The Parameterized Complexity of Welfare Guarantees in Schelling Segregation

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ABSTRACT

Schelling's model considers k types of agents each of whom needs to select a vertex on an undirected graph, where every agent prefers neighbor agents of the same type. We are motivated by a recent line of work that studies solutions that are optimal with respect to notions related to the welfare of the agents. We explore the parameterized complexity of computing such solutions. We focus on the well-studied notions of social welfare and Pareto optimality, alongside the recently proposed notions of group-welfare optimality and utility-vector optimality.

KEYWORDS

Schelling; Parameterized Complexity; Welfare Maximization

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1 INTRODUCTION

Residential segregation is a phenomenon that is observed in many residential areas around the globe. As a result of de-facto segregation, people group together forming communities based on traits such as race and ethnicity, and residential areas become noticeably divided into segregated neighborhoods. Half a century ago, Schelling [18] proposed a simple agent-based model to study how residential segregation emerges from individuals' perceptions.

At a high level, Schelling's model works as follows. There are two types of agents, say red and blue, each of whom is placed on a unique node on a graph. Agents are aware of their neighborhood; agents of the same type are considered "friends" and those of the opposite type "enemies". An agent is happy with their location if and only if the fraction of friends in their neighborhood is at least τ, where $\tau \in [0, 1]$ is a tolerance parameter. Schelling proposed a random process that starts from a random initial assignment and agents who are unhappy in their current neighborhood relocate to a different, random, empty node, whilst happy agents stay put. It is expected that when agents are not tolerant towards a diverse neighborhood, $\tau > \frac{1}{2}$, these dynamics will converge to a segregated assignment. However, Schelling's experimentation on grid graphs showed that even when agents are in favour of integration, i.e. $\tau \approx \frac{1}{3}$, the final assignment will be segregated.

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Since Schelling's model was proposed, his work has been the subject of many empirical studies in sociology [10], in economics [19, 20], and more recently in computer science. For example Barmpalias et al. [2, 3, 4, 5] and Immorlica et al. [14] analyze Schelling's model on a grid graph with its original random dynamics. A different line of work studies Schelling *games*, such as [1, 6, 15, 17] and there are many variations of this [7, 9, 13, 16].

More recently, Bullinger et al. [8] studied assignments with certain *welfare guarantees* for the agents and the computational complexity of computing them. In Schelling's model high social welfare translates to high segregation. However, there are certain scenarios where segregation essentially captures the effectiveness of an allocation of agents over a network. As an example, think of the nodes of the graph as the resources of an organization, the edges as compatibility and interference between the resources, and the types of agents as different working groups, or skilled workers. Under this point of view, "segregation" is desirable, since we have better utilization of both the available workers and resources. For this reason, the welfare guarantees studied by [8] are the focus of this paper, albeit under the prism of parameterized complexity.

In parameterized algorithmics [11], the running-time of an algorithm is studied with respect to a parameter $k \in \mathbb{N}_0$ and input size n. The most favorable outcome is an FPT (*fixed-parameter tractable*) algorithm, running in time $f(k) \cdot n^{O(1)}$, where f is a computable function. A less favorable, but still positive, outcome is an XP algorithm, which is an algorithm running in time $O(n^{f(k)})$. Finally, showing that a problem is W[1]-hard rules out the existence of an FPT algorithm under the well-established assumption that W[1] $\neq$ FPT.

2 MODEL

Given two vectors $\mathbf{x}, \mathbf{y}$ of length n, we say that $\mathbf{x}$ *weakly dominates* $\mathbf{y}$ if $\mathbf{x}(i) \geq \mathbf{y}(i)$ for every $i \in [n]$; $\mathbf{x}$ *strictly dominates* $\mathbf{y}$ if at least one of the inequalities is strict.

A *Schelling instance* $\langle G, A\rangle$, consists of an undirected graph $G = (V, E)$ and a set of *agents* A, where $|A| \leq |V|$. Every agent has a *type*, or *color*. When there are only two colors available, we assume that $A = R \cup B$, where R contains red agents and B contains blue agents. We denote $r = |R|$ and $b = |B|$. Agents i and j are *friends* if they have the same color; otherwise, they are *enemies*. For any agent $i \in A$ we use $F(i)$ to declare the set of his friends.

