.

Theorem 3. *Let $d_{max} = \max_{i \in N}\{d_i\}$ and $d_{min} = \min_{i \in N}\{d_i\}$. The approximation ratio of Mechanism 4 is*

$$\rho = \begin{cases} \max\{2, \frac{d_{max}}{d_{min}}\} & \text{if } d_{min} \leq \frac{1}{2} \\ \frac{1}{d_{min}} & \text{otherwise} \end{cases} \tag{2}$$

for minimizing the social cost.

Proof. For the case of $d_{min} \leq \frac{1}{2}$, first we consider the total cost achieved by a pair of agents where agent i is on the left of the y and agent $n-i+1$ is on the right of y. Without loss of generality we assume that $d_i \leq d_{n-i+1}$. Within interval $[x_i, x_{n-i+1}]$, we observe that x_i achieves the minimum total cost and $\min\{x_i + d_i, x_{n-i+1}\}$



(a) $sc_{i,j}(x_i, \mathbf{r}) < 1$ (b) $sc_{i,j}(x_i, \mathbf{r}) \geq 1$ (c) $d_{min} > 1/2$

Figure 2: Three cases of a pair of agents where $j = n - i + 1$

achieves the maximum total cost. Then we have $sc_{i,n-i+1}(y, \mathbf{r}) \leq sc_{i,n-i+1}(\min\{x_i + d_i, x_{n-i+1}\}, \mathbf{r})$ where $sc_{i,j}$ is total cost of agent i and j. As Figure 2(a) shows, if $sc_{i,n-i+1}(x_i, \mathbf{r}) < 1$, we have

$$\begin{aligned} sc_{i,n-i+1}(\min\{x_i + d_i, x_{n-i+1}\}, \mathbf{r}) &\leq \frac{d_{n-i+1}}{d_i} sc_{i,n-i+1}(x_i, \mathbf{r}) \\ &\leq \frac{d_{max}}{d_{min}} sc_{i,n-i+1}(x_i, \mathbf{r}). \end{aligned}$$

Otherwise, as Figure 2(b) shows, $sc_{i,n-i+1}(\min\{x_i+d_i, x_{n-i+1}\}, \mathbf{r}) \leq 2$ since there are two agents, which is equivalent to

$$sc_{i,n-i+1}(\min\{x_i + d_i, x_{n-i+1}\}, \mathbf{r}) \leq 2 sc_{i,n-i+1}(x_i, \mathbf{r}).$$

Then we have the social cost

$$sc(y, \mathbf{r}) \leq \sum_{i \leq \lfloor \frac{n}{2} \rfloor} sc_{i,n-i+1}(\min\{x_i + d_i, x_{n-i+1}\}, \mathbf{r}).$$

Let y^* be the optimal output, we have

$$sc(y^*, \mathbf{r}) \geq \sum_{i \leq \lfloor \frac{n}{2} \rfloor} sc_{i,n-i+1}(x_i, \mathbf{r}).$$

Therefore, the approximation ratio $\rho = \frac{sc(y,\mathbf{r})}{sc(y^*,\mathbf{r})} \leq \max\{\frac{d_{max}}{d_{min}}, 2\}$.

However, the approximation can be better than 2 in some cases. The facility location achieving the approximation 2 is a place where both agents have costs 1, while for the case of $d_{min} > \frac{1}{2}$, there is no such a location since $x_i + d_i > 1/2$ and $x_{n-i+1} - d_{n-i+1} < 1/2$. Therefore, as Figure 2(c) shows, Mechanism 4 gives $\frac{1}{d_{min}}$-approximation ratio for total cost of a pair of agents, implying the approximation ratio for the social cost is $\frac{1}{d_{min}}$. □

Then we turn to lower bounds, which are proved by using profiles with the same threshold.

