Hedonic Games With Friends, Enemies, and Neutrals: Resolving Open Questions and Fine-Grained Complexity

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ABSTRACT

We investigate verification and existence problems for prominent stability concepts in hedonic games with friends, enemies, and optionally with neutrals [8, 15]. We resolve several (long-standing) open questions [4, 15, 19, 22] and show that for friend-oriented preferences, under the *friends and enemies* model, it is coNP-complete to verify whether a given agent partition is (strictly) core stable, while under the *friends, enemies, and neutrals* model, it is NP-complete to determine whether an individual stable partition exists. We further look into natural restricted cases from the literature, such as when the friends and enemies relationships are symmetric, when the initial coalitions have bounded size, when the vertex degree in the friendship graph (resp. the union of friendship and enemy graph) is bounded, or when such graph is acyclic or close to being acyclic. We obtain a complete (parameterized) complexity picture regarding these cases.

KEYWORDS

Hedonic games; Friends and enemies; Core stable; Nash stable; Individually stable; Parameterized complexity

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1 INTRODUCTION

Hedonic games, introduced by Dréze and Greenberg [9], are coalition formation games where each agent's preferences over possible coalitions (i.e., subsets of agents) depend only on the members in the respective coalitions. The goal is to partition the agents into disjoint coalitions which are "stable". Typical stability concepts include *(strict) core stability*, *Nash stability*, and *individual stability* [2, 3, 5, 11, 12, 16, 18, 21]. Briefly put, a partition is *core* stable if no subset S of agents can strictly improve by joining S, and it is *strictly core* stable if no subset S of agents can weakly improve by joining S whereas at least one agent can strictly improve. The partition is *Nash* stable if it is *individually rational* (i.e., no agent prefers to be alone), and no agent envies another coalition (i.e., prefers to be in this coalition rather than her own). It is *individually* stable if it is individually rational, and no agent envies another coalition and this coalition is fine with accepting her.

The existence of a stable partition and the computational complexity of determining whether such partition exists depends on the representation of the preferences of each agent [22]. To simplify the representation of the preferences, Dimitrov et al. [8] introduce the so-called hedonic games with friends and enemies, where there is a directed graph on the agents (the so-called *friendship graph*) such that an agent x considers another agent y a friend if there is an arc from x to y; otherwise, x considers y an enemy. Depending on whether more friends or fewer enemies are preferred, Dimitrov et al. distinguish between *friend-oriented* and *enemy-oriented* preferences. Under *friend-oriented* preferences, when comparing two coalitions, an agent prefers the one with more friends, and for the same number of friends, she prefers fewer enemies, while under *enemy-oriented* preferences, an agent prefers the coalition with fewer enemies, and for the same number of enemies, she prefers more friends. Recently, Brandt et al. [6] show that it is NP-complete to determine the existence of a Nash stable partition under friend-oriented preferences. Dimitrov et al. [8] show that under friend-oriented preferences, there is always a strictly core stable partition (which is hence core stable and individually stable), and under the enemy-oriented preferences, a core stable partition always exists. However, the computational effort to find these partitions is different: Under the friend-oriented preferences, the strongly connected components in the friendship graph form a strictly core stable partition and can be found in linear time, whereas under the enemy-oriented preferences, it is NP-hard to find a core stable partition [20, 22] and beyond NP to find a strictly core stable partition [19]. One question that has remained open for a decade asks what the complexity of the core verification in the friend-oriented case is [4, 15, 19, 22]; it was conjectured to be polynomial-time solvable by Woeginger [22].

Ota et al. [15] extend the model of Dimitrov et al. by also allowing agents to be *neutral* to other agents who do not impact the preferences, and show that the same approach of Dimitrov et al. gives rise to a core stable partition under friend-oriented preferences. They leave open the complexity of verifying core stable partitions. Barrot et al. [4] show that this model may not admit individually stable partitions and leave open the complexity of determining whether

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Table 1: FE$^{\mathsf{S}}$-(S)CoreV refers to the problems CoreV and StrictCoreV in FE$^{\mathsf{S}}$. See Section 2 for the definitions of Δ, κ, and f. "$\clubsuit$" means hardness holds even for planar graphs (and for symmetric preferences, albeit with a larger, constant max degree [T2]). "$\dagger$" (resp. "$\spadesuit$") means hardness holds even for planar graphs (resp. symmetric preferences) . "$\blacklozenge$" (resp. "$\ddagger$") means hardness holds even when the enemy graph is acyclic (resp. $\kappa + \Delta + \mathsf{f}$ is a constant). "$\diamond$" (resp. "$\circ$") means polynomial even if $\mathsf{f} = 2$ (resp. only the friendship graph is acyclic). * Brandt et al. [6] recently showed it to be NP-h. However, their reduction does not bound the maximum degree or the feedback arc set and it is not planar.

restrictions \ always exists?	FE$^{\mathsf{S}}$-(S)CoreV	FEN$^{\mathsf{S}}$-CoreV	FEN$^{\mathsf{S}}$-StrictCoreV	FE$^{\mathsf{S}}$-NashEx	FEN$^{\mathsf{S}}$-NashEx	FEN$^{\mathsf{S}}$-IndividEx
	yes	no	no	no	no	no
Δ	coNP-c$^{\clubsuit}$ [T1]	coNP-c$^{\spadesuit}$ [T12]	coNP-c$^{\spadesuit}$ [T12]	NP-c$^{\dagger *}$ [T3]	NP-c$^{\blacklozenge}$ [T13]	NP-c$^{\blacklozenge}$ [T14]
f	coNP-c [T4]	coNP-c [T11]	coNP-c [T11]	NP-c [T9]	NP-c$^{\blacklozenge}$ [T13]	NP-c$^{\blacklozenge}$ [T14]
$\Delta + \mathsf{f}$	coNP-c [T5]	coNP-c‡ [T11]	coNP-c‡ [T11]	NP-c [T9]	NP-c$^{\blacklozenge}$ [T13]	NP-c$^{\blacklozenge}$ [T14]
κ	coW[1]-h$^{\spadesuit}$, XP [T6]	coNP-c$^{\spadesuit}$ [T12]	coNP-c$^{\spadesuit}$ [T12]	– –	– –	– –
$\mathsf{f} + \kappa$	FPT [T8]	coNP-c‡ [T11]	coNP-c‡ [T11]	– –	– –	– –
$\Delta + \kappa$	FPT [T7]	coNP-c$^{\spadesuit\ddagger}$ [T12]	coNP-c$^{\spadesuit\ddagger}$ [T12]	– –	– –	– –
symm.	coNP-c$^{\clubsuit}$ [T1]	coNP-c$^{\spadesuit}$ [T12]	coNP-c$^{\spadesuit}$ [T12]	P [O2]	P [O2]	P [O2]
DAG	P [P2]	P [P3]	coNP-c [T11]	P$^{\diamond}$ [T9]	P [T10]	P$^{\circ}$ [T10]

one exists. As far as we know, Nash stability has not been studied in the context with neutrals.

Both models, with or without neutrals, are a restriction of *hedonic games with additive preferences* where it is NP-complete to decide whether a Nash stable or individually stable partition exists [21], and it is Σ_2^{P}-complete to decide whether a core stable or strictly core stable partition exists [17, 23].