An *assignment* $\mathbf{v} = (v(1), \dots, v(|A|))$ for the Schelling instance $\langle G, A\rangle$ maps every agent in A to a vertex $v \in V$, such that every vertex is occupied by at most one agent. Here, $v(i) \in V$ is the vertex of G that agent i occupies. For any assignment $\mathbf{v}$ and any agent $i \in A$, $N_i(\mathbf{v})$ denotes the set of neighbors of $v(i) \in V$ that are occupied under $\mathbf{v}$. Let $f_i(\mathbf{v}) = |N_i(\mathbf{v}) \cap F(i)|$ and let $e_i(\mathbf{v}) = |N_i(\mathbf{v})| - f_i(\mathbf{v})$ be respectively the numbers of neighbors of agent i who are his friends and his enemies under $\mathbf{v}$. The *utility* of agent i under assignment $\mathbf{v}$, denoted $u_i(\mathbf{v})$, is 0 if $|N_i(\mathbf{v})| = 0$, and is defined as

$$u_i(\mathbf{v}) = \frac{f_i(\mathbf{v})}{|N_i(\mathbf{v})|} = \frac{f_i(\mathbf{v})}{f_i(\mathbf{v}) + e_i(\mathbf{v})}$$

if $|N_i(\mathbf{v})| \neq 0$. The *social welfare* of $\mathbf{v}$ is the sum of the utilities of all agents, formally $\mathrm{SW}(\mathbf{v}) = \sum_{i \in A} u_i(\mathbf{v})$. For $X \in \{R, B\}$ we denote $\mathrm{SW}_X(\mathbf{v}) = \sum_{i \in X} u_i(\mathbf{v})$.

We use $\mathbf{u}(\mathbf{v})$ to denote the vector of length $|A|$ that contains the utilities of the agents under $\mathbf{v}$, sorted in non-increasing order. Similarly, let $\mathbf{u}_X(\mathbf{v})$ denote the corresponding vector of utilities of the agents in $X \in \{R, B\}$. An assignment $\mathbf{v}$ is *utility-vector dominated* by $\mathbf{v}'$ if $\mathbf{u}(\mathbf{v}')$ strictly dominates $\mathbf{u}(\mathbf{v})$; $\mathbf{v}$ is *group-welfare dominated* by $\mathbf{v}'$ if $\mathrm{SW}_X(\mathbf{v}') \geq \mathrm{SW}_X(\mathbf{v})$, where $X \in \{R, B\}$, and at least one of the inequalities is strict. An assignment $\mathbf{v}$ is:

- *welfare optimal*, denoted WO, if for every other assignment $\mathbf{v}'$ we have $\mathrm{SW}(\mathbf{v}) \geq \mathrm{SW}(\mathbf{v}')$;
- *Pareto optimal*, denoted PO, if and only if there is no $\mathbf{v}'$ such that $\mathbf{u}_X(\mathbf{v}')$ weakly dominates $\mathbf{u}_X(\mathbf{v})$ for $X \in \{R, B\}$ and at least one of the dominations is strict;
- *utility-vector optimal*, denoted UVO, if it is not utility-vector dominated by any other assignment;
- *group-welfare optimal*, denoted GWO, if it is not group-welfare dominated by any other assignment;
- *perfect*, denoted Perfect, if every agent gets utility 1.

The notions UVO and GWO were introduced by Bullinger et al. [8] where the following were proven.

Proposition 1. *If an assignment $\mathbf{v}$ is WO, then it is UVO, GWO, and PO. If $\mathbf{v}$ is UVO or GWO, then it is PO.*

Observation 1. *If Schelling instance $\langle G, A\rangle$ admits a Perfect assignment, then every PO assignment is Perfect.*

In this paper we study the complexity of ϕ-Schelling, where $\phi \in \{\mathrm{WO}, \mathrm{PO}, \mathrm{GWO}, \mathrm{UVO}, \mathrm{Perfect}\}$. In other words, given a Schelling instance $\langle G, A\rangle$, we study the problem of finding an assignment $\mathbf{v}$ satisfying the given optimality notion.