Theorem 4. *Let $D = \{d\}$. There does not exist any strategyproof mechanism with an approximation ratio less than*

$$\begin{cases} \frac{3}{2} & 0 < d < \frac{2}{5} \\ \frac{1}{d} - 1 & \frac{2}{5} \leq d < \frac{1}{2} \\ 2 - \frac{1}{2d} & \frac{1}{2} \leq d < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2d} & \frac{2}{3} \leq d < 1 \end{cases}.$$

Proof. In the following proof, we will use a profile $\mathbf{x}'$ with two agents at 0 and another two agents at 1. In this profile, the optimal solution will be 0 or 1 with the social cost of 2.

Case 1: $0 < d < 2/5$. Consider a location profile $\mathbf{x}$ with one agent at 0, one agent at d, and another two agents at 1. Assume for contradiction that there exists a strategyproof mechanism f with approximation ratio less than $\frac{3}{2}$. For the location profile $\mathbf{x}$, the optimal facility location is at 1 and has the social cost of 2. Let y be the output of mechanism f. Due to the approximation ratio, the social cost for y is less than 3, which implies $y > 1 - \frac{d}{2}$. Let y' denote mechanism f's output for the location profile $\mathbf{x}'$. The social cost for y' should be less than 3 and without loss of generality we assume that $y' < \frac{d}{2}$. Then we can reach a contradiction to the strategyproofness since the agent at d can be better off by lying to the location of 0 due to $|d - y'| < d < \frac{2}{5}$ and $y - d > (1 - \frac{d}{2}) - d = 1 - \frac{3}{2}d > \frac{2}{5}$.

Case 2: $2/5 \leq d < 1/2$. We reuse the profiles in the first case. For the location profile $\mathbf{x}$, the optimal facility location is at 1 and has the social cost of 2. Let y be the output of mechanism f. Due to the approximation ratio, the social cost for y is less than $\frac{2}{d} - 2$, which implies $y > 2d$. Let y' be mechanism f's output for the location profile $\mathbf{x}'$. The social cost for y' should be less than $\frac{2}{d} - 2$ and without loss of generality we assume that $y' < 1 - 2d$. Then we can reach a contradiction to the strategyproofness since the agent at d can be better off by lying to the location of 0 due to $|d - y'| < d$ and $y - d > 2d - d = d$.

Case 3: $1/2 \leq d < 2/3$. In this case, we use new profiles. Consider a location profile $\mathbf{x}$ with one agent at 0, one agent at $1 - d$, and another two agents at 1. Assume for contradiction that there exists a strategyproof mechanism f with approximation ratio less than $2 - \frac{1}{2d}$. For the location profile $\mathbf{x}$, the optimal facility location is at 1. Let y be the output of mechanism f. Due to the approximation ratio, the social cost for y is less than $4 - \frac{1}{d}$, which implies $y > 2 - 2d$. Let y' denote mechanism f's output for the location profile $\mathbf{x}'$. The social cost for y' should be less than $4 - \frac{1}{d}$, which implies that $y' < d - \frac{1}{2}$ or $y' > \frac{3}{2} - d$. Because the profile is symmetric, without loss of generality we assume that $y' < d - \frac{1}{2}$. Then we can reach a contradiction to the strategyproofness since the agent at $1 - d$ can be better off by lying to the location of 0 due to $\frac{y+d-1}{d} > \frac{1-d-y'}{d}$.

Case 4: $2/3 \leq d < 1$. We use the same notations as the third case. By the approximation ratio of f, the social cost for y is less than $1 + \frac{1}{d}$. Therefore, $y > d$ by the definition of social cost. Let y' denote the facility location output by mechanism f. The social cost for y' should be less than $1 + 1\frac{1}{d} < \frac{2}{d}$ which implies that $y' < 1 - d$ or $y' > d$. Because the profile is symmetric, without loss of generality we assume that $y' < 1 - d$. By $\frac{y+d-1}{d} > \frac{1-d-y'}{d}$, the agent at $1 - d$ can be better off by lying to 0 which is a contradiction to the strategyproofness. □

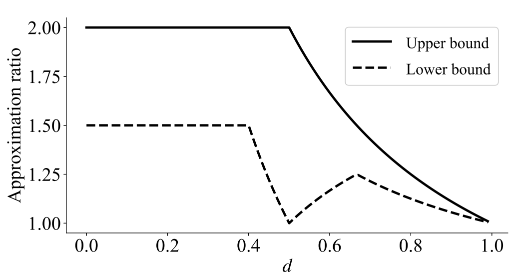


Figure 3: Upper bound and Lower bound for uniform threshold d.