In this paper, we focus on the friend-oriented model and resolve long-standing open questions by showing that all mentioned problems whose complexity was unknown are in fact intractable (either coNP- or NP-complete). In particular, we refute Woeginger's conjecture [19, 22] and show that verifying core stable partitions is *not* polynomial-time solvable unless P=NP. To understand the true causes of the intractability results and to explore the line between easy and hard cases, we further look into interesting restricted scenarios such as planar or acyclic graphs, and natural parameters such as maximum degree Δ and feedback arc set number f of the input graph, and also the size κ of the largest coalition in a given partition. We analyze and obtain a complete picture of fine-grained complexity of both the verification and existence problems with respect to the four stability concepts and under friend-oriented preferences. Our results are given in Table 1. We summarize our main contributions as follows.

- First and foremost, we establish that it is coNP-complete to decide whether a given partition is core stable or strictly core stable, even in the case without neutrals (see Theorem 1), and it is NP-complete to decide whether an individually stable partition exists in the case with neutrals (see Theorem 14).
  The first result has both theoretical and practical significance: (1) The reduction is based on a novel friendship gadget, which may be of independent interest for other hardness reductions for hedonic games; (2) It not only showcases a rare complexity situation where verification is much harder than searching, but it can also be served as a complexity barrier against manipulation; e.g., when an agent or a subset of agents want to know if it is beneficial to maintain the status quo rather than to deviate, they essentially need to solve the coNP-hard verification question.
- Second, we show that assuming the friends and enemies relationship graph to be acyclic (DAG) almost always ensures polynomial-time solvability. The strict core verification problem with neutrals is the only exception. Moreover, we obtain complexity dichotomies with regards to the distance to being a DAG, the so-called feedback arc set number f. We note that DAGs or relationship graphs with small f occurs, for instance, for authors when the friendships are based on the popularity of authors. A prominent author is followed by many other authors, whereas, this relation is often asymmetric and ordering the authors according to their popularity can yield a small f.
- Third, strengthening known and own results, we show that assuming the relationship graph to be planar (e.g., when the agents are located on the plane) or sparse (i.e., the maximum degree Δ in the relationship graph is bounded since each agent typically only knows a few other agents) does not lower the complexity.
- Finally, for the verification problem where a partition is given, we show that under the friends and enemies model, if the initial coalitions have small constant sizes, then the problem can be solved in polynomial time, i.e., an XP algorithm wrt. κ, but this parameter alone cannot yield fixed-parameter (FPT) algorithms under standard complexity theoretic assumptions. Combining with f or Δ, we obtain FPT algorithms. Note that the algorithm for the combined parameter (κ, f) is based on a reduction to Directed Subgraph Isomorphism where the pattern graph is of size $O(\kappa^2)$. Our crucial observation further reduces it to the case where the pattern graph is indeed a directed in-tree, enabling us to design an algorithm with desired running time. The problem is much harder when neutrals are present; both core verification problems remain coNP-hard even if $\kappa + \Delta + \mathsf{f}$ is a constant.

Paper structure. In Section 2, we define the model and relevant concepts, the central problems, and parameters. In Sections 3 and 4, we consider the model without neutrals and with neutrals, respectively. We conclude in Section 5. Due to space constraints, proofs of the results and additional materials marked with (★) are deferred to the full version of the paper [?].

2 BASIC DEFINITIONS AND FUNDAMENTALS

Given an integer t, let $[t]$ denote the set $\{1, 2, \dots, t\}$. Given a directed graph G and a vertex v, the sets $N^+_G(v)$ and $N^-_G(v)$ denote the out- and in-neighborhood of v. An instance of Hedonic Games consists of a set V of agents and for each agent a preference order (with possibly ties) over non-empty agent subsets, called *coalitions*, which contains her. In this paper, we focus on a natural and simple variant of Hedonic Games where each agent regards every other agent either good (i.e., a friend), or bad (i.e., an enemy), or neutral such that agents' preferences are friends oriented. Formally, we are given a set of agents V and two directed graphs on V, called *friendship* graph G^{g} and *enemy* graph G^{b} with *disjoint* arc sets, such that an agent i regards another agent j as friend (resp. enemy) whenever G^{g} (resp. G^{b}) contains the arc (i, j); i considers j neutral if neither G^{g} nor G^{b} contains (i, j).

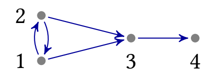
For each agent $i \in V$, the preference order $\succeq_i$ of i is derived as follows: For two coalitions S and T containing i, agent i *(strictly) prefers* S to T, written as $S \succ_i T$, if (i) either $|N^+_{G^{\mathrm{g}}}(i) \cap S| > |N^+_{G^{\mathrm{g}}}(i) \cap T|$, (ii) or $|N^+_{G^{\mathrm{g}}}(i) \cap S| = |N^+_{G^{\mathrm{g}}}(i) \cap T|$ and $|N^+_{G^{\mathrm{b}}}(i) \cap S| < |N^+_{G^{\mathrm{b}}}(i) \cap T|$. Agent i is *indifferent* between S and T, written as $S \sim_i T$, if $|N^+_{G^{\mathrm{g}}}(i) \cap S| = |N^+_{G^{\mathrm{g}}}(i) \cap T|$ and $|N^+_{G^{\mathrm{b}}}(i) \cap S| = |N^+_{G^{\mathrm{b}}}(i) \cap T|$. Agent i *weakly prefers* S to T if $S \succ_i T$ or $S \sim_i T$. Note that the number of neutral agents in the coalition does not affect agent's preferences regarding that coalition.

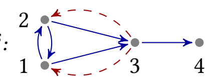
We call V the *grand* coalition. A *coalition structure* Π of V is a partition of V into disjoint coalitions, i.e., the coalitions V' in Π are pairwise disjoint and $\bigcup_{V' \in \Pi} V' = V$. We will use coalition structure and partition interchangeably. Given a coalition structure Π of V and an agent $i \in V$, let $\Pi(i)$ denote the coalition which contains i. A coalition W is *strictly blocking* a coalition structure Π if every agent $i \in W$ strictly prefers W to $\Pi(i)$, and it is *weakly blocking* Π if every agent $i \in W$ weakly prefers W to $\Pi(i)$ and at least one agent $i \in W$ strictly prefers W to $\Pi(i)$.

We use *FEN*$^{\mathrm{s}}$ to denote the Hedonic Games variant with friends, enemies, and neutrals, and use *FE*$^{\mathrm{s}}$ to denote the restricted variant of FEN$^{\mathrm{s}}$ where no agent is neutral to any other agent, i.e., $(i, j) \in E(G^{\mathrm{g}}) \cup E(G^{\mathrm{b}})$ holds for all distinct agents i and j. Note that the superscript $^{\mathrm{s}}$ refers to simple and is added to distinguish from the abbreviation FEN used in the literature [12]. For FE$^{\mathrm{s}}$, we follow the convention in the literature and only specify the friendship relation. Due to this, in the remainder of the paper, we assume that an FE$^{\mathrm{s}}$ instance consists of the friendship graph only.

(Strictly) core stable coalition structures, Nash and individual stability. Let Π be a coalition structure. We say that Π is *core stable* (resp. *strictly core stable*) if no coalition is strictly (resp. weakly) blocking Π. Clearly, by definition, a strictly core stable partition is also a core stable one. We call Π *Nash stable* if it is *individually rational* (i.e., no agent prefers to be alone) and no agent *envies* another coalition (i.e., no agent x and coalition C in Π exist such that x prefers $C \cup \{x\}$ to her own). It is *individually stable* if it is individually rational, and no agent x and coalition $C \in \Pi$ form a *blocking tuple* (i.e., x envies C and each agent $j \in C$ weakly prefers $C \cup \{x\}$ to C).