3 OUR RESULTS

A full version containing all proofs can be found in [12].

3.1 Two Types

Theorem 2. *Assuming* P $\neq$ NP, *there is no poly-time algorithm for ϕ-Schelling, for $\phi \in \{$WO, PO, UVO, GWO$\}$, even when $b = 1$.*

On the positive side, we can easily get an XP algorithm for Perfect-Schelling parameterized by b.

Theorem 3. *For Perfect-Schelling there is an* XP*-algorithm parameterized by b.*

Theorem 4. *Deciding whether a Schelling instance admits a perfect assignment is* W[1]*-hard when parameterized by $r + b$.*

Since we show that the problem is hard, the XP-algorithm from Theorem 3 is actually the best we can hope for. The combination of Theorem 4, Proposition 1, and Observation 1, gives us the following corollary.

Corollary 5. *There is no* FPT *algorithm for ϕ-Schelling when parameterized by $r+b$, for $\phi \in \{$Perfect, WO, PO, UVO, GWO$\}$, unless* FPT = W[1].

In light of these negative results, we turn our attention to instances where the structure of G is restricted.

Theorem 6. *Assuming* P $\neq$ NP, *there is no poly-time algorithm for WO-Schelling and GWO-Schelling on cubic graphs, even if $r + b = |V|$.*

Theorems 4 and 6 show that we cannot hope for an efficient algorithm, at least for WO and GWO, just by parameterizing only by $r + b$ or only by the maximum degree Δ. We complement this with the following result.

Theorem 7. *There is an* FPT *parameterized by $r + b + \Delta$ for ϕ-Schelling, for every $\phi \in \{$WO, PO, UVO, GWO$\}$. Moreover, ϕ-Schelling admits a kernel with at most $O(\Delta^2 \cdot r^2 \cdot b^2)$ many vertices.*

3.2 Multiple types

Now, we depart from the standard model and study Schelling instances with multiple types, denoted SchellingM.

Theorem 8. *Let $\mathcal{G}$ be an arbitrary class of connected graphs that contains at least one graph of size s for every $s \in \mathbb{N}$. Deciding whether a Schelling instance with multiple types admits a perfect assignment is* NP*-hard and* W[1]*-hard when parameterized by agent-types, even when every connected component of G is in $\mathcal{G}$.*

Corollary 9. *Deciding whether a Schelling instance with multiple types admits a perfect assignment is* NP*-hard and* W[1]*-hard when parameterized by agent-types, even if G is a tree.*

Again, using Proposition 1 and Observation 1, we can get the following corollary.

Corollary 10. *For every $\phi \in \{$WO, PO, UVO, GWO$\}$, assuming* P $\neq$ NP, *there is no polynomial time algorithm for ϕ-SchellingM even when G is a tree. Moreover, assuming* FPT $\neq$ W[1], *there is no* FPT *algorithm for ϕ-SchellingM parameterized by agent-types, even when G is a tree.*

The following result provides an algorithm for ϕ-SchellingM that matches the lower bound from Corollary 10.

Theorem 11. *There is an $|A|^{O(k \cdot \mathrm{tw}(G))} \cdot |V(G)|$ time algorithm for ϕ-SchellingM, $\phi \in \{$WO, PO, GWO, UVO$\}$, where k is the number of agent-types.*

It then follows from the running time of the algorithm that SchellingM is actually FPT when parameterized by treewidth plus the number of agents.

Corollary 12. *There is an* FPT *algorithm for ϕ-SchellingM when parameterized by treewidth and the number of agents, for every $\phi \in \{$WO, PO, GWO, UVO$\}$.*

Finally, while Corollary 10 implies that we cannot obtain an FPT algorithm if the number of agent-types is part of the parameter, we can point out a specific case that can be solved in FPT time with a very minor modification of our algorithm.

Corollary 13. *When the number of types is constant, Perfect-SchellingM admits an* FPT *algorithm parameterized by treewidth.*

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