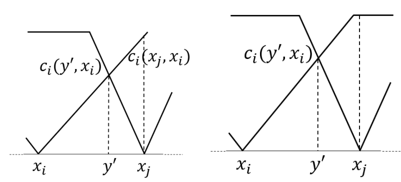
Figure 3 shows the upper and lower bounds for uniform threshold d. Interestingly, putting the facility at the median location can minimize the social cost in the classic setting [16], but cannot minimize the social cost in our model, even if all agents have the same thresholds. Next, we investigate the maximum cost.

4.2 Maximum Cost

For the maximum cost, a classic mechanism which outputs the leftmost agent's location is considered. We observe that the classic leftmost mechanism has a larger approximation ratio when d_{min} is very small. For instance, there is one agent with threshold d_{max} at 0 and the other one with threshold d_{min} at d_{min}. The leftmost mechanism achieves the maximum cost of 1 while the optimal is $\frac{d_{min}}{d_{min}+d_{max}}$, then the approximation ratio is $\frac{d_{min}+d_{max}}{d_{min}}$. Hence, we need to design a new mechanism that leverages the threshold information.

Mechanism 5. *Let* $d_{min} = \min_{i \in N}\{d_i\}$. *Put the facility at* $y = x_j$ *where* $d_j = d_{min}$. *If there are more than one* j, *choose the leftmost one.*

Theorem 5. *Mechanism 5 is strategyproof and has an approximation ratio of 2 for minimizing the maximum cost.*



(a) $x_i + d_i > x_j$ (b) $x_i + d_i \leq x_j$

Figure 4: Two cases of cost functions about agent i and agent j in the model with upper thresholds

Proof. For agents whose thresholds are not d_{min}, they cannot change the facility location. Hence, they have no incentive to misreport their locations. For agents whose thresholds are d_{min}, they cannot be on the left of x_j, then the facility location cannot be moved unless they move to the left of x_j, which makes the facility move farther away. Hence, they have no incentive to misreport their locations. Therefore, Mechanism 5 is strategyproof. Then we show the approximation ratio.

Without loss of generality, we assume that $mc(y, \mathbf{r}) = c(y, r_i)$ and $x_i < x_j$. As Figure 4 shows, there are 2 cases, $x_i + d_i > x_j$ and $x_i + d_i \leq x_j$. If $x_i + d_i > x_j$, we can find a location y' satisfying $c(y', r_i) = c(y', r_j)$. Then we can reuse Lemma 1 to show $c(x_j, r_i) \leq (1 + \frac{d_j}{d_i})c(y', r_i)$. Let y^* be the optimal solution. We have the approximation ratio

$$\rho = \frac{mc(y, \mathbf{r})}{mc(y^*, \mathbf{r})} \leq \frac{c_i(x_j, x_i)}{c_i(y', x_i)} \leq 1 + \frac{d_j}{d_i} = 1 + \frac{d_{min}}{d_i} \leq 2.$$

If $x_i + d_i \leq x_j$, we have $\frac{1}{d_i}d(x_i, x_j) \geq 1$ and $\frac{\frac{1}{d_i}d(x_j, x_i)}{c_i(y', x_i)} \leq 2$, implying an approximation ratio of

$$\rho = \frac{mc(y, \mathbf{r})}{mc(y^*, \mathbf{r})} \leq \frac{c_i(x_j, x_i)}{c_i(y', x_i)} < \frac{\frac{1}{d_i}d(x_j, x_i)}{c_i(y', x_i)} \leq 1 + \frac{d_j}{d_i} \leq 2.$$

Therefore, Mechanism 5 has an approximation ratio of 2. □

We can reuse the lower bound 2 in the classic setting [16], implying that our mechanism is the best possible.