Example 1. *The graph on the left (with blue arcs only) is an instance of FE*$^{\mathrm{s}}$*, where each arc specifies the friendship relation. The coalition structure, derived from the strongly connected components,* $\Pi_1 = \{\{1, 2\}, \{3\}, \{4\}\}$ *is strictly core stable, but not Nash stable since* 3 *wants to join* $\{4\}$*. Indeed, there is no Nash stable solution. The graph on the right (blue arcs indicating friends while red arcs enemies) is an instance of FEN*$^{\mathrm{s}}$*. The coalition structure* $\Pi_2 = \{\{1, 2\}, \{3, 4\}\}$ *is strictly core stable and Nash stable.*

FE$^{\mathrm{s}}$*:*


FEN$^{\mathrm{s}}$*:*


The following relation is known from the literature [5, 8, 15].

Proposition 1. *(i) Every strictly core stable coalition structure is individually stable and core stable.*
(ii) Nash stability implies individual stability.
(iii) For FE$^{\mathrm{s}}$*, a strictly core stable coalition structure always exists and it can be found in linear time.*
(iv) For FEN$^{\mathrm{s}}$*, a core stable coalition structure always exists and it can be found in linear time.*

Central problems. We are interested in the following core verification problems.

FEN$^{\mathrm{s}}$-CoreV (resp. FE$^{\mathrm{s}}$-CoreV)
Input: An FEN$^{\mathrm{s}}$ instance $(V, G^{\mathrm{g}}, G^{\mathrm{b}})$ (resp. FE$^{\mathrm{s}}$ instance (V, G^{g})), and a coalition structure Π on V.
Question: Is Π core stable?

We define FEN$^{\mathrm{s}}$-StrictCoreV and FE$^{\mathrm{s}}$-StrictCoreV accordingly when we instead ask whether Π is strictly core stable. All four problems are contained in coNP since checking whether a coalition is blocking a coalition structure can be done in polynomial time.

By definition, it is straightforward that verifying Nash stability or individual stability is polynomially solvable. Hence, we look into the existence questions, which are contained in NP.

FEN$^{\mathrm{s}}$-NashEx (resp. FEN$^{\mathrm{s}}$-IndividEx)
Input: An FEN$^{\mathrm{s}}$ instance $I = (V, G^{\mathrm{g}}, G^{\mathrm{b}})$.
Question: Does I admits a Nash stable (resp. individually stable) coalition structure?

We define FE$^{\mathrm{s}}$-NashEx accordingly for the FE$^{\mathrm{s}}$ case. We assume basic knowledge of parameterized complexity and refer to the following textbooks [7, 14] for more details.

Graph structures and parameters. We investigate the (parameterized) complexity of the above problems and focus on restricted instances. Given an instance $I = (V, G^{\mathrm{g}}, G^{\mathrm{b}})$, we define the following parameters:

- Max degree Δ: For FEN$^{\mathrm{s}}$, it is defined as $\max_{i \in V} |N^+_{G^{\mathrm{g}}+G^{\mathrm{b}}}(i) \cup N^-_{G^{\mathrm{g}}+G^{\mathrm{b}}}(i)|$, while for FE$^{\mathrm{s}}$, it is defined as $\max_{i \in V} |N^+_{G^{\mathrm{g}}}(i) \cup N^-_{G^{\mathrm{g}}}(i)|$ since $G^{\mathrm{g}} + G^{\mathrm{b}}$ is a complete digraph.
- Max coalition size κ: It is defined as the size of the largest coalition in the coalition structure from the input.
- Feedback arc set number f: For FEN$^{\mathrm{s}}$, f is the smallest number of arcs deleting which makes $G^{\mathrm{g}} + G^{\mathrm{b}}$ acyclic, while for FE$^{\mathrm{s}}$, f is the smallest number of arcs deleting which makes G^{g} acyclic,

We say that I has *symmetric preferences* if each arc in G^{g} and G^{b} is bi-directional (see the arcs $(1, 2)$ and $(2, 1)$ in Example 1). It contains *acyclic* graph (*DAG*) if the union $G^{\mathrm{g}} + G^{\mathrm{b}}$ is acyclic for the $\mathrm{FEN}^{\mathrm{s}}$ model and G^{g} is acyclic for the FE^{s} model, respectively.

3 THE FRIENDS AND ENEMIES MODEL

In this section, we focus on the FE^{s} model [8]. First of all, we settle the complexity of problems regarding (strict) core verification and Nash existence, and show that they are intractable and remain so even for very restricted cases such as sparse graphs and symmetric preferences. For the hardness reductions, we use the following NP-complete problem:

Planar-X3C

Input: A $3\hat{n}$-element set $\mathcal{X} = [3\hat{n}]$ and a collection $\mathcal{C} = (C_1, \ldots, C_{\hat{m}})$ of 3-element subsets of X such that each element $i \in X$ appears in either two or three members in $\mathcal{C}$ and that the associated *element-linked graph* is planar.

Question: Does $\mathcal{C}$ contain an *exact cover* for X, i.e., a subcollection $\mathcal{K} \subseteq \mathcal{C}$ such that each element of X occurs in exactly one member of $\mathcal{K}$?

Herein, given a Planar-X3C instance $I = (X, \mathcal{C})$, the associated *element-linked graph* of I is a graph $G(I) = (U \uplus W, E)$ on two partite vertex sets $U = \{u_i \mid i \in X\}$ and $W = \{w_j \mid C_j \in \mathcal{C}\}$ such that $E = \{\{u_i, w_j\} \mid i \in C_j\} \cup \{\{u_i, u_{i+1}\} \mid i \in [3\hat{n} - 1]\} \cup \{\{u_1, u_{3\hat{n}}\}\}$ is planar. We call the vertices in U and W the *element-vertices* and the *set-vertices*, respectively. We also call the cycle induced by the element-vertices the *element-cycle*. For notational convenience, for each element $i \in [3\hat{n}]$, let $\mathcal{C}(i) := \{C_j \in \mathcal{C} \mid i \in C_j\}$ denote the sets which contain element i.

Dyer and Frieze [10] show that the NP-completeness of Planar-X3C remains even if the planar embedding of the associated element-linked graph satisfies the following:

$$\text{for all } u_i \in U \text{ there are at most two vertices } w_j \text{ such that } i \in C_j \text{ and both lie } \textit{inside} \text{ or } \textit{outside} \text{ of the element-cycle.} \quad (\heartsuit)$$

Hence, for notational convenience and based on this planar embedding, we partition $\mathcal{C}$ into two disjoint subcollections: $\mathcal{C}^{out} := \{C_j \in \mathcal{C} \mid w_j \text{ lies outside of the element-cycle }\}$ and $\mathcal{C}^{in} := \mathcal{C} \setminus \mathcal{C}^{\mathrm{out}}$. In the hardness proofs, we will utilize this fact to construct appropriate gadgets which do not exceed the desired maximum vertex degree.

Theorem 1. *FE^{S}-CoreV and FE^{S}-StrictCoreV are* coNP*-complete even for planar friendship graphs and $\Delta = 4$.*

Proof sketch. We show hardness for both problems via the same reduction. Let $I = ([3\hat{n}], \mathcal{C})$ denote an instance of Planar-X3C with $\mathcal{C} = \{C_1, \ldots, C_{\hat{m}}\}$, and let $G(I) = (U \cup W, E)$ denote the associated element-linked planar graph. Recall that there exists a planar embedding of $G(I)$ which satisfies ($\heartsuit$) such that $\mathcal{C}^{\mathrm{out}}$ and $\mathcal{C}^{\mathrm{in}}$ partition the set family $\mathcal{C}$ into two disjoint subfamilies according to this embedding.

