Axiomatic Analysis of Medial Centrality Measures

Wiktoria Kośny
University of Warsaw
Warsaw, Poland
wiktoria.kosny@gmail.com

Oskar Skibski
University of Warsaw
Warsaw, Poland
oskar.skibski@mimuw.edu.pl

ABSTRACT

We perform the first axiomatic analysis of medial centrality measures. These measures, also called betweenness-like centralities, assess the role of a node in connecting others in the network. We focus on a setting with one target node and several source nodes. We consider three classic medial centrality measures adapted to this setting: Betweenness Centrality, Stress Centrality and Random Walk Betweenness Centrality. While Betweenness and Stress Centralities assume that the information in the network follows shortest paths, Random Walk Betweenness Centrality assumes it moves randomly along the edges. We develop the first axiomatic characterizations of all three measures. Our analysis shows that Random Walk Betweenness, while conceptually different, shares several common properties with classic Betweenness and Stress Centralities.

CCS CONCEPTS

• **Information systems → Social networks**; • **Mathematics of computing → Graph theory**.

KEYWORDS

Social Networks; Centrality Measures; Betweenness; Axiomatic Approach

ACM Reference Format:
Wiktoria Kośny and Oskar Skibski. 2023. Axiomatic Analysis of Medial Centrality Measures. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), London, United Kingdom, May 29 – June 2, 2023*, IFAAMAS, 9 pages.

1 INTRODUCTION

Identifying key elements in complex interconnected systems is one of the fundamental challenges of network science. More than a hundred such methods, called *centrality measures*, have been proposed [6, 14]. In this paper, we focus on *medial centralities* [5] which, along with the (radial) distance-based centralities and feedback centralities, constitute one of the major classes of centrality measures. They have been used in many applications, including the analysis of protein interaction networks [15], identifying gatekeepers in the social or covert networks (i.e., nodes with the ability to control information flow) [7] or assessing monitoring capabilities in computer networks [9].

Medial centralities, also called *betweenness-like centralities*, assess a node by the role it plays as an intermediary in a network. Their canonical example is Betweenness Centrality—arguably, one of the three most important centrality measures. Betweenness Centrality and other classic medial centralities first assess the role in connecting two nodes, a source and a target, and then sum these values over all pairs of nodes. In the case of Betweenness and Stress Centralities the number of shortest paths going through a node is counted. In turn, Random Walk Betweenness Centrality focuses on the visits of the random walk.

The choice of a suitable centrality measure for a specific goal out of multiple similar concepts is often hard. On one hand, real-world situations are often hard to translate to graph notions. In particular, in the citation network it is not clear whether we should focus on the shortest paths or the random walk. On the other hand, intuitive interpretations of centralities are often misleading. To give an example, Betweenness Centrality is often highly correlated with the degree, as high degree imposes a high number of shortest paths a node is on. Resulting from that, centrality measures are often chosen based on their popularity rather than on their fit for the application. This misleads the analysis and leads to poorly funded conclusions.

That is why, in recent years, the axiomatic approach has gained popularity in centrality analysis [2, 4, 23]. In this approach, simple properties called *axioms* are defined that highlight specific behaviors of a measure. A carefully designed set of axioms allows one to uniquely characterize a measure. Such results deepen the understanding of centrality measures and highlight the differences and similarities between them. Also, if axioms are natural and based on simple graph operations, they allow a practitioner to test whether a specific property is desirable in the application at hand.

There are several papers that apply the axiomatic approach to distance-based centralities [13, 21] and even more that concern feedback centralities [8, 28] (see Related Work for details). Medial centralities, however, have not been studied using the axiomatic approach. In particular, no axiomatization of Betweenness Centrality and its variants has been proposed to date.

The main reason for the lack of such results is the complex nature of betweenness-like measures. In fact, it is especially hard to find any graph operations that do not change these centralities and such operations are usually the basis of axiomatic characterizations. Adding even a single edge in a graph may completely change the structure of the set of shortest paths and, as a consequence, values of medial centralities based on shortest paths. As a result, out of dozens of axioms proposed in the literature, Betweenness Centrality satisfies only few simplest.

To cope with this challenge, in this paper we focus on a setting with one target node and arbitrary many source nodes. In this way, we concentrate on the key aspect of medial centralities which allows us to better identify similarities and differences between them. At the same time, the setting is simpler to analyze which

Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), A. Ricci, W. Yeoh, N. Agmon, B. An (eds.), May 29 – June 2, 2023, London, United Kingdom.

allows us to obtain strong axiomatic results which was not possible for the general model so far.

Moreover, centrality analysis in our setting with one target node has several natural applications. In the World Wide Web, it can indicate the role of websites in directing users to a specific page. In the financial network, centralities can identify top intermediaries responsible for transferring money to a specific bank account. In the communication networks, they can assess the role in controlling the flow of information going to one specific entity.

We consider three classic medial measures adapted to this setting: Betweenness, Stress and Random Walk Betweenness and create the first axiom system that enables us to characterize all of them.

To this end, first, we ask the question: what are the properties satisfied by all three centralities? We identify four such properties. *Locality* states that the importance of a node does not depend on separate parts of the network. *Additivity* imposes that the centrality is additive in respect to node weights. *Node Redirect* states that merging out-twins does not affect centralities of other nodes. Finally, *Target Proxy* says that if all paths to the target goes through one specific node, then it can be considered a target. While Locality and Additivity can be considered general axioms that should be satisfied by all reasonable centrality measures, Node Redirect and Target Proxy capture similarities between considered measures (in particular, they are not satisfied by measures based on flow).

Furthermore, we propose two axioms that are specific for centralities based on shortest-paths. *Symmetry* concerns the scenario where there is only one source node. The axiom states that in such a case, the importance would not change if the graph is reversed and the source replaced with the target. Now, *Direct Link Domination* states that if a node has a link to the target, its other edges may be deleted. Interestingly, these two axioms are not satisfied by Random Walk Betweenness.

Finally, we propose two borderline axioms that specify the centrality for a trivial graph with only two nodes, a source and a target, and k edges from the source to the target. Specifically, Atom 1-1 states the centrality of both nodes equal one and Atom k-k states they are equal k. There are two goals of these axioms. First of all, they highlight the difference in the way medial centralities treat multiple edges between nodes. Betweenness Centrality and Random Walk Betweenness ignore them and assign value 1 to both nodes. In turn, Stress Centrality assigns value k. Second of all, they serve as a boundary case to pinpoint specific values of a measure and preclude other measures that only differ by scalar multiplication.

