Fair Chore Division under Binary Supermodular Costs

Extended Abstract

Siddharth Barman
Indian Institute of Science
Bengaluru, India
barman@iisc.ac.in

Vishnu V. Narayan
Tel Aviv University
Tel Aviv-Yafo, Israel
narayanv@tauex.tau.ac.il

Paritosh Verma
Purdue University
West Lafayette, United States
paritoshverma97@gmail.com

ABSTRACT

We study the problem of dividing indivisible chores among agents whose costs (for the chores) are supermodular set functions with binary marginals. Such functions capture complementarity among chores, i.e., they constitute an expressive class wherein the marginal disutility of each chore is either one or zero, and the marginals increase with respect to supersets. In this setting, we study the broad landscape of finding fair and efficient chore allocations. In particular, we establish the existence of (i) EF1 and Pareto efficient chore allocations, (ii) MMS-fair and Pareto efficient allocations, and (iii) Lorenz dominating chore allocations. Furthermore, we develop polynomial-time algorithms—in the value oracle model—for computing the chore allocations for each of these fairness and efficiency criteria. Complementing these existential and algorithmic results, we show that in this chore division setting, the aforementioned fairness notions, namely EF1, MMS, and Lorenz domination are incomparable: an allocation that satisfies any one of these notions does not necessarily satisfy the others.

Additionally, we study EFX chore division. In contrast to the above-mentioned positive results, we show that, for binary supermodular costs, Pareto efficient allocations that are even approximately EFX do not exist, for any arbitrarily small approximation constant.

KEYWORDS

Fair Division; Chores; Envy-freeness up to one good; Pareto optimal

ACM Reference Format:
Siddharth Barman, Vishnu V. Narayan, and Paritosh Verma. 2023. Fair Chore Division under Binary Supermodular Costs: Extended Abstract. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), London, United Kingdom, May 29 – June 2, 2023*, IFAAMAS, 3 pages.

1 INTRODUCTION

The question of dividing indivisible items among a set of agents in a fair manner is a pervasive problem in many domains. Popular notions of fairness in the field of discrete fair division include envy-freeness up to one good (EF1), envy-freeness up to any good (EFX), and the maximin share guarantee (MMS). Both existential and algorithmic guarantees for these and related fairness notions have been extensively studied in recent years; see e.g., [3, 16].

In this research direction, a majority of results focus on the fair division of *goods*, which correspond to items that, when allocated, induce non-negative values among the agents. Notably, the complementary settings of fair division of *chores* (which model negatively valued items or tasks) are relatively under-explored. While the definitions of familiar fairness criteria (such as EF1 and MMS) extend quite directly, the conditions under which a fair chore division exists do not mirror the goods' case.

In fact, important known results for the goods setting do not directly extend to the chores setup. For example, the influential work of Caragiannis et al. [13] studies the fair division of goods when the agents have additive valuations and establishes that, in this context, there always exists an allocation that is both EF1 and Pareto efficient. In particular, they show that an allocation of goods that maximizes Nash welfare among the agents achieves these fairness and efficiency goals. By contrast, in the context of chores, the existence of allocations that are simultaneously EF1 and Pareto efficient has remained a challenging open problem. Such allocations have only recently been shown to exist for a specific subclass of valuation functions, namely *bivalued* additive valuations [15, 19]. Consequently, it is clear that the fair division of chores presents a new set of technical challenges, and the study of chore division is an important thread of research in discrete fair division.

We contribute to the recent literature on chore division by focusing on settings wherein the agents' costs (disutilities) for the chores have *binary marginals*. Specifically, an agent i's cost function, c_i, is said to have binary marginals (equivalently, is said to be dichotomous) if the marginal value of the chore t relative to any subset S is either zero or one, i.e., $c_i(S \cup \{t\}) - c_i(S) \in \{0, 1\}$. In the complementary context of goods, an abundance of papers consider agents with dichotomous valuations (e.g. [10, 22]), since such valuations model agent preferences in several real-world settings, such as kidney exchanges [26] and housing allocations [8].