For brevity's sake, define $L = 27\hat{n} - 1$; we note that the desired blocking coalition will be of size $L+1$. For each element $i \in [3\hat{n}]$, create three *element agents* x_i, s_i, t_i, and a set of $L+1$ private *friendship* agents x_i^z, $z \in \{0, \ldots, L\}$; these $L+1$ agents comprise the friendship gadget of element i. The number of the friendship agents will ensure that we indeed have an exact cover. For each set $C_j \in \mathcal{C}$ and each element $i \in C_j$, create two *set agents* c_j^i and d_j^i. To connect the elements with the sets, for each element $i \in [3\hat{n}]$, create two groups

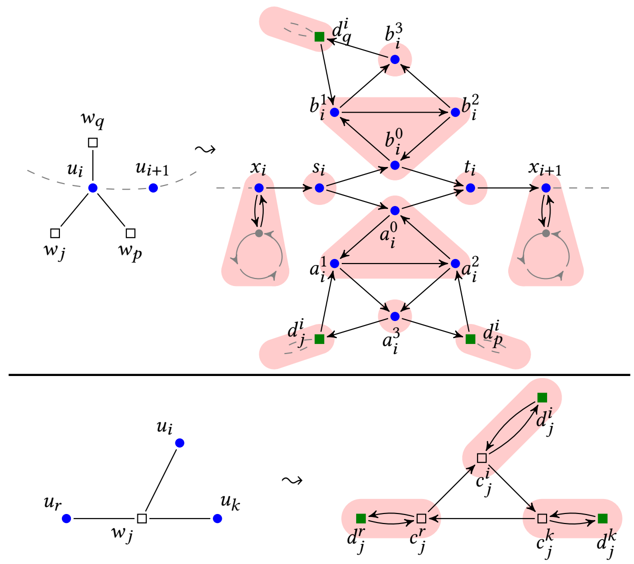


Figure 1: Gadgets for Theorem 1. The red areas indicate the coalitions in the initial partition Π. Upper part: An element-gadget, where $\mathcal{C}(i) = \{C_j, C_p, C_q\}$ such that $C_j, C_p \in \mathcal{C}^{\mathrm{out}}$ (indicated by the dashed line). The private friendship gadget for x_i (resp. x_{i+1}) is depicted as a gray directed cycle. Lower part: A set-gadget, where $C_j = \{i, k, r\}$. The red areas indicate the coalitions in the initial partition Π.

of connection gadgets (one for each side of the element-cycle) with a total of eight agents called $a_i^z, b_i^z, z \in \{0, 1, 2, 3\}$, which serve as selector agents. We remark that agents a_i^3 and b_i^3 serve as connectors and will be friends with the agents corresponding to the sets which contain i. This completes the construction of the agents. In total, we have agent set $V := \{x_i, x_i^0, \ldots, x_i^L, s_i, t_i, a_i^z, b_i^z \mid i \in [3\hat{n}], z \in \{0, 1, 2, 3\}\} \cup \{c_j^i, d_j^i, c_j^k, d_j^k, c_j^r, d_j^r \mid C_j \in \mathcal{C} \text{ with } C_j = \{i, k, r\}\}$. Next, we describe the friendship graph G^{g}; its planar embedding is based on the planar embedding of $G(I)$.

- Starting from the planar embedding of $G(I)$, we replace each element vertex $u_i \in U$ with the corresponding element agent x_i and their friendship agents x_i^z, $z \in \{0, \ldots, L\}$, with bidirectional arcs (x_i, x_i^0), (x_i^0, x_i) and directed cycle (x_i^z, x_i^{z+1}) ($z \in \{0, \ldots, L\}$, $z+1$ taken modulo $L+1$). For each edge $\{u_i, u_{i+1}\}$ on the element-cycle, we replace it with the arcs (x_i, s_i), (s_i, a_i^0), (s_i, b_i^0), (a_i^0, t_i), (b_i^0, t_i), (t_i, x_{i+1}); let $i+1 = 1$ if $i = 3\hat{n}$. Further, for each $(i, z) \in [3\hat{n}] \times \{0, 1, 2\}$, we add the arcs (a_i^z, a_i^{z+1}), (b_i^z, b_i^{z+1}) ($z+1$ taken modulo 3), (a_i^1, a_i^3), (a_i^2, a_i^3), and (b_i^1, b_i^3), (b_i^2, b_i^3).
- For each set $C_j \in \mathcal{C}$ with $C_j = \{i, k, r\}$ and $i < k < r$, we replace the corresponding set vertex in the planar embedding with a directed subgraph consisting of the following arcs: (c_j^i, d_j^i), (d_j^i, c_j^i), (c_j^k, d_j^k), (d_j^k, c_j^k), (c_j^r, d_j^r), (d_j^r, c_j^r), (c_j^i, c_j^k), (c_j^k, c_j^r), (c_j^r, c_j^i).
- For each edge $\{u_i, w_j\} \in E(G(I))$, we replace this edge according to where the corresponding set vertex lies in the planar embedding. If $C_j \in \mathcal{C}^{\mathrm{out}}$, then we add the arcs (a_i^3, d_j^i), (d_j^i, a_i^z), where z is deterministically fixed to either 1 or 2 so as to maintain the

planarity. Analogously, if $C_j \in C^{\mathsf{in}}$, then we add arcs (b_i^3, d_j^i), (d_j^i, b_i^z); again z is deterministically fixed to either 1 or 2 so as to maintain the planarity. Note that by (♡), adding arcs (d_j^i, v_i^z) ($v \in \{a, b\}, z \in [2]$) preserves planarity.

See Figure 1 for an illustration. To complete the construction, define the initial coalition structure $\Pi := \{\{x_i, x_i^0, x_i^1, \dots, x^L\}, \{s_i\}, \{t_i\} \mid i \in [3\hat{n}]\} \cup \{\{v_i^0, v_i^1, v_i^2\}, \{v_i^3\} \mid (i, v) \in [3\hat{n}] \times \{a, b\}\} \cup \{\{c_j^i, d_j^i\} \mid \{u_i, w_j\} \in E(G(I))\}$.

The general idea for the reduction is as follows: For each element $i \in [3\hat{n}]$ we created an element-gadget (consisting of several element agents). The element-gadgets are "connected" via appropriate selector gadgets so they correspond to the element-cycle in the associated element-linked graph $G(I)$. This ensures that a weakly blocking coalition will need to contain dedicated agents which correspond to all elements. For each set $C_j \in C$ we created a set-gadget (again consisting of several set agents). We connected an element-gadget to a set-gadget via a communication gadget if and only if the corresponding element is contained in the corresponding set. This ensures that an agent in a set-gadget is in a blocking coalition if and only if the element agents "contained" in the set are in the blocking coalition as well. Note that this already gives us a covering for the elements. To have an exact cover, we introduced a novel friendship gadget which can never participate in any blocking coalition but shall ensure that its associated agent will never join a blocking coalition that is too large. This sets an upper limit on the size of a blocking coalition.

One can verify that the graph is planar. The proofs for having maximum degree 4 and the correctness are deferred to the full version. □

Next, we show that restricting the preferences to be symmetric does not lower the complexity. The ideas of the reduction are fairly similar to that for Theorem 1.

Theorem 2 (★). *FE^S-CoreV and FE^S-StrictCoreV are* coNP*-complete even if the preferences are symmetric and the friendship graph is planar and* $\Delta = 8$.

Next, strengthening the result by Brandt et al. [6], we show that finding a Nash stable solution is NP-hard even if the friendship graph has constant maximum degree and is planar.

Theorem 3 (★). *FE^S-NashEx is* NP*-complete even if the friendship graph is planar and* $\Delta = 9$.

3.1 Algorithms and refined complexity for FE

We start with some simple polynomial-time algorithms for core verification.