In our main result, we show that Betweenness Centrality is uniquely characterized by four common axioms (Locality, Additivity, Node Redirect, Target Proxy), two shortest-paths specific axioms (Symmetry, Direct Link Domination) and Atom 1-1. Specifically, if a centrality measures satisfies all seven axioms, then it must be equal to Betweenness Centrality. On top of that, we show that if we replace Atom 1-1 with Atom k-k, then we obtain an axiomatization of Stress Centrality. Most of these axioms are new, only two out of eight (Locality and Node Redirect) are known axioms adapted to our setting.

Furthermore, we show how this axiomatization can be extended for Random Walk Betweenness Centrality. To this end, we use five previous axioms (Locality, Additivity, Node Redirect, Target Proxy and Atom 1-1) and add two axioms from a recent axiomatization of PageRank [26]: *Edge Swap* and *Edge Multiplication*. This is possible, as Random Walk Betweenness Centrality can be considered a border case of PageRank of a modified graph without the damping factor (see Preliminaries for details).

Axiom	B^t	S^t	RWB^t
Locality	✓	✓	✓
Additivity	✓	✓	✓
Node Redirect	✓	✓	✓
Target Proxy	✓	✓	✓
Symmetry	✓	✓	✗
Direct Link Domination	✓	✓	✗
Atom 1-1	✓	✗	✓
Atom k-k	✗	✓	✗
Edge Swap	✗	✗	✓
Edge Multiplication	✗	✗	✓

Table 1: Axiomatic characterizations of medial centralities.

As a result, we obtain a joint axiomatic characterization of three classic medial centralities that highlights their differences as well as similarities. See Table 1 for a summary.

1.1 Related Work

The axiomatic approach to centrality measure was initiated by the work of Sabidussi [19] and Nieminen [18] who characterized which functions are centrality measures. The proposed axioms were inspired mostly by distance-based centralities that is why medial centralities violates most of them.

Later on, Boldi and Vigna [4] proposed three axioms and checked that out of the standard centrality measures only Harmonic Centrality satisfies all three of them. Betweenness Centrality, the only medial centrality considered, violates all three axioms (it satisfies one axiom only under additional assumptions).

A more popular way to apply the axiomatic approach to centrality analysis is creating an axiomatic characterization of a measure or the whole class of measures. Our work contributes to this line of research. A few papers have considered distance-based centralities [13, 21, 24] and game-theoretic centralities [22, 23]. Most papers, however, focused on feedback centralities: measures in which the importance of a node is defined based on the number and the importance of its neighbors (or direct predecessors in directed graphs). Specifically, van den Brink and Gilles [25] axiomatized β-measure, Altman and Tennenholtz [1] the Seeley index (a simplified version of PageRank) and Kitti [16] eigenvector centrality. More recently, Dequiedt and Zenou [8] and Wąs and Skibski [27] created a joint axiomatization of eigenvector and Katz centrality for undirected and directed graphs, respectively. Also, Wąs and Skibski [26] created an axiomatization of PageRank.

Since centrality measures from different classes vary significantly, Betweenness and Stress Centralities satisfy only a few of the simplest axioms from these papers (e.g., they satisfy Anonymity that states isomorphic nodes have equal centralities). However, when restricted to one target node, they both satisfy Node Redirect [26] which is a meaningful axiom; hence, we use it in our paper.

In turn, Random Walk Betweenness satisfies several axioms proposed in the axiomatization of PageRank [26]. That is why our last result is related to this axiomatization. We use 3 axioms proposed in [26] and show they can be combined with our other axioms to obtain the axiomatization of Random Walk Betweenness. On a high level, our proof has a similar structure to the proof of the PageRank axiomatization. However, since most of the axioms are new and we consider a different setting the main part of the proof is also new.

2 PRELIMINARIES

In our work, we consider weighted directed multigraphs with possible self-loops. This model generalizes to unweighted and undirected graphs and could be used to represent the World Wide Web, social or financial networks, among others.

A graph is a pair $G = (V, E)$, where V is the set of nodes and E is the multiset of edges, that is, ordered pairs of nodes $(v, u) \in V \times V$. We will denote the number of occurrences of an element e in the multiset E by $m_e(E)$. An edge (v, u) is *outgoing* from the node v, which is the *start* of the edge and is *incoming* to the node u, which is the *end* of the edge. If $v = u$, the edge is a *self-loop*.

Let $\Gamma^+_v((V, E)) = \{(v, u) \in E\}$ denote the multiset of edges outgoing from the node v (including self-loops) and let its cardinality be the *out-degree* of the node v. Let $N^+_v((V, E)) = \{u \in V : (v, u) \in E\}$ denote the set of *direct successors* of v. Similarly, let $\Gamma^-_v((V, E)) = \{(u, v) \in E\}$ denote the multiset of edges incoming to the node v (including self-loops) and let its cardinality be the *in-degree* of the node v. Let $N^-_v((V, E)) = \{u \in V : (u, v) \in E\}$ denote the set of *direct predecessors* of v. Also, let $\Gamma_v(G) = \Gamma^+_v(G) \cup \Gamma^-_v(G)$.

Two nodes v and u are *out-twins* if they have the same outgoing edges, that is for every $w \in V$ there holds $m_{(v,w)}(E) = m_{(u,w)}(E)$.

A *path* $p = (e_1, e_2, ..., e_k)$ of *length* k, is a sequence of edges of the graph (i.e., $e_1, \dots, e_k \in E$) in which every edge starts with a node with which the previous edge ended, that is for every $i \in \{1, ..., k-1\}$ there are some nodes $v, u, w \in V$ such that $e_i = (v, u)$, $e_{i+1} = (u, w)$. The start of the first edge is the *start of the path* and the end of the last edge is *the end of the path*. A cycle is a path that starts and ends in the same node.