5 AGENTS WITH BOTH THRESHOLDS

In this section, we consider the case where each agent has both thresholds. The case when only one threshold exists does give an initial picture of the general setting with both thresholds. For example, our results can be applied when the first threshold is very close to 0 or the second threshold is very close to 1. For the other two extremes when both thresholds are very big or both are very small, actually it does not matter where the facility is built since the cost is mostly the same. The case when two types of agents co-exist, i.e., $D_i \in \{(a,1)|a \in [0,1]\} \cup \{(0,b)|b \in [0,1]\}$, is also a special case of this setting since agents with one threshold are special cases of agents with both thresholds.

5.1 Social Cost

In this subsection, we combine two mechanisms (Mechanism 1 and Mechanism 4) we designed for the social cost in the previous two sections.

Mechanism 6. *If $\min_i\{d_{i1}\} \geq \frac{1}{2}$, put the facility at $\frac{1}{2}$. Otherwise let $Y = \{x_i \pm d_{i1} | i = 1, 2, ..., n\}$ and y_i denote the ith element in Y. Without loss of generality, assume that $y_1 \leq y_2 \leq ... \leq y_{2n}$. Put the facility at y_n.*

Theorem 6. *Mechanism 6 is strategyproof and has an approximation ratio*

$$\rho = \begin{cases} \max\{2, \frac{\Delta d_{max}}{d_{min}}\} & \text{if } \Delta d_{min} \leq \frac{1}{2} \\ \frac{1}{\Delta d_{min}} & \text{otherwise} \end{cases}$$

for the social cost where $\Delta d_{max} = max\{d_{i2} - d_{i1}\}$ and $\Delta d_{min} = min\{d_{i2} - d_{i1}\}$.

Proof. Mechanism 6 satisfies the strategyproofness trivially. Then we focus on the approximation ratio.

If $\min_i\{d_{i1}\} \geq \frac{1}{2}$, all agents have costs of 0, implying our mechanism is optimal.

For the remaining two cases, first we can observe that there are three kinds of agents, $x_i + d_{i1} < y$, $x_i - d_{i1} \leq y \leq x_i + d_{i1}$ and $x_i - d_{i1} > y$. Note that the cost of the second kind of agents is 0, and the number of the first kind of agents and the third kind of agents is the same (It may not be equal for some cases, but we can reclassify some agents with $x_i + d_{i1} = y$ to the first or with $x_i - d_{i1} = y$ to the third to make it equal). Therefore, the way we choose a pair of agents is one from the first and the other one from the third. Then the following proof is similar as Theorem 3.

We consider the total cost achieved by a pair of agents i and j with $x_i + d_{i1} < y$ and $x_j - d_{j1} > y$. Without loss of generality we assume that $\Delta d_i \leq \Delta d_j$ where $\Delta d_i = d_{i2} - d_{i1}$. Within interval $[x_i + d_{i1}, x_j - d_{j1}]$, we observe that $x_i + d_{i1}$ achieves the minimum social cost and $\min\{x_i + d_{i2}, x_j - d_{j1}\}$ achieves the maximum social cost. Then we have $sc_{i,j}(y, \mathbf{r}) \leq sc_{i,j}(\min\{x_i + d_{i2}, x_j - d_{j1}\}, \mathbf{r})$ where $sc_{i,j}$ is the total cost of agent i and j.