Proposition 2 (★). *For each of the following cases, FE^S-CoreV and FE^S-StrictCoreV are polynomial-time solvable.*
(i) The friendship graph is a acyclic.
(ii) $\Delta = 2$.
(iii) The preferences are symmetric and $\Delta = 4$.

While both verification problems are trivial if the friendship graph is acyclic, we show that interestingly even one feedback arc makes the problem intractable.

Theorem 4 (★). *FE^S-CoreV and FE^S-StrictCoreV are* coNP*-complete even if* $\mathsf{f} = 1$*, and each agent has at most* 3 *friends.*

Additionally bounding the maximum degree Δ does not help to break down the complexity.

Theorem 5 (★). *FE^S-CoreV and FE^S-StrictCoreV are* coNP*-complete, even if* $\mathsf{f} = 2$ *and* $\Delta = 5$.

Next, we observe that checking whether a specific strictly blocking coalition exists can be done in linear time. This result will be useful for designing further algorithms.

Lemma 1 (★). *Given a coalition structure* Π *with maximum coalition size* κ*, in linear time, we can either find a blocking coalition where every agent obtains strictly more friends than in* Π*, or conclude that each weakly (resp. strictly) blocking coalition has size at most* κ.

Proof sketch. Call a coalition a *wonderfully blocking coalition* if every agent in it has strictly more friends than in Π. We observe that if no coalitions are wonderfully blocking, then in any blocking (resp. weakly blocking) coalition U', there is an agent who has the same number of friends, so she cannot get more enemies than in Π, implying that $|U'| \leq \kappa$. Hence, checking whether wonderfully blocking coalitions exist in the desired time completes the proof.

For each agent $v \in V$, let $f_\Pi(v)$ denote the number of friends she has in Π, i.e., $f_\Pi(v) = |N^+_{G^{\mathsf{g}}}(v) \cap \Pi(v)|$, and let $r(v) = f_\Pi(v) + 1$. Let U be a hypothetical wonderfully blocking coalition. Then, each agent $v \in U$ needs at least $r(v)$ friends in U. Now, if there are agents in the input with out-degree less than $r(v)$, then we delete them since they cannot be included in U. Then, we recursively delete the agents v that have less than $r(v)$ out-neighbors in the resulting friendship graph. We repeat this process as long as there is an agent v with out-degree less than $r(v)$. If this procedure terminates with some agents remaining, then they form a wonderfully blocking coalition; otherwise, there can be none. The correctness proof and the running time are deferred to the full version. □

Based on Lemma 1, core verification is polynomial-time solvable if the largest initial coalition has bounded size. However, this result cannot be improved to obtain fixed-parameter tractability.

Theorem 6 (★). *FE^S-CoreV and FE^S-StrictCoreV are in* XP *and* coW[1]*-hard wrt.* κ*; hardness remains even for symmetric preferences.*

Combining κ with Δ, we obtain a fixed-parameter algorithm, based on random separation.

Theorem 7 (★). *FE^S-CoreV and FE^S-StrictCoreV are* FPT *wrt.* (κ, Δ).

Based on the observation below, we obtain a color-coding based fixed-parameter algorithm for the combined parameter (κ, f). To this end, we call a coalition in a given coalition structure Π a *singleton* (resp. *non-singleton*) coalition if it has size one (resp. larger than one). Accordingly, an agent is a *singleton* (resp. *non-singleton*) agent (wrt. Π) if she is in a singleton (resp. non-singleton) coalition.

Observation 1 (★). *If* Π *is core stable, then there are at most* $\kappa \cdot \mathsf{f}$ *non-singleton agents.*

Theorem 8. *FE^S-CoreV and FE^S-StrictCoreV are* FPT *wrt.* (κ, f).

Proof sketch. Let (V, G^{g}) be an instance of FE^{s}and Π an initial coalition structure. The algorithm has two phases. First, we preprocess the instance so that each non-trivial blocking coalition has at most κ agents and the reduced instance excludes some undesired cycles. Second, we further reduce the instance to one which is acyclic and observe that any non-trivial blocking coalition must "contain" an in-tree of size $O(\kappa^2)$. Hence, for each possible in-tree we can use color-coding to check whether it exists in FPT-time. In the following we use V_{S} and V_{NS} to denote the singleton and non-singleton agents, respectively.

The first phase consists of the following polynomial-time steps:

(P1) Check whether Π contains a coalition U such that $G^{g}[U]$ is not strongly connected. If yes, then return NO since the strongly connected subgraph corresponding to the sink component in $G^{g}[U]$ is strictly blocking Π.

(P2) If $G^{g}[V_{\mathrm{S}}]$ contains a cycle, then the singletons agents on the cycle is strictly blocking Π, so return NO.

(P3) By Lemma 1, check in linear time whether there is a blocking (resp. weakly blocking) coalition of size greater than κ.

The second phase is as follows: For each subset $B_{\mathrm{NS}} \subseteq V_{\mathrm{NS}}$ of size $k' \le \kappa$ and each size b with $k' \le b \le \kappa$, we check whether there exists a blocking coalition of size b which contains all non-singletons from B_{NS} and exactly $b - |B_{\mathrm{NS}}|$ singletons; note that after phase one, we only need to focus on coalitions of size at most κ and can assume that $|V_{\mathrm{NS}}| \le \kappa \cdot \mathrm{f}$. We return YES if and only if no pair (B_{NS}, b) can be extended to a blocking coalition (i.e., Algorithm 1 returns NO for all (B_{NS}, b)).

Given (B_{NS}, b), the task reduces to searching for the $b - |B_{\mathrm{NS}}|$ missing singleton agents, assuming that such an extension is possible. To achieve this, we reduce to searching for an in-tree of size $O(\kappa^2)$ in a directed acyclic graph (DAG) H, which using color coding, can be done in $f(\kappa) \cdot |H|^{O(1)}$ time where f is some computable function. First of all, if $|B_{\mathrm{NS}}| = b$, then we check whether B_{NS} is blocking (resp. weakly blocking) in polynomial time and return B_{NS} if this is the case; otherwise we continue with a next pair (B_{NS}, b). In the following, let B be a hypothetical blocking coalition of size $b > |B_{\mathrm{NS}}|$ which consists of B_{NS} and $b - |B_{\mathrm{NS}}|$ singleton agents. The searching has two steps.

(C1) Construct a search graph $\hat{G}^{g}$ from G^{g}. Based on G^{g} we construct a DAG $\hat{G}^{g}$, where we later search for the crucial part of the blocking coalition. We compute the minimum number $r(a_i)$ of singleton-friends each non-singleton agent a_i in B_{NS} should obtain from B by checking how many friends she initially has. Let $n_i(\Pi)$ and $n_i(B_{\mathrm{NS}})$ denote the number of friends agent a_i has in $\Pi(a_i)$ and B_{NS}, respectively. If $n_i(\Pi) < n_i(B_{\mathrm{NS}})$, then let $r(a_i) := 0$. Otherwise for CoreV, let

$$r(a_i) := \begin{cases} n_i(\Pi) + 1 - n_i(B_{\mathrm{NS}}), & \text{if } b \ge |\Pi(a_i)| \\ n_i(\Pi) - n_i(B_{\mathrm{NS}}), & \text{otherwise.} \end{cases}$$