A node u is *reachable* from a node v if there is a path that starts with v and ends with u. The *distance* from v to u, denoted by $dist_{v,u}(G)$, is the length of the shortest path that starts with v and ends with u. The number of shortest paths from v to u in G is denoted by $\sigma_{v,u}(G)$ and the number of shortest paths from v to u in G that contain w is denoted by $\sigma_{v,u}(G, w)$.

2.1 Our Setting

In this paper, we treat graphs as the information networks. We will assume there is one *target node*, denoted by t, which is the destination of all data traveling through the network. As mentioned in the introduction, other interpretations include users travelling towards some page in the Web or money transferred to a specific bank account.

We will restrict our attention to graphs in which the target node t is reachable from every node; the set of all such graphs will be denoted by $\mathcal{G}_t$.

To specify which nodes are the sources of information and how much information they send, we will consider node weight functions $b : V \to \mathbb{R}_{\geq 0}$. The simplest case is when there is a single source: $s \in V$. To describe such situations, we denote by $\mathbb{1}^s$ a node weight function such that $(\mathbb{1}^s)(s) = 1$ and $(\mathbb{1}^s)(v) = 0$ for $v \in V - \{s\}$. The multiplication of a weight function b by a constant $x \in \mathbb{R}_{\geq 0}$ is defined as $(x \cdot b)(v) = x \cdot b(v)$ for $v \in V$. The addition of two weight functions $b : V \to \mathbb{R}_{\geq 0}$, $b' : V' \to \mathbb{R}_{\geq 0}$ is defined as $(b + b')(v) = b(v) + b'(v)$ for every $v \in V \cap V'$, $(b + b')(v) = b(v)$ for every $v \in V - V'$ and $(b + b')(v) = b'(v)$ otherwise.

Let us define several operations on (node-)weighted graphs. The *sum* of two graphs is obtained by summing the corresponding node sets, edge multisets and weight functions. Formally, for two weighted graphs (G, b), (G', b') with $G = (V, E)$, $G' = (V', E')$ we have:

$$(G, b) + (G', b') = (V \cup V', E + E', b + b'),$$

where $E + E'$ denotes the sum of multisets E, E'.

We will also use a shorthand notation for adding and deleting edges E' from the graph $G = (V, E)$: we define $G + E' = (V, E + E')$ and $G - E' = (V, E - E')$.

For two different nodes $v, u \in V$ we define *merging* and *redirecting*. *Merging* v into u deletes v from the graph and moves its weight and all outgoing and incoming incident edges to u. Formally: $M_{v \to u}(G, b) = ((V - \{v\}, E'), b')$, where $E' = E - \Gamma_v(G) + \{(f_{v \to u}(w), f_{v \to u}(w')) : (w, w') \in \Gamma_v(G)\}$ where $f_{v \to u}(v) = u$ and $f_{v \to u}(w) = w$ for $w \in V - \{v\}$ and $b'(u) = b(v) + b(u)$ and $b'(w) = b(w)$ for $w \in V - \{v, u\}$. Now, *redirecting* v into u deletes outgoing edges of v (except for self-loops) and merges v into u: $R_{v \to u}(G, b) = M_{v \to u}((V, E - (\Gamma^+_v(G) - \Gamma^-_v(G))), b)$.

2.2 Centralities

A *centrality measure* F is a function that for every node v in a (node-)weighted graph (G, b) assigns a non-negative real value, denoted by $F_v(G, b)$.

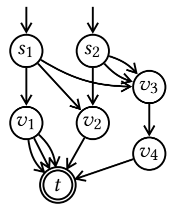
In our setting, centrality measures capture how often a node conveys the data to the target node. Hence, we will consider centrality measures parameterized by the target node t defined on the class of graphs $\mathcal{G}_t$. We will call them *t-centrality measures*. We will consider three t-centrality measures which are direct counterparts of the classic measures for the general setting.

The simplest and chronologically the first medial centrality measure was proposed by Shimbel [20] under the name *Stress Centrality*. This measure simply counts the number of shortest paths that go through a specific node. We adjust Stress Centrality to our setting by fixing the target node of all paths and considering paths from different sources with different weights.

Definition 2.1. For a graph $G = (V, E) \in \mathcal{G}_t$ with weights b, *t-Stress Centrality* of node $v \in V$ is defined as follows:

$$S^t_v(G, b) = \sum_{s \in V} b(s) \cdot \sigma_{s,t}(G, v).$$

Betweenness Centrality [11], the most widely used medial centrality, can be considered a relative version of Stress Centrality. In Betweenness Centrality the number of shortest paths from s to t that goes through v is divided by the total number of shortest paths from s to t. We define t-Betweenness Centrality accordingly.



Node	B^t	S^t	RWB^t
s_1	1.00	4.00	1.00
s_2	1.00	1.00	1.00
v_1	0.75	3.00	0.33
v_2	1.25	2.00	0.67
v_3	0.00	0.00	1.00
v_4	0.00	0.00	1.00
t	2.00	5.00	2.00

Figure 1: An illustration of the t-centralities. Nodes s_1 and s_2 are the only sources, as indicated by the incoming arrows, i.e., $b = \mathbb{1}^{s_1} + \mathbb{1}^{s_2}$. Node t, marked by a double line, is the target node. The values of t-Betweenness, t-Stress and t-Random Walk Betweenness of all nodes are presented in the table on the right-hand side.

Definition 2.2. For a graph $G = (V, E) \in \mathcal{G}_t$ with weights b, *t-Betweenness Centrality* of node $v \in V$ is defined as follows:

$$B^t_v(G, b) = \sum_{s \in V} b(s) \cdot \frac{\sigma_{s,t}(G, v)}{\sigma_{s,t}(G)}.$$

The standard Betweenness and Stress Centralities for an unweighted graph are the sum of the t-Betweenness and t-Stress Centralities, respectively, over all target nodes $t \in V$ with unit node weights.

Betweenness and Stress Centralities are based on the underlying assumption that the information travels the network through shortest paths. Such an assumption makes sense in settings in which the whole structure is known beforehand and the process is optimized. The opposite approach is proposed in *Random Walk Betweenness Centrality*.

Assume an information packet starts from a source node s and moves randomly through the network. In each step, it chooses one of the outgoing edges of the node it is in, uniformly at random, and moves along this edge. To measure the role in transferring the information to one specific target node t, node t is treated as an absorbing node in which all packets end their travel (technically, this is achieved by deleting outgoing edges of t). Now, the role in connecting nodes s to t in the random-walk version of Betweenness Centrality is defined as the expected number of times a packet visits a specific node.

