Measuring a Priori Voting Power - Taking Delegations Seriously

Extended Abstract

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ABSTRACT

In this paper, we introduce new power indices to measure the criticality of voters involved in different elections where delegations play a key role, namely, two variants of the proxy voting setting and a liquid democracy setting. We argue that our power indices are natural extensions of the Penrose-Banzhaf index in classic simple voting games; we show that recursive formulas can compute these indices for weighted voting games in pseudo-polynomial time; and we provide numerical results to illustrate how introducing delegation options modifies the voting power of voters.

KEYWORDS

Voting Power, Interactive Democracy, Voting Games

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1 INTRODUCTION

Voting games have been extensively used to study the a priori voting power of voters participating in an election [10], i.e., the power granted solely by the rules governing the election process. Notably, these measures do not consider the nature of the bill nor the affinities between voters. The class of I-power measures (e.g., the Penrose-Banzhaf measure [3, 17]) notably quantify how likely a voter will be influential in the decision's outcome. In simple voting games, an assembly of voters must make a collective decision on a proposal, and each voter may either support or oppose the proposal. The Penrose-Banzhaf measure can be presented as follows: voters are assumed to vote independently from one another; a voter is as likely to vote in favour or against the proposal. It then measures the probability that a voter can alter the election's outcome given this probabilistic model on the other voters.

Simple voting games have been extended in several directions to take into account more realistic frameworks that are more diverse and complex. For example, taking into account abstention [11], several levels of approval [12], or coalition structures [16]. Hence, new power indices have been designed to understand voters' criticality in these frameworks better. However, election frameworks with delegations have been largely unexplored so far with respect to a priori voting power. Yet, frameworks such as proxy-voting [15, 19] or liquid democracy [4, 5] have received increasing interest from the AI community due to their ability to provide a more flexible and engaging voting process. While proxy voting allows agents to choose a proxy from a list of representatives who will vote on their behalf [1, 2, 7, 13, 15, 19], liquid democracy further allows these delegations to be transitively delegated [6, 9, 14, 20]. Hence, studying these new frameworks thoroughly via their distribution of a priori voting power is an interesting research direction. [1]

2 MODELS

Let V be a set of n voters taking part in a binary election to decide if some proposal should be accepted or not. Each voter may vote directly (either for (1) or against (-1) the proposal) or delegate their vote to another voter.

Definition 1. *A* delegation partition *of a set V is a map D from V to the possible ballots $\{-1, 1\} \cup V$ such that for all $v \in V$, $D(v) \neq v$. We denote by D^-, D^+, and D^v the inverse images of $\{-1\}$, $\{1\}$ and $\{v\}$ for each $v \in V$ under D.*

In contrast, a direct-vote partition divides the voters such that each partition cell corresponds to a possible voting option. We allow for abstentions, which correspond to situations where a delegator cannot find a voting delegatee to represent them.

Definition 2. *A* direct-vote partition *of a set V is a map T from V to the votes $\{-1, 0, 1\}$. We let T^-, T^0, and T^+ denote the inverse images of $\{-1\}$, $\{0\}$ and $\{1\}$ under T, respectively.*

A delegation partition D naturally induces a direct-vote partition (denoted T_D) by resolving the delegations. First, we let voters in D^-, and D^+ also be in T^-, and T^+, respectively. Thereafter, for some $\circ \in \{-, +\}$, if $v' \in D^v$ and $v \in T^\circ$, then $v' \in T^\circ$. This continues until no more voters can be added to T^+ or T^-. The remaining unassigned agents in T abstain and thus are in T^0. With this procedure, agents receive their delegate's vote unless their delegation leads to a cycle.

Next we define a partial ordering $\leq$ among direct-vote partitions: if T_1 and T_2 are two direct-vote partitions of V, we let: $T_1 \leq T_2 \Leftrightarrow T_1(x) \leq T_2(x), \forall x \in V$.

Definition 3. *A ternary (resp. binary) voting rule is a map W from the set $\{-1, 0, 1\}^n$ (resp. $\{-1, 1\}^n$) of all direct-vote partitions (resp. all direct-vote partitions without abstention) of V to $\{-1, 1\}$ satisfying the following conditions:*

(1) $W(\mathbb{1}) = 1$ *and* $W(-\mathbb{1}) = -1$ *where* $\mathbb{1} = (\underbrace{1, \ldots, 1}_{\times n})$*;*

(2) *Monotonicity:* $T_1 \leq T_2 \Rightarrow W(T_1) \leq W(T_2)$.

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[1] The complete version of this work (including full proofs) can be found in [8].

Weighted voting games (WVGs). WVGs make it possible to express ternary voting rules compactly. In a WVG, there is a quota $q \in (0.5, 1]$ and a map $w : V \longrightarrow \mathbb{N}_{>0}$ assigning each voter a positive weight. Given a set $S \subseteq V$, we set $w(S) = \sum_{i \in S} w(i)$. In a WVG, we have that $W(T(V)) = 1$ iff $w(T^+) > q \times w(T^+ \cup T^-)$.