For StrictCoreV, the first if-condition is $b > |\Pi(a_i)|$ instead of $b \ge |\Pi(a_i)|$. After the computation, we check whether some non-singleton agent $a_i \in B_{\mathrm{NS}}$ has $r(a_i) > b - |B_{\mathrm{NS}}|$. If a_i is such an agent, then she will not weakly prefer B to $\Pi(a_i)$ since there are not enough friends for her, so we continue with a next pair (B_{NS}, b). Now, we construct $\hat{G}^{g}$. First, duplicate the vertices in B_{NS} as $\hat{B}_{\mathrm{NS}} := \{a_i^z \mid (a_i, z) \in B_{\mathrm{NS}} \times [r(a_i)]\}$. The vertex set and the arc set of $\hat{G}^{g}$ are defined as $\hat{V} := \hat{B}_{\mathrm{NS}} \cup V_{\mathrm{S}} \cup \{t\}$, where t is an artificial sink, and $\hat{E} := \{(a_i^z, s) \mid (a_i, s) \in E(G^{g}) \cap (B_{\mathrm{NS}} \times V_{\mathrm{S}}), z \in [r(a_i)]\} \cup E(G^{g}[V_{\mathrm{S}}]) \cup \{(s,t) \mid (s, a_i) \in E(G^{g}) \cap (V_{\mathrm{S}} \times B_{\mathrm{NS}})\}$, respectively. Now let $\hat{G}^{g} := (\hat{V}, \hat{E})$. Briefly put, we remove all arcs that are not incident to the singletons, and redirect every arc from a singleton to a non-singleton vertex in B_{NS} to the artificial sink t. Note that $\hat{G}^{g}$ is acyclic since by (P2) no singleton agents induce a cycle.

(C2) Search for a tree structure in $\hat{G}^{g}$. Observe that in $G^{g}[B]$, each non-singleton agent $a_i \in B_{\mathrm{NS}}$ has at least $r(a_i)$ singleton friends and each singleton agent in $B \setminus B_{\mathrm{NS}}$ has at least one friend. Equivalently, in the modified induced subgraph $\hat{G}^{g}[B]$, each non-singleton agent in $\hat{B}_{\mathrm{NS}}$ (resp. singleton agent in $B \setminus \hat{B}_{\mathrm{NS}}$) has at least one out-arc. Then, $\hat{G}^{g}[B]$ contains an in-tree T_B on vertex set $\hat{B}_{\mathrm{NS}} \cup (B \setminus B_{\mathrm{NS}}) \cup \{t\}$ such that

(t1) every vertex $a_i^z \in \hat{B}_{\mathrm{NS}}$ has exactly one out-neighbor and this out-neighbor is a singleton vertex such that no two non-singletons a_i^z and a_i^j ($z \ne j$) share the same out-neighbor,

(t2) every singleton vertex in $V(T_B) \setminus (\hat{B}_{\mathrm{NS}} \cup \{t\})$ has exactly one out-neighbor and this out-neighbor is either the root t or some singleton vertex, and

(t3) t does not have any out-neighbors.

Observe that T_B has exactly $|\hat{B}_{\mathrm{NS}}| + |B \setminus B_{\mathrm{NS}}|$ arcs. Since $\hat{G}^{g}$ is acyclic and the artificial sink t does not have any out-arcs, by the above conditions, T_B must be a directed in-tree with root at t. In particular, T_B is connected. For ease of reasoning, let us call an in-tree T *good* if there exists a subset B' of V_{S} with $b - |B_{\mathrm{NS}}|$ vertices such that T is a directed graph on $\hat{B}_{\mathrm{NS}} \cup B' \cup \{t\}$ and satisfies condition (t1)–(t3) above, replacing the name T_B with T.

If $\hat{G}^{g}$ contains a good in-tree T', then the vertices in B_{NS} and the singleton vertices in T' forms a desired blocking coalition. By applying the color-coding algorithm of Alon et al. [1], we can already search for a good in-tree in FPT-time. For the sake of completeness and to better analyze the running time, we show how to combine color-coding with a polynomial-time algorithm to search for it.

We describe the approach via Algorithm 1. By Naor et al. [13], in line 1, we compute in $f(\kappa) \cdot |V_{\mathrm{S}}|^{O(1)}$ time a family $\mathcal{F}$ of coloring functions (aka. *perfect Hash family*) from V_{S} to $[b - |B_{\mathrm{NS}}|]$ which guarantees to contain a *good* coloring function. Here, a function $\chi\colon V_{\mathrm{S}} \to [b - |B_{\mathrm{NS}}|]$ is called *good* (wrt. B) if it assigns to each singleton vertex in $B \cap V_{\mathrm{S}}$ a distinct color from $[b - |B_{\mathrm{NS}}|]$; see the full version for more details on this. Hence, in line 2 we iterate through each coloring χ in $\mathcal{F}$. Note that if χ is good for B, then after coloring the vertices in V_{S} according to χ, there must exist a good in-tree on vertices $\hat{B}_{\mathrm{NS}} \cup [b - |B_{\mathrm{NS}}|] \cup \{t\}$ as well. Hence, in line 3, we iterate through all good in-trees T.

For ease of reasoning, we also use color to refer to a vertex in $[b - |B_{\mathrm{NS}}|]$, and given a subset $S' \subseteq V_{\mathrm{S}}$ let $\chi(S') = \{\chi(s) \mid s \in S'\}$. In lines 5–9, we iterate through each singleton vertex v in the topological order $\tau(V_{\mathrm{S}})$ in $\hat{G}^{g}$ (recall that $\hat{G}^{g}$ is a DAG), and check whether it has enough in-neighbors whose colors match the in-neighbors of its color $\chi(v)$ in the tree T (line 7). More specifically, we check whether v has singleton in-neighbors of colors C_v indicated by the in-neighbors of $\chi(v)$ in T and whether it has the same non-singleton in-neighbors as its color $\chi(v)$ in T. If yes, then we use S_v to store a subset of such singleton in-neighbors for v (line 8). Otherwise, assuming that χ is good, v cannot be used for the blocking coalition, so we delete it from the search graph. Note that

ALGORITHM 1: (C2) Searching for T_B in $\hat{G}^{\mathrm{g}}$ given B_{NS} and b

1 $\mathcal{F} \leftarrow (|V_{\mathrm{S}}|, b - |B_{\mathrm{NS}}|)$-perfect Hash family on the universe V_{S}
2 **foreach** *coloring* $\chi \in \mathcal{F}$ *with* $\chi \colon V_{\mathrm{S}} \to [b - B_{\mathrm{NS}}]$ **do**
3 **foreach** *Good in-tree* T *on vertex set* $\hat{B}_{\mathrm{NS}} \cup [b - |B_{\mathrm{NS}}|] \cup \{t\}$ **do**
4 $B_{\mathrm{S}} \leftarrow \emptyset$; $\tau(V_{\mathrm{S}}) \leftarrow$ a topological order of V_{S} in $\hat{G}^{\mathrm{g}}$
5 **foreach** $v \in \tau(V_S)$ **do**
6 $C_v \leftarrow N^-_T(\chi(v)) \cap [b - |B_{\mathrm{NS}}|]$
7 **if** $C_v \subseteq \chi(N^-_{\hat{G}^{\mathrm{g}}}(v) \cap V_{\mathrm{S}})$ *and* $N^-_T(\chi(v)) \cap \hat{B}_{\mathrm{NS}} = N^-_{\hat{G}^{\mathrm{g}}}(v) \cap \hat{B}_{\mathrm{NS}}$ **then**
8 let $S_v \subseteq N^-_{\hat{G}^{\mathrm{g}}}(v) \cap V_{\mathrm{S}}$ s.t. $|S_v| = |C_v|$ and $\chi(S_v) = C_v$
9 **else** $\hat{G}^{\mathrm{g}} \leftarrow \hat{G}^{\mathrm{g}} - v$;
10 **if** $N^-_T(t) \subseteq \chi(N^-_{\hat{G}^{\mathrm{g}}}(t))$ **then**
11 let $S_t \subseteq N^-_{\hat{G}^{\mathrm{g}}}(t)$ s.t. $|S_t| = |N^-_T(t)|$ and $\chi(S_t) = N^-_T(t)$
12 $B_{\mathrm{S}} \leftarrow S_t$; $Q \leftarrow S_t$
13 **while** $Q \neq \emptyset$ **do**
14 $P \leftarrow \bigcup_{v \in Q} S_v$
15 $B_{\mathrm{S}} \leftarrow B_{\mathrm{S}} \cup P$; $Q \leftarrow P$
16 **if** $|B_{\mathrm{S}}| = b - |B_{\mathrm{NS}}|$ **then return** $B_{\mathrm{S}} \cup B_{\mathrm{NS}}$;
17 **return** NO, i.e., (B_{NS}, b) cannot be extended to a blocking coalition