If $sc_{i,j}(x_i + d_{i1}, \mathbf{r}) < 1$, we have

$$\begin{aligned} sc_{i,j}(\min\{x_i + d_{i2}, x_j - d_{j1}\}, \mathbf{r}) &\leq \frac{\Delta d_j}{\Delta d_i} sc_{i,j}(x_i + d_{i1}, \mathbf{r}) \\ &\leq \frac{\Delta d_{max}}{\Delta d_{min}} sc_{i,j}(x_i + d_{i1}, \mathbf{r}). \end{aligned}$$

Otherwise, $sc_{i,j}(\min\{x_i+d_{i2}, x_j-d_{j1}\}, \mathbf{r}) \leq 2$ since there are two agents and then $sc_{i,j}(\min\{x_i + d_{i2}, x_j - d_{j1}\}, \mathbf{r}) \leq 2sc_{i,n-i+1}(x_i + d_{i1}, \mathbf{r})$.

Further, we have the approximation ratio

$$\rho = \frac{sc(y, \mathbf{r})}{sc(y^*, \mathbf{r})} \leq \max\{\frac{\Delta d_{max}}{\Delta d_{min}}, 2\}.$$

However, the approximation can be better than 2 in some cases. The facility location achieving the approximation of 2 is a place where both agents have costs of 1, while for the case of $\Delta d_{min} > \frac{1}{2}$ there is no such location since $x_i + d_{i2} > 1/2$ and $x_j - d_{j2} < 1/2$. Therefore, Mechanism 8 gives $\frac{1}{\Delta d_{min}}$ approximation for the total cost of a pair of agents, implying the approximation ratio for the social cost is $\frac{1}{\Delta d_{min}} \leq \max\{\frac{\Delta d_{max}}{\Delta d_{min}}, 2\}$. □

Because the agents with upper thresholds are the agents with lower thresholds 0, the lower bound can be adapted here, using Δd instead of d, where all agents have the same gap Δd between the lower threshold and the upper threshold.

5.2 Maximum Cost

Mechanism 7. *Put the facility at $y = \min\{\min_i\{x_i + d_{i1}\}, 1\}$.*

Theorem 7. *Let $\Delta d_{max} = max\{d_{i2} - d_{i1}\}$ and $\Delta d_{min} = min\{d_{i2} - d_{i1}\}$. Mechanism 7 is strategyproof and has an approximation ratio of $1 + \frac{\Delta d_{max}}{\Delta d_{min}}$ for the max cost.*

Proof. Agents with positive cost must be on the right of y and can only make the facility move to the left, so it is strategyproof. Assume that we put the facility at $x_j + d_{j1}$. If all agents have cost 0, then the approximation ratio is 1. Without loss of generality suppose agent k achieves the max cost, then there exists a location $y' \in (x_j + d_{j1}, x_k - d_{k1})$ achieving the minimum max cost of agents j and k, which is a lower bound of the optimal max cost. Then we can use the similar argument as Lemma 1 to prove $\frac{c_j(x_j+d_{j1}, x_j)}{c_j(y', x_j)} = 1 + \frac{d_{k2}-d_{k1}}{d_{j2}-d_{j1}}$, implying the approximation ratio is at most $1 + \frac{\Delta d_{max}}{\Delta d_{min}}$. □

We can reuse the lower bound 2 in the classic setting [16] for minimizing the maximum cost.

6 CONCLUSION AND FUTURE WORK

We develop a formal model for the facility location game with thresholds, and study the case where agents only have lower/upper thresholds. For each setting, we design strategyproof mechanisms for minimizing the social cost and the maximum cost. Finally, we extend our results to the case where agents have both thresholds.

Naturally, there are many potential future directions for the facility location game with thresholds in mechanism design. An immediate direction is to tighten the gaps between the lower and upper bounds of our results. It is also interesting to study the setting where agents can misreport their thresholds. For instance, an undemanding agent may benefit by misreporting that she is demanding. Moreover, we mainly focus on the agents with one threshold (Section 3 and Section 4), we believe that the case where the agents with two thresholds deserves more long-term research.

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