Formally, let us denote by $\mathbb{P}(\omega_{G,s}(k) = v)$ the probability that the random walk $\omega_{G,s}$ on graph G that starts in node s after k steps will be at node v. We define t-Random Walk Betweenness Centrality as follows.

Definition 2.3. For a graph $G = (V, E) \in \mathcal{G}_t$ with weights b, *t-Random Walk Betweenness Centrality* of node $v \in V$ is defined as follows:

$$RWB^t_v(G, b) = \sum_{s \in V} \sum_{k=0}^{\infty} b(s) \cdot \mathbb{P}(\omega_{G-\Gamma^+_t(G),s}(k) = v).$$

We note that t-Random Walk Betweenness Centrality is a counterpart of the measure defined by Blöchl et al. [3] for directed graphs and not of the measure defined by Newman [17] for undirected graphs under the same name.

Example 2.4. (t-Centralities) Consider a graph from Figure 1. Let us analyze the role of the intermediary nodes v_1, v_2, v_3, v_4 in transmitting data from the source nodes s_1, s_2 to the target node t according to the three medial centralities.

There are four shortest paths from s_1 to t: three paths through v_1 and one path through v_2. There is one shortest path from s_2 to t and it goes through v_2. As a result, t-Stress ranks node v_1 highest, as it is the only node which is on three of the relevant shortest paths. In turn, t-Betweenness ranks node v_2 highest, as it is on all of the shortest paths from s_2 and on 1/4 of the shortest paths from s_1.

Now, consider t-Random Walk Betweenness. The random walk that starts in the node s_1 reaches nodes v_1, v_2, v_3 with the probability 1/3; hence, it also reaches node v_4 with probability 1/3. The random walk that starts in the node s_2 reaches nodes v_3 and v_4 with the probability 2/3, node v_2 with the probability 1/3 and cannot reach node v_1. Hence, t-Random Walk Betweenness ranks nodes v_3 and v_4 highest.

3 AXIOMATIZATION OF T-BETWEENNESS CENTRALITY

We will now present the set of simple properties that we will use to uniquely characterize t-Betweenness Centrality. First, we will present four common axioms satisfied by t-Betweenness, t-Stress as well as t-Random Walk Betweenness: *Locality*, *Additivity*, *Node Redirect* and *Target Proxy*. Then, we will introduce two new axioms satisfied only by centralities based on shortest paths, t-Stress and t-Betweenness, namely *Symmetry* and *Direct Link Domination*. Finally, we introduce a borderline axiom named *Atom* 1-1, satisfied by t-Betweenness and t-Random Walk Betweenness.

3.1 Common Axioms

We start with axioms satisfied by all three t-centralities. For an illustration see Figure 2.

Axiom 1. *(Locality) For every two graphs $G = (V, E)$, $G' = (V', E') \in \mathcal{G}_t$ with weights b, b' such that $V \cap V' = \{t\}$ and node $w \in V - \{t\}$:*

$$F^t_w((G, b) + (G', b')) = F^t_w(G, b)$$

and $F^t_t((G, b) + (G', b')) = F^t_t(G, b) + F^t_t(G', b')$.

Axiom 2. *(Additivity) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b, b' and node $w \in V$:*

$$F^t_w(G, b + b') = F^t_w(G, b) + F^t_w(G, b').$$

Axiom 3. *(Node Redirect) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b and out-twins $v, u \in V - \{t\}$:*

$$F^t_v(R_{u\to v}(G, b)) = F^t_v(G, b) + F^t_u(G, b)$$

and $F^t_w(R_{u\to v}(G, b)) = F^t_w(G, b)$ for every $w \in V - \{v, u\}$.

Axiom 4. *(Target Proxy) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b such that $\Gamma^-_t(G) = \{(v, t)\}$ and $\Gamma^+_v(G) = \{(v, t)\}$, $b(t) = 0$ and node $w \in V - \{t\}$:*

$$F^v_w(M_{t\to v}(G, b)) = F^t_w(G, b).$$

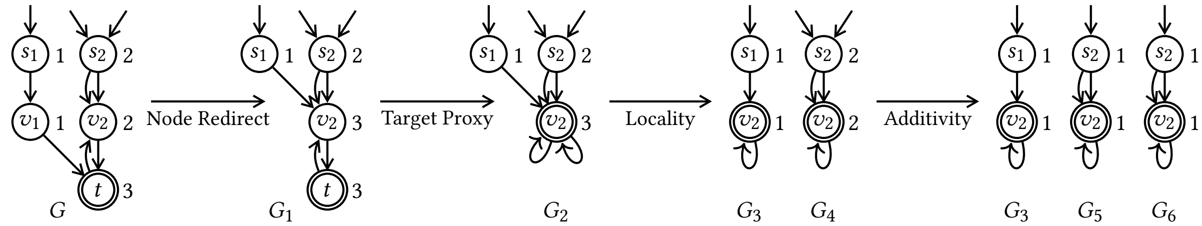


Figure 2: An illustration of the first four axioms. In the initial graph G, nodes s_1 and s_2 are the only sources, s_2 with twice as much weight: $b = \mathbb{1}^{s_1} + 2 \cdot \mathbb{1}^{s_2}$ and node t is the target. The values of t-Betweenness Centrality are placed right to the nodes.

Locality describes the operation of joining two networks with the common target. This may occur for example in the computer network, when two separate subnetworks are connected by one server. The axiom states that the centrality of all nodes other than the target does not change and the centrality of the target is the sum of its centralities from both separate graphs. This means that the behavior of data packets is not affected by the existence of another independent part of the network with the same target.

We can also look from the other perspective and say that *Locality* describes splitting the network at the cut vertex. When the cut vertex is the target node, centralities in each component it joins are independent. A similar axiom under the same name was proposed for undirected graphs in [23] where separate connected components are considered.

Additivity states that the centrality treated as a function of the node weights is additive. This means that the packets travel independently from each other, they do not collide in any way.