the order $\tau(V_{\mathrm{S}})$ ensures that we do not mistakenly store a singleton vertex which cannot be used later on. Finally, in line 10, we check whether the in-neighbors of t contain enough singletons with appropriate colors. If yes, we use the stored set of singleton vertices S_v to iteratively collect all vertices in S_v from root t to leaves; note that in an in-tree, the root is the sink and the leaves are the sources. We return the set B_{S} if it contains exactly $b - |B_{\mathrm{NS}}|$ singletons, and return NO if no iteration gives a desired set B_{S}. The correctness proof and running time analysis are deferred to the full version. $\square$

Next, we determine the existence of Nash stable partitions and show a dichotomy result, strengthening a result by Brandt et al. [6].

Theorem 9 ($\star$). *$\mathrm{FE^S}$-NashEx is polynomial-time solvable if* $\mathrm{f} \leq 2$, *whereas it is* NP*-hard even if* $\mathrm{f} = 3$ *and* $\Delta = 5$.

Proof sketch. We only show the first part and give a linear-time algorithm for $\mathrm{f} \leq 2$. Let (V, G^{g}) be an instance of $\mathrm{FE^S}$-NashEx. For ease of reasoning, call an agent a *sink agent* if she does not have any friend in G^{g}; otherwise call her a *non-sink* agent. Let S denote the set of all sink agents. First, we put each sink agent into a singleton coalition. After that, if there is a non-sink agent who has *only* sink friends, then we return NO. If every non-sink agent that has a sink friend also has at least two non-sink friends, then we return YES. Otherwise there is an agent v who has at least one sink friend and only one non-sink friend w. We place v and w in a size-two coalition $\{v, w\}$ if the friendship relation is symmetric (i.e., w also considers v a friend), otherwise we stop and return NO. Finally, we put all remaining agents in the same coalition. If the obtained partition is Nash stable, then we output YES, otherwise NO.

For correctness, observe that all sink agents must be in singleton coalitions in any Nash stable partition. If there is an agent v who has only sink friends, there is no way to place her into a coalition which contains at least one of her friends. However, because v has at least one sink friend, she will envy the singleton coalition of the friend, and we cannot have a Nash stable partition. If every non-sink agent, who has a sink friend, has at least two non-sink friends, the following partition $\{\{s\} \mid s \in S\} \cup \{\{V \setminus S\}\}$ is Nash stable: Clearly, each agent $v \in V \setminus S$ that has a sink friend has at least 2 friends in $V \setminus S$ but at most one friend in any $\{s\}$, where $s \in S$. If $v \in V \setminus S$ has no sink friends, then all of v's friends are by construction in $V \setminus S$.

Assume there is a vertex v, who has a sink friend s but only one non-sink friend w. Then by the above, in any Nash stable partition Π, $\Pi(s) = \{s\}$ but v has one friend and no enemies in $\{s\}$. This means, in Π, we have $\Pi(v) = \{v, w\}$. Hence, if the friendship is not symmetric, then there is no Nash stable solution. Finally, if the algorithm assigned v and w together, then (v, w, v) is a cycle and must contain a feedback arc. Every remaining non-sink agent must have a friend in any Nash stable solution. This implies that they must be in a coalition that contains a cycle. Since there is only one feedback arc left in $G^{\mathrm{g}}[V \setminus (S \cup \{v, w\}]$, the remaining agents must be in the same coalition in a Nash stable partition. Therefore, if there is a Nash stable solution, then the algorithm finds one and otherwise outputs NO. The analysis of the running time is straightforward and deferred to the full version. $\mathrm{FE^S}$-NashEx is polynomial-time solvable if $\mathrm{f} \leq 2$, whereas it is NP-hard even if $\mathrm{f} = 3$ and $\Delta = 5$. $\square$

4 THE MODEL WITH NEUTRALS

In this section, we consider the model with neutrals [15]. First, we observe that for acyclic friendship graphs, checking core stability is easy since it is equivalent to checking individual rationality.

Proposition 3 ($\star$). *For acyclic friendship graphs, $\mathrm{FEN^S}$-CoreV is linear-time solvable.*

For acyclic graphs, IndividEx and NashEx can be solved by a clever greedy algorithm operated on the reverse topological order.

Theorem 10 ($\star$). *If the friendship graph (resp. the union graph) is acyclic, then every $\mathrm{FEN^S}$-instance admits an individually stable (resp. Nash stable) partition, which can be found in polynomial time.*

Proof Sketch. Let $(V, G^{\mathrm{g}}, G^{\mathrm{b}})$ be an $\mathrm{FEN^S}$-instance. We first consider $\mathrm{FEN^S}$-IndividEx and assume that G^{g} is acyclic and thus has a topological order $v_1, \ldots, v_n$ of V. The algorithm proceeds as follows: Iterate over V in the reverse topological order $v_n, \ldots, v_1$. In each step, check whether there exists a coalition U where v_i has at least one friend and no one in U considers her an enemy. If no such coalition exists, then v_i starts a new coalition. Otherwise, let v_i join the most preferred coalition U among all such coalitions.

Now, we turn to $\mathrm{FEN^S}$-NashEx and assume that $G^{\mathrm{g}} \cup G^{\mathrm{b}}$ is acyclic and thus has a topological order $v_1, \ldots, v_n$ of V. The algorithm proceeds as follows: We iterate over V in the reverse topological order $v_n, \ldots, v_1$. In each step we let the current agent v_i join her most preferred existing coalition, or in the case when v_i has no friends in any, to start a new one.

The correctness of both algorithms relies on iterating through the agents in the topological order. Each of the agents selects in her turn her most preferred feasible coalition, and due to the ordering no agent will change her choice about her most preferred coalition later in the execution. The details of the correctness and the running time are deferred to the full version. $\square$

It is know that for symmetric and additive separable preferences, Nash stable partitions always exist [5]. For the FEN$^{\mathrm{S}}$-model, we can even find one in linear time, which consists of singletons who do not have any friends and the remaining agents in a grand coalition.

Observation 2 (★). *For symmetric friendship relations, Nash (and hence individually) stable partitions can be found in linear time.*

The following complements Proposition 3 regarding f and show that both core verification problems remain hard even if f, Δ and κ are bounded.

Theorem 11 (★). *FEN$^{\mathrm{S}}$-CoreV (resp. FEN$^{\mathrm{S}}$-StrictCoreV) is* coNP-*complete even if* $\Delta = 12$, $\kappa = 3$, *and* $\mathrm{f} = 1$ *(resp.* $\mathrm{f} = 0$*).*

Unlike the Nash and individual stability, symmetric preferences do not help in reducing the complexity.