The majority of our results focus on the case where the agents have *supermodular* cost functions. Supermodular functions have received considerable attention in the economics literature; notably, the use of supermodularity to express complementarity in agents' preferences dates back to the works of Edgeworth, Pareto, and Fisher [27].[1] Specifically, an agent i's cost function c_i (i.e., disutilities for the chores) is supermodular if it has increasing marginals: $c_i(T \cup \{a\}) - c_i(T) \geq c_i(S \cup \{a\}) - c_i(S)$, for all subsets $S \subseteq T$ and all chores $a \notin T$. Increasing marginals are a well-suited assumption for chores, since taking on a new task is increasingly likely to raise one's cost due to the burdens of multitasking and frequent task-switching. Binary supermodular functions model, for instance, the costs associated with page caching, or contexts in which only the first few trials of some software (or delivery service) are free.

Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), A. Ricci, W. Yeoh, N. Agmon, B. An (eds.), May 29 – June 2, 2023, London, United Kingdom. © 2023 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org). All rights reserved.

[1]While we focus on chore division under supermodular costs, prior works have complementarily addressed fair division of goods with supermodular valuations; see, e.g., [8, 13].

2 OUR RESULTS

We study the problem of finding fair and efficient allocations of indivisible chores among agents with binary supermodular cost functions. Our work develops several results on the existence and computability of EF1, MMS, Pareto efficient (PO), and Lorenz dominating allocations. Specifically, we show that, for these cost functions, (i) an allocation that is EF1 and PO, (ii) an allocation that is PO and in which every agent receives its minimax share (MMS), and (iii) a Lorenz dominating allocation, always exist and can be computed in polynomial time (given value-oracle access to the cost functions). These results constitute some of the first positive chore-division guarantees for standard fairness and economic-efficiency notions in discrete fair division.

In the current context of chore division with binary supermodular costs, we also show that certain pairs of these guarantees, such as Lorenz domination and EF1 (or MMS along with EF1), are *incomparable*, i.e., an allocation that satisfies one of these fairness criterion does not necessarily bear the other. This is in contrast to some well-known results for goods. For instance, in the complementary case of goods division with binary submodular valuations, *any* Lorenz dominating allocation is EF1 [4]. It is interesting to note that while some positive results carry forward from the goods setting to the case of chores, others are negated.

We identify binary supermodular costs as a relevant function class for which EF1 and PO allocations of chores are guaranteed to exist. A natural follow-up question is whether this guarantee can be strengthened: do EFX and PO allocations always exist in the current context? We answer this question in the negative by showing a significantly stronger negative result: for binary supermodular costs, PO and β-EFXk allocations do not exist for any $\beta \in (0, 1]$ and for any $k \geq 1$, even when the cost functions are identical.

Complementing this negative result and focusing on fairness alone, we present positive results towards the existence of EFX for chores. We show algorithmically that when the agents have *identical* cost functions, an EFX allocation always exists. Notably, this result only requires the (identical) cost function to be monotonic. Our algorithm, that we call Add-and-Fix, provides an alternate proof of the existence of EFX chore allocations for the identical valuations case.[2] In fact, for any monotonic cost function $c(\cdot)$ that is integer-valued (i.e., $c(S) \in \mathbb{Z}_{\geq 0}$ for all subsets S), Add-and-Fix runs in pseudo-polynomial time. Consequently, for identical cost functions with binary marginals, we obtain a polynomial-time algorithm for finding EFX chore allocations.

Additional Related Work. Over the past decades, the fair division problem has emerged as a central and influential topic at the interface of mathematical economics and computer science. A collection of early fair-division results study the problem of achieving *envy-freeness* [18], where no agent prefers the bundle of another. Most of these classic works focus on the divisible setting, where a heterogeneous item (a *cake*) can be fractionally divided to create an allocation; see e.g. [11, 28].

Indivisible items. Over the preceding few years, the research focus has evolved towards studying the fair division of *indivisible goods*, where each good has to be allocated integrally to one agent. Since envy-freeness cannot be necessarily achieved in such combinatorial settings, the goal here is to obtain any of a variety of its relaxations and other approximate fairness criteria, including envy-freeness up to one item (EF1) [12, 24], envy-freeness up to any item (EFX) [13, 25], and the maximin share (MMS) guarantee [12, 23].