Node Redirect formalizes the intuition that if the packet from two nodes has the same possible further routes, then redirecting one of these nodes into the other would not change the centralities of other nodes. Moreover, the centrality of the combined node will be the sum of centralities of both nodes in the original graph. This axiom was proposed in [26], but in our version we do not allow redirecting of (and to) the target node.

Target Proxy can be understood in the following way: if every path to the target node goes through a *proxy*, v, then the role in transferring data to the target is the same as the role in transferring data to node v. In the context of the Internet network, this axiom can be interpreted as the *layered system* REST constraint [10]. It is a rule for designing web API which states that the communication between the clients and the target server should not be affected if the target is hidden behind a firewall, proxy or a load balancer.

3.2 Shortest-Paths Axioms

Let us now present two axioms specific for centralities based on shortest-paths. See Figure 3 for an illustration.

Axiom 5. *(Symmetry) For every graph $G = (V, E) \in \mathcal{G}_t$, source node $s \in V$ and node $w \in V$ such that $G' = (V, \{(u, v) : (v, u) \in E\}) \in \mathcal{G}_s$:*

$$F^s_w(G', \mathbb{1}^t) = F^t_w(G, \mathbb{1}^s).$$

Axiom 6. *(Direct Link Domination) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b such that $(v, t), (v, u) \in E$, $u \neq t$ and node $w \in V$:*

$$F^t_w(G - \{(v, u)\}, b) = F^t_w(G, b).$$

Symmetry states that if there is only one source node, then reversing the graph and swapping source and target nodes does not change centralities in the graph. Note that this operation applies only if the reversed graph belongs to $\mathcal{G}_s$, i.e., node s which is the target in the reversed graph is reachable from every node. It is suitable when data packets would be transferred through the same paths from the target to the source in the reversed network. This is true if data packets go through the shortest paths. However, it is not the case for the random walk, for which some paths would be used more often in the reversed network than in the original one.

Direct Link Domination states that if from v there is a direct connection to t, then we can delete other outgoing edges of v. This captures the assumption that a node which can send a data packet directly to the target will do so. For example, when sending parcels through the transportation network it is natural to anticipate that the logistics company will choose the direct connection whenever it is possible. At the same time, the axiom does not impose any restrictions on the behavior of packets in other nodes.

3.3 Atom Axiom

We conclude with a simple atom axiom.

Axiom 7. *(Atom 1-1) For every node s, natural number $k \in \mathbb{N}_+$ and graph $G = (\{s, t\}, k \cdot \{(s, t)\})$:*

$$F^t_s(G, \mathbb{1}^s) = 1 = F^t_t(G, \mathbb{1}^s).$$

Atom 1-1 specifies the centrality in a simple graph with two nodes, the source s and the target t, and k edges from s to t. It states that both nodes should have centrality equal to 1, as this is the amount of information packets both nodes are responsible

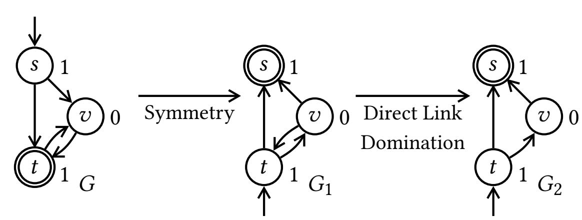


Figure 3: An illustration of Symmetry and Direct Link Domination. Node s is the only source: $b = \mathbb{1}^s$ and node t is the target. The values of t-Betweenness Centrality are placed right to the nodes.

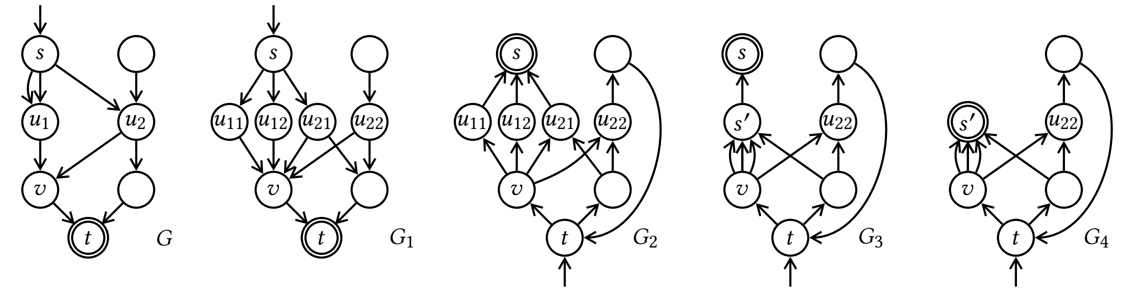


Figure 4: The key inductive step from the proof of Theorems 3.1 and 4.1. Assume node v is at distance at least 2 from s. We transform the original graph G in a way that the distance from the source to the target is shortened, but the centrality of v remains unchanged. First, using Node Redirect, we create copies of direct successors of s, each with the original outgoing edges, but only one incoming edge (G_1). Then, using Symmetry (and No Target Outlet), we reverse the graph (G_2). Next, using Node Redirect, we merge copies with an edge to s into one node s' (G_3). Finally, using Target Proxy, we merge s and s' (G_4).

for. Hence, the number of edges does not matter. This is the case, for example, when edges represent links between webpages: if all links from one website point to the other, their number does not matter. Atom 1-1 is satisfied by t-Betweenness and t-Random Walk Betweenness.

The following theorem contains the axiomatic characterization of t-Betweenness. [1]

Theorem 3.1. *A t-centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Atom* 1-1 *if and only if it is t-Betweenness Centrality.*

Sketch of the proof. It is easy to check that t-Betweenness Centrality satisfies the axioms listed in Theorem 3.1. Hence, in what follows, we will show that this set of axioms uniquely characterizes a centrality measure.

We begin by showing three simple properties.

- *(Anonymity)*: Any node other than the target can be renamed without changing any centralities.
- *(Target Self-Loop)*: Deleting a self-loop of the target does not change any centralities.
- *(No Target Outlet)*: Deleting any outgoing edge of the target does not change any centralities.

No Target Outlet is an especially useful property as it allows us to add edges from the target t to all other nodes in the graph before using Symmetry. In this way, we make sure that after reversing the graph from each node there will be a path to the new target, possibly through t, which is the requirement of the Symmetry axiom.