Theorem 12 (★). *FEN$^{\mathrm{S}}$-CoreV (resp. FEN$^{\mathrm{S}}$-StrictCoreV) is* coNP-*complete even when the preferences are symmetric,* $\Delta = 7$ *(resp.* $\Delta = 26$*), and* $\kappa = 3$ *(resp.* $\kappa = 4$*).*

The following two theorems complement Theorem 10 regarding f and show that determining Nash (resp. individually) stable partitions remain hard even if both f and Δ are bounded.

Theorem 13 (★). *FEN$^{\mathrm{S}}$-NashEx remains* NP-*complete even if* $\Delta = 9$ *and* $\mathrm{f} = 1$, *and both the friendship and the enemy graphs are respectively acyclic. .*

Theorem 14. *FEN$^{\mathrm{S}}$-IndividEx remains* NP-*complete even if* $\Delta = 18$ *and* $\mathrm{f} = 1$ *such that the friendship graph has one feedback arc and the enemy graph is acyclic.*

Proof. We reduce from Planar-X3C; we will not use the planarity property though. Let $I = ([3\hat{n}], \mathcal{C})$ be an instance of Planar-X3C, where $\mathcal{C} = \{C_1, \ldots, C_{\hat{m}}\}$. Without loss of generality, assume that $\hat{n}$ is odd. For each element $i \in [3\hat{n}]$, create an element agent a_i, a leader agent u_i, and two follower agents u_i^1, u_i^2; define $U_i = \{u_i, u_i^1, u_i^2\}$. For each set $C_j \in \mathcal{C}$, create a set agent c_j. The friendship (resp. enemy) graph contains the following arcs, where $i+1$ is taken as $(i \bmod 3\hat{n}) + 1$: $E(G^{\mathrm{g}}) = \{(u_i^z, u_i) \mid (i, z) \in [3\hat{n}] \times [2]\} \cup \{(a_i, a_{i+1}) \mid i \in [3\hat{n}]\} \cup \{(a_i, c_j) \mid i \in [3\hat{n}]$ and $j \in [\hat{m}]$ with $i \in C_j\} \bigcup \{(a_i, u_i^1), (a_i, u_i^2), (a_i, u_{i+1}^1), (a_i, u_{i+1}^2)\}$. $E(G^{\mathrm{b}}) = \{(u_i, u_{i+1}) \mid i \in [3\hat{n} - 1]\} \cup \{(u_1, u_{3\hat{n}})\} \bigcup \{(a_i, u_{i+1}) \mid i \in [3\hat{n}]\} \cup \{(c_j, c_t) \mid j, t \in [\hat{m}] \colon j < t \wedge C_j \cap C_t \neq \emptyset\} \cup \{(u_i, c_j), (u_{i+1}, c_j), (u_{i+2}, c_j) \mid (i, j) \in [3\hat{n}] \times [\hat{m}] \colon i \in C_j\}$. The construction is illustrated in Figure 2, and satisfies:

Claim 14.1 (★). *The constructed instance satisfies that* $\Delta = 18$, *the friendship and enemy graphs are respectively acyclic, and* $\mathrm{f} = 1$.

It remains to show that I admits an exact cover if and only if the constructed instance has an individually stable partition. For the "if" part, let Π be an individually stable partition. We first observe:

Claim 14.2 (★). *(i) For each* $i \in [3\hat{n}]$, *agents* u_i, u_i^1, u_i^2 *must be together in* Π.

(ii) For each $i \in [3\hat{n}]$, *agent* a_i *must have at least two friends in* Π *and at most one of them can be a set agent* c_j.

(iii) There is no $i \in [3\hat{n}]$ *such that* a_i *and* U_i *are together in* Π.

(iv) There is no $i \in [3\hat{n}]$ *s.t.* a_i *and* $U_{(i \bmod 3\hat{n})+1}$ *are together* Π.

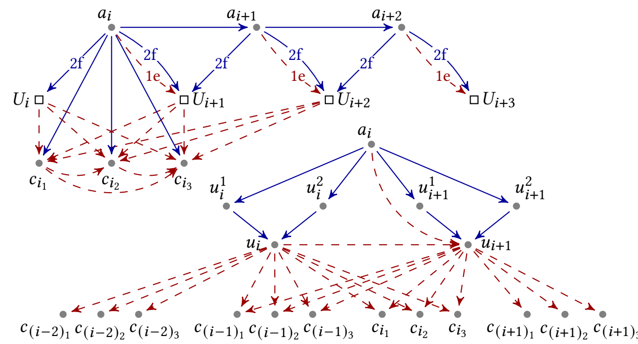


Figure 2: Illustration for the proof of Theorem 14. Top: An overview of the reduction, where solid blue (resp. dashed red) arcs indicate friends (resp. enemies). For each $z \in [3\hat{n}]$, let $C_{z_1}, C_{z_2}, C_{z_3}$ denote the sets containing it such that $z_1 < z_2 < z_3$. Bottom: Description of the arcs related to $U_i = \{u_i, u_i^1, u_i^2\}$ and $U_{i+1} = \{u_{i+1}, u_{i+1}^1, u_{i+1}^2\}$.

By Claim 14.2(iii)–(iv), for each $i \in [3\hat{n}]$, agent a_i does not have any friends from U_i or U_{i+1}. By Claim 14.2(ii) a_i must have at least two friends and at most one of them can be a set agent. Therefore, we obtain that in Π, each agent a_i, $i \in [3\hat{n}]$ must have exactly one set agent friend and a_{i+1} as friends. This implies that $a_1, \ldots, a_{3n}$ are all together and each of them has exactly one set agent friend in the coalition. Hence, the sets corresponding to those set agents must form an exact cover.

For the "only if" part, let $\mathcal{K}$ be an exact cover. Then, define a coalition $P = \{c_j \mid C_j \in \mathcal{K}\} \cup \{a_i \mid i \in [3\hat{n}]\}$. We claim that $\Pi = \bigcup_{i \in [3\hat{n}]} \{U_i\} \cup \{P\} \cup \bigcup_{C_j \in \mathcal{C} \setminus \mathcal{K}} \{\{c_j\}\}$, consisting of $3\hat{n} + \hat{m} + 1$ coalitions, is Nash stable, and hence individually stable. Since c_j has no friends in G for all $j \in [\hat{m}]$, and they have no enemies in Π, they do not envy any coalition. Similarly, u_i has no friends in G for any $i \in [3\hat{n}]$ and she has no enemies in Π. For all $i \in [3\hat{n}]$, $z \in [2]$, agent u_i^z has all of her friends and no enemies in Π, so she does not envy any coalition. Finally, for each $i \in [3\hat{n}]$, a_i has two friends and no enemies in Π, and among the other six friends of a_i, two of them are alone (the two other set agents c_j with $i \in C_j$), two of them are in U_i, and two of them are in U_{i+1}. Therefore, a_i envies none of the coalitions. □

5 CONCLUSION AND FUTURE WORK

We resolved many complexity questions from the literature under the FE$^{\mathrm{S}}$ as well as the FEN$^{\mathrm{S}}$ model, and significantly extended previous work for these two models. As an immediate open question, we do not know the complexity of FE$^{\mathrm{S}}$-CoreV (resp. FE$^{\mathrm{S}}$-StrictCoreV) $\Delta = 6$ for the symmetric case. For the case of $\Delta = 3$ with not necessarily symmetric preferences, we can show that FE$^{\mathrm{S}}$-CoreV remains coNP-hard, even if f = 1, between the submission and the publication of the paper. We conjecture that Last but not least, it would be interesting to know how the refined complexity for the variant with enemy aversion [8] behaves.

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