Voting games with delegations. We let V be divided into a set of n_v delegatees V_v, and a set of $n_d = n - n_v$ delegators $V_d = V \setminus V_v$. In the proxy voting setting, V_v is given in the input and has been previously determined, e.g., by an election or sortition. Settings PV_α and PV_β differ in the options available to agents in V_d. While in PV_α, each delegator can only delegate to one of the delegatees, in PV_β, they can also vote directly. In both settings, voters in V_v may only vote directly. The third setting is Liquid Democracy (LD), in which delegations are transitive, and delegators can delegate to delegatees through other delegators. The set of delegatees is not a priori fixed in this setting. While in the LD setting, all delegation-partitions may arise (referring to them as *acceptable*), the PV_α and PV_β settings can only lead to specific delegation-partitions called *PV_α-partitions* and *PV_β-partitions.*

Definition 4. *Given a non-empty set $V_v \subseteq V$, a* PV_α-partition *(resp. PV_β-partition) P is a delegation partition D for which $D(v) \in \{-1, 1\}$ if $v \in V_v$, and $D(v) \in V_v$ (resp. $D(v) \in V_v \cup \{-1, 1\}$) otherwise.*

Let $\mathcal{D}^\alpha$, (resp. $\mathcal{D}^\beta$, $\mathcal{D}^{ld}$) denote the set of PV_α-partitions (resp. PV_β-partitions, delegation partitions). The PV_α, PV_β, and LD settings induce three models where delegations play a significant role and are increasingly permissive. We measure *a priori* voting power in these settings. As with the intuition behind the standard Penrose-Banzhaf index, we also invoke the principle of insufficient reason.

That is, without information about the voters or the nature of the proposal, we assume they are equally likely to vote in favour or against the proposal. Moreover, in ignorance of any concurrence or opposition of interests between voters, we assume that all choices of voters' for their delegation are equally likely and that the voters' behaviours are independent. These assumptions lead to the following probabilistic model: in the LD (resp. PV_β) setting, each voter (resp. voter in V_d) may vote with probability p_v or delegate to another voter with probability $p_d = 1 - p_v$. In PV_α, this is predetermined by V_v. In all three models, if a voter votes, they are equally likely to vote in favour of or against the proposal. In the proxy settings (resp. LD setting), if a voter delegates, they may delegate to any member of V_v (resp. any other voter) with probability $1/n_v$ (resp. $1/(n-1)$).

3 OUR RESULTS

We want to measure how critical a voter is in determining the outcome. Given our probabilistic models on acceptable delegation partitions, we consider the probability that a voter can change the outcome decided by a voting rule.

Definition 5. *Given a set V of voters, a voting rule W (and a set $V_v \subseteq V$ of delegatees in the PV settings), the* PV_α, PV_β, *and* LD *Penrose-Banzhaf measures, respectively $\mathcal{M}_i^\alpha(W)$, $\mathcal{M}_i^\beta(W)$, $\mathcal{M}_i^{ld}(W)$ of voter $i \in V$ are defined as:*

$$\mathcal{M}_i^\gamma(W) = \sum_{D \in \mathcal{D}^\gamma} \mathbb{P}(D) \frac{W(T_{D_i^+}) - W(T_{D_i^-})}{2}.$$

with $\gamma \in \{\alpha, \beta, ld\}$, where $\mathbb{P}(D)$ is the probability of the delegation partition D occurring, and D_i^+ (resp. D_i^-) is identical to D with the only possible difference being that i supports (resp. opposes) the proposal.

We make some remarks. 1) The definition of the measure is different for voters in V_d in the PV_α setting as their ability to be critical depends on if there are two proxies with different votes. A full discussion on this point has been deferred to [8]. 2) Our power measures correspond to the probability that the voter is critical: $\mathbb{P}(i \text{ is critical}) = \mathcal{M}_i^\gamma(W)$ for $\gamma \in \{\alpha, \beta, ld\}$. 3) Observe that our power measures extend the standard Penrose-Banzhaf measure (consider $V_v = V$ or $p_d = 0$) and that they are not normalized (i.e., summing over the agents does not yield 1). The corresponding voting power indices can be found by normalizing over voters.

On the computational aspect, definition 5 requires summing on all acceptable delegation partitions. In [8], we provide more compact formulas by grouping voters making similar choices. Despite this, the exact computation of these measures is #P-hard due to the fact that they extend the standard Penrose-Banzhaf measure [18]. More positively, we show that in WVGs, they can be computed in pseudo-polynomial time, similarly to the Penrose-Banzhaf measure. This is proven using a dynamic programming approach.

Theorem 1. *Given a WVG with weight function w and quota-ratio q, a set of voters $V_v \subseteq V$ (for the PV settings), and a voter i, measures $\mathcal{M}_i^\alpha$, $\mathcal{M}_i^\beta$, and $\mathcal{M}_i^{ld}$ can be computed in pseudo-polynomial time.*

4 SOME COMMENTS ON NUMERICAL TESTS

We performed numerical tests on our power measures to test their impact and the relationships between their parameters. We observed several noticeable trends. First, there is a *flattening effect* on the power measures as p_d increases. By this, we mean that the difference between the lowest and highest measure of power in the WVG (for any agent) becomes smaller. This flattening, in our LD setting, is due to all voters having the same available voting actions, no matter their weights. Notably, there cannot be dummy agents when $p_d > 0$, as for any agent, the delegation partition where all other voters delegate to them has a positive probability. Secondly, in the LD setting, we see that when the probability of delegating increases, so does the probability of being critical. When p_d increases, the number of direct voters decreases, and the expected total weight of a voting agent increases. Hence, they are more likely to be critical when they vote directly. In the PV settings, as expected, only the criticality of voters in V_v increases with p_d, while it decreases for voters in V_d. Third, in the PV settings, we observed that the criticality of voters in V_v (resp. V_d) decreases (resp. increases) with $|V_v|$. Indeed, when there are few delegatees, these voters are likely to receive more delegations. On the other hand, when the number of delegatees increases, delegators have more options.

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