For now, let us concentrate on a graph with one source s with weight 1, i.e., $b = \mathbb{1}^s$. We proceed by induction on the distance from the source s to the target t: $dist_{s,t}(G)$. In the base case we consider $dist_{s,t}(G) \leq 2$:

- If $dist_{s,t}(G) = 0$, then the only source is also the target. In such a case, we present the graph as the sum of two graphs: the original graph with weight of t changed to zero and the second graph with only one node t with unitary weight. From Locality and Additivity we get that $F^t_t(G, \mathbb{1}^t) = F^t_t((\{t\}, \{\}), \mathbb{1}^t)$ and $F^t_v(G, \mathbb{1}^t) = 0$ for $v \in V - \{t\}$. Now, the centrality of t in graph $(\{t\}, \{\})$ with weights $\mathbb{1}^t$ can be determined based on the Atom axiom by using Target Proxy and Target Self-Loop.
- If $dist_{s,t}(G) = 1$, then the only source and the target are connected by at least one edge. Here, using Node Redirect and Locality we decompose graph G into the original graph with weight of s changed to zero and a graph with two nodes, source s and target t, and k edges from s to t, with unitary weight of s. Centralities in the first graph all equal zero from Additivity and centralities in the second graph are known from the Atom axiom.
- If $dist_{s,t}(G) = 2$, then we know that at least one of the successors of s is a predecessor of t. First, we use Node Redirect to split each successor of s into several nodes so that each has only one incoming edge. Then, using Symmetry and Node Redirect for the reversed graph we split predecessors of t so that each copy has only one outgoing edge. In the resulting graph all nodes which are both successors of s and predecessors of t are isomorphic, which allows us to deduce that they have equal centralities. To argue what are the centralities of other nodes, we merge isomorphic nodes using Node Redirect and use Target Proxy to obtain the case where the distance from s to t equals one.

Let us discuss the inductive step. Fix graph G with $dist_{s,t}(G) \geq 3$ and some node v. Since

$$dist_{s,v}(G) + dist_{v,t}(G) \geq dist_{s,t}(G) \geq 3,$$

we either have $dist_{s,v}(G) \geq 2$ or $dist_{v,t}(G) \geq 2$. Let us assume the former; in the other case, we reverse the graph and based on Symmetry proceed in the same way. Now, we show that we can transform the graph in a way that the distance from the source to the target decreases, but the centrality of v remains unchanged. We present this key construction in Figure 4.

So far, we have considered only one source. If there are multiple sources, by using Additivity we split the graph into several copies,

[1] The extended version of our paper with the full proofs can be found at https://arxiv.org/abs/2112.01892.

each with a single unitary source:

$$F^t_v(G, b) = \sum_{s \in V} b(s) \cdot F^t_v(G, \mathbb{1}^s).$$

Based on that, we show that if a centrality measure satisfies Atom 1-1, then it is t-Betweenness Centrality. This concludes the proof. □

4 AXIOMATIZATION OF T-STRESS CENTRALITY

To characterize t-Stress Centrality, we propose the following modification of Atom 1-1:

Axiom 8. *(Atom k-k) For every node s, natural number $k \in \mathbb{N}_+$ and graph $G = (\{s, t\}, k \cdot \{(s, t)\})$:*

$$F^t_s(G, \mathbb{1}^s) = k = F^t_t(G, \mathbb{1}^s).$$

As Atom 1-1, *Atom k-k* specifies the centrality in a graph with two nodes, the source s and the target t, and k edges from s to t. In such a case, Atom k-k states that both nodes have centrality equal to the number of edges between them. In particular, the nodes' assessment increases when there are more edges between the source and the target. Such an approach makes sense if we assume the information packets can be duplicated and sent through each edge. Then, if we measure not the relative importance of nodes, but the absolute number of packets that go through the node we get these values. Out of three centralities considered by us, Atom k-k is satisfied only by t-Stress Centrality.

Now, we show that replacing Atom 1-1 by Atom k-k in the axiomatization of t-Betweenness Centrality results in the axiomatization of t-Stress Centrality.

Theorem 4.1. *A t-centrality satisfies Locality, Additivity, Node Redirect, Target Proxy, Symmetry, Direct Link Domination and Atom* k-k *if and only if it is t-Stress Centrality.*

The proof of Theorem 4.1 is analogous to the proof of Theorem 3.1. Specifically, we prove that t-Stress Centrality satisfies the axioms and then using the reasoning from the proof of Theorem 3.1 we show it is a unique such measure.

5 AXIOMATIZATION OF T-RANDOM WALK BETWEENNESS

So far, we have considered two centrality measures based on shortest paths. In this section, we present the axiomatization of their random-walk counterpart: t-Random Walk Betweenness.

To this end, we adapt to our setting two axioms from the axiomatization of PageRank in [26]: *Edge Swap* and *Edge Multiplication.* Both axioms are satisfied by t-Random Walk Betweenness Centrality, but not by t-Betweenness nor t-Stress Centralities.

Axiom 9. *(Edge Swap) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b and edges $(v, v'), (u, u') \in E$ such that $v, u \neq t$, $F_v(G, b) = F_u(G, b)$, $|\Gamma^+_v(G)| = |\Gamma^+_u(G)|$, $G' = G - \{(v, v'), (u, u')\} + \{(v, u'), (u, v')\} \in \mathcal{G}_t$ and node $w \in V$:*

$$F^t_w(G', b) = F^t_w(G, b).$$

Axiom 10. *(Edge Multiplication) For every graph $G = (V, E) \in \mathcal{G}_t$ with weights b, number $k \in \mathbb{Z}_{\geq 0}$ and nodes $v, w \in V$:*

$$F^t_w(G + k \cdot \Gamma^+_v(G), b) = F^t_w(G, b).$$

Edge Swap states that swapping ends of two outgoing edges of nodes with equal centralities and out-degrees does not affect the centrality of any node. This means that if information packets are equally often in two nodes, then it does not matter from which of them a node has an incoming edge. The only difference between our version of the axiom and the original one from [26] is the fact that swapped edges cannot start in the target node.

Edge Multiplication states that creating additional copies of the outgoing edges of a node does not affect the centrality of any node. This means that it is not the total number of edges that matters, but their proportions.

Now, replacing Symmetry and Direct Link Domination by Edge Swap and Edge Multiplication results in the axiomatization of t-Random Walk Betweenness. Interestingly, our t-Random Walk Betweenness axiomatization is very similar to the axiomatization of t-Betweenness, despite these centralities being based on completely different models of transmission.

Note that Additivity, while satisfied by t-Random Walk Betweenness, is implied by other axioms (Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Atom 1-1). Hence, it does not appear in the theorem statement.

Theorem 5.1. *A t-centrality satisfies Locality, Node Redirect, Target Proxy, Edge Swap, Edge Multiplication and Atom* 1-1 *if and only if it is t-Random Walk Betweenness.*

Sketch of the proof. It is easy to check that t-Random Walk Betweenness satisfies the axioms listed in Theorem 5.1. Hence, in what follows, we will show that this set of axioms uniquely characterizes a centrality measure.

We begin by considering simple graphs with two nodes: the source s and the target t (which has 0 weight).

- First, we show that if there is only one edge, from s to t, then the centralities of both nodes are equal $b(s)$.
- Second, we show that outgoing edges from t does not affect the centralities of s and t.

Building upon this, we show that if we add to a graph a new node v with several edges $\{v, t\}$, then its centrality will be equal to its weight and centralities of nodes other than t will not change. This operation, that we call *new node technique*, is used frequently in the remainder of the proof. In particular, we use it to show that every node v without incoming edges has centrality equal to its weight. Furthermore, assuming such v has k outgoing edges, if we

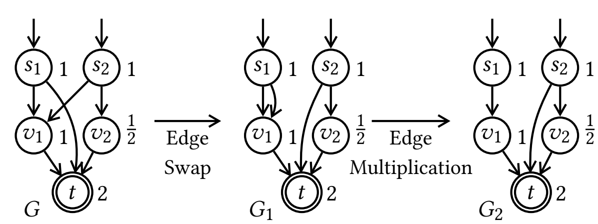


Figure 5: An illustration of Edge Swap and Edge Multiplication. Nodes s_1 and s_2 are the only sources: $b = \mathbb{1}^{s_1} + \mathbb{1}^{s_2}$ and node t is the target. The values according to t-Random Walk Betweenness are placed right to the nodes.

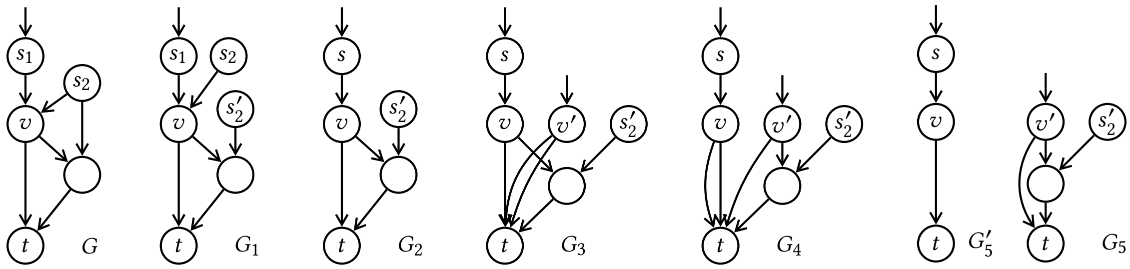


Figure 6: The key inductive step from the proof of Theorem 5.1. We transform the original acyclic graph G in a way that the number of edges is decreased, but the centrality of v remains unchanged. First, we create copies of nodes without incoming edges, each with one outgoing edge (G_1). Then, using Node Redirect, we merge copies of direct predecessors of v into one node s (G_2). Now, using the new node technique, we add a node v' with the same number of outgoing edges as v, all to t, and the weight equal to the centrality of v (G_3). Next, we use Edge Swap to exchange the outgoing edges of v with the outgoing edges of v' (G_4). Finally, using Locality and Edge Multiplication, we split the graph in two graphs, one which has one edge less than G (G_5) and another simple graph for which centralities follow from previous lemmas (G_5').

replace it with k copies, each with one outgoing edge and weight $b(v)/k$, then the centralities of other nodes will not change.

In the next part of the proof, we show that on acyclic graphs the centrality is equal to t-Random Walk Betweenness. The proof proceeds by induction on the number of edges.

- If $|E| \leq 1$, then the thesis follows from previous lemmas.
- If $|E| \geq 2$, but all edges ends in t, then this result follows from Locality.
- If $|E| = 2$, and not all edges ends in t, then we have: $G = (\{s, v, t\}, \{(s, v), (v, t)\})$. In such a case, centralities of s and v can be deduced by using Target Proxy and previous lemma concerning graph with only one edge. The centrality of t can be obtained using Target Proxy, Edge Swap and Locality.
- If $|E| \geq 3$, but not all edges end in t, then there exists a node other than the target with incoming edges. Let v be the topologically greatest node (first node in the topological order of nodes). Because it is topologically greatest, all its predecessors have no incoming edges. Now, we show that we can transform a graph in a way that the number of edges decreases, but the centrality of v remains unchanged. This construction is described in Figure 6.

Acyclic graphs are the basis for the induction on the number of cycles which constitutes the top level of the proof.

Now, assume that a graph contains some cycles. We can also assume that the target node does not have any outgoing edges, as otherwise they can be deleted without changing any centralities.

Let us discuss the inductive step of the induction on the number of cycles: Fix a node v that belongs to at least one cycle. Using again the new node technique we add a node v' with the same number of outgoing edges as v, all to t, and the weight equal to the centrality of v. Now, using Edge Swap, we swap all outgoing edges of v with all outgoing edges of v'. In the new graph, nodes v and v' do not belong to any cycles, hence the number of cycles has decreased. This concludes the proof. $\square$

On the top level, our proof has a similar structure to the proof of the axiomatization of PageRank [26]. However, there are several differences. First of all, three out of six axioms (Locality, Target Proxy and Atom 1-1) have not appeared in the axiomatization of PageRank. Second of all, these axioms which have appeared have additional constraints, excluding their use on the target node. Third, in our paper, we restrict ourselves to graphs in which the target node is reachable which makes many constructions from the original proof impossible. As a result, five out of nine our lemmas for Random Walk Betweenness do not have direct counterparts in the mentioned paper. Moreover, even some lemmas that have counterparts have vastly different proofs (in particular, key lemma presented in Figure 6).

6 CONCLUSIONS

We proposed the first axiomatization of three medial centralities: Betweenness, Stress and Random Walk Betweenness. We focused on a setting with one target node and arbitrarily many source nodes. This allowed us to focus on the key aspect of these measures. We specified several properties which are satisfied by all three centrality measures. Also, we proposed axioms specific for Betweenness and Stress and determined axioms specific for Random Walk Betweenness which highlights the differences between these two approaches. Our characterization not only deepens the understanding of centrality measures, but also could help in choosing a centrality measure for a specific application at hand.

Our work can be extended in many ways. The ultimate goal would be to create an axiomatization of considered measures in a setting with arbitrary many targets. This is, however, challenging as it precludes the use of axioms that do not apply to target nodes. Also, some axioms (e.g., Node Redirect) are not satisfied by Betweenness, Stress and Random Walk Betweenness in their general form with multiple targets.

Another interesting question is how to include in the axiom system other, less popular medial centralities, such as Flow Betweenness Centrality [12]. Interestingly, Flow Betweenness Centrality satisfies several of the axioms that we considered (in particular, Locality, Additivity, Symmetry and Atom k-k) although it is based on yet another model of transition. Finally, undirected or edge-weighted graphs can be considered.

ACKNOWLEDGMENTS

Wiktoria Kośny and Oskar Skibski were supported by the Polish National Science Centre Grant No. 2018/31/B/ST6/03201.

REFERENCES

[1] Alon Altman and Moshe Tennenholtz. 2005. Ranking systems: the PageRank axioms. In *Proceedings of the 6th ACM Conference on Electronic Commerce (ACM-EC)*. 1–8.

[2] Francis Bloch, Matthew O. Jackson, and Pietro Tebaldi. 2019. Centrality Measures in Networks. (2019). Available at SSRN 2749124.

[3] Florian Blöchl, Fabian J. Theis, Fernando Vega-Redondo, and Eric O'N. Fisher. 2011. Vertex centralities in input-output networks reveal the structure of modern economies. *Physical Review E* 83, 4 (2011), 046127.

[4] Paolo Boldi and Sebastiano Vigna. 2014. Axioms for centrality. *Internet Mathematics* 10, 3-4 (2014), 222–262.

[5] Stephen P. Borgatti and Martin G. Everett. 2006. A graph-theoretic perspective on centrality. *Social Networks* 28(4) (2006), 466–484.

[6] Ulrik Brandes and Thomas Erlebach. 2005. *Network analysis: Methodological foundations*. Springer-Verlag.

[7] Thayne Coffman, Seth Greenblatt, and Sherry Marcus. 2004. Graph-based technologies for intelligence analysis. *Commun. ACM* 47, 3 (2004), 45–47.

[8] Vianney Dequiedt and Yves Zenou. 2017. Local and consistent centrality measures in parameterized networks. *Mathematical Social Sciences* 88 (2017), 28–36.

[9] Shlomi Dolev, Yuval Elovici, and Rami Puzis. 2010. Routing betweenness centrality. *Journal of the ACM (JACM)* 57, 4 (2010), 1–27.

[10] Roy Thomas Fielding. 2000. *Architectural Styles and the Design of Network-based Software Architectures. Chapter 5: Representational State Transfer (REST)*. Ph.D. Dissertation. University of California, Irvine.

[11] Linton C. Freeman. 1977. A set of measures of centrality based on betweenness. *Sociometry* 40, 1 (1977), 35–41.

[12] Linton C. Freeman, Stephen P. Borgatti, and Douglas R. White. 1991. Centrality in valued graphs: A measure of betweenness based on network flow. *Social Networks* 13, 2 (1991), 141–154.

[13] Manuj Garg. 2009. Axiomatic foundations of centrality in networks. (2009). Available at SSRN 1372441.

[14] Matthew O. Jackson. 2008. *Social and Economic Networks*. Vol. 3. Princeton university press.

[15] Maliackal Poulo Joy, Amy Brock, Donald E Ingber, and Sui Huang. 2005. High-betweenness proteins in the yeast protein interaction network. *Journal of Biomedicine and Biotechnology* 2005, 2 (2005), 96.

[16] Mitri Kitti. 2016. Axioms for centrality scoring with principal eigenvectors. *Social Choice and Welfare* 46, 3 (2016), 639–653.

[17] Mark E. J. Newman. 2005. A measure of betweenness centrality based on random walks. *Social Networks* 27, 1 (2005), 39–54.

[18] UJ Nieminen. 1973. On the centrality in a directed graph. *Social Science Research* 2, 4 (1973), 371–378.

[19] Gert Sabidussi. 1966. The centrality index of a graph. *Psychometrika* 31, 4 (1966), 581–603.

[20] Alfonso Shimbel. 1953. Structural parameters of communication networks. *Bulletin of Mathematical Biophysics* 15, 4 (1953), 501–507.

[21] Oskar Skibski. 2023. Closeness centrality via the Condorcet principle. *Social Networks* 74 (2023), 13–18.

[22] Oskar Skibski, Tomasz P. Michalak, and Talal Rahwan. 2018. Axiomatic Characterization of Game-Theoretic Centrality. *Journal of Artificial Intelligence Research* 62 (2018), 33–68.

[23] Oskar Skibski, Talal Rahwan, Tomasz P. Michalak, and Makoto Yokoo. 2019. Attachment centrality: Measure for connectivity in networks. *Artificial Intelligence* 274 (2019), 151–179.

[24] Oskar Skibski and Jadwiga Sosnowska. 2018. Axioms for Distance-Based Centralities. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*. 1218–1225.

[25] René van den Brink and Robert P. Gilles. 2000. Measuring domination in directed networks. *Social Networks* 22, 2 (2000), 141–157.

[26] Tomasz Wąs and Oskar Skibski. 2020. Axiomatic Characterization of PageRank. *arXiv preprint arXiv:2010.08487* (2020).

[27] Tomasz Wąs and Oskar Skibski. 2018. An Axiomatization of the Eigenvector and Katz Centralities. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*. 1258–1265.

[28] Tomasz Wąs and Oskar Skibski. 2021. An Axiom System for Feedback Centralities. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI)*, Zhi-Hua Zhou (Ed.). 443–449.