Pareto efficiency. While most of the results mentioned above consider the problem of achieving fairness alone, a central desideratum is to seek fairness with economic efficiency. A standard notion of efficiency in mathematical economics is that of Pareto efficiency or Pareto optimality (PO), in which no agent can be made strictly better off without making at least one other agent worse off in the process. The work of Caragiannis et al. [13] shows that when the agents have additive valuations over the goods, an allocation that maximizes Nash welfare is simultaneously EF1 and PO. However, computing a Nash welfare maximizing allocation is NP-hard. Barman et al. [5] bypass this hardness by showing that an EF1 and PO allocation (of goods) can be directly computed in pseudo-polynomial time for agents with additive valuations.

Binary marginals. Binary marginals have received substantial attention in the fair division literature for goods (see e.g. [4, 9, 22]). Darmann and Schauer [14] develop an efficient algorithm to find a Nash Welfare maximizing allocation for binary additive valuations, while Barman and Verma [7] present an approximation algorithm for the same problem for the more general class of binary XOS valuations. Truthful mechanisms for fair division of goods under binary additive valuations [20] and binary submodular valuations [4] have also been developed in prior works. Babaioff et al. [4] and Benabbou et al. [8] show the existence and polynomial-time computability of allocations that are EFX and PO (and, hence, EF1 and PO) for the class of binary submodular (or, equivalently, matroid-rank) valuations. For this valuation class, Babaioff et al. [4] establish existence, efficient computation, and fairness implications of Lorenz dominating allocations.

Chore division. While the definitions of envy-freeness, MMS and Pareto optimality extend directly from the goods case to the chores setting, the fairness criteria of EF1 and EFX are typically defined via the removal of a chore from the envying agent's bundle (rather than a good from the envied agent's bundle). For approximately-MMS-fair division of chores, several results are known, including an $\frac{11}{9}$-approximation factor [21] and a $\frac{44}{43}$-impossibility bound [17] for additive agents. As mentioned previously, while allocations that are EF1 and PO are known to exist in a variety of settings for goods, in the chore division case such existence results are only known for bivalued instances [15, 19]. We refer the reader to the recent survey by Amanatidis et al. [2] for a comprehensive overview of the discrete fair division literature. Details of the results mentioned above appear in the full version of the paper [6].

[2]A result of Plaut and Roughgarden [25] shows that a leximin-type solution (that they call leximin++) is EFX for identical valuations in the case of goods. It is straightforward to extend this result to show the existence of EFX for chores (see, e.g., [1]). However, we show that the leximin++ solution is NP-hard to compute even when the agents have identical costs with binary marginals. By contrast, our algorithm obtains an EFX allocation in polynomial time in this setting.

ACKNOWLEDGMENTS

Siddharth Barman gratefully acknowledges the support of a SERB Core research grant (CRG/2021/006165). Paritosh Verma was supported by the NSF CAREER award CCF-2144208. Vishnu V. Narayan was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 866132), by an Amazon Research Award, and by the NSF-BSF (grant number 2020788).

REFERENCES

[1] Martin Aleksandrov. 2018. Almost Envy Freeness and Welfare Efficiency in Fair Division with Goods or Bads. *arXiv preprint arXiv:1808.00422* (2018).

[2] Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A Voudouris, and Xiaowei Wu. 2022. Fair division of indivisible goods: A survey. *arXiv preprint arXiv:2208.08782* (2022).

[3] Haris Aziz, Bo Li, Herve Moulin, and Xiaowei Wu. 2022. Algorithmic fair allocation of indivisible items: A survey and new questions. *arXiv preprint arXiv:2202.08713* (2022).

[4] Moshe Babaioff, Tomer Ezra, and Uriel Feige. 2021. Fair and truthful mechanisms for dichotomous valuations. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 35. 5119–5126.

[5] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. 2018. Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*. 557–574.

[6] Siddharth Barman, Vishnu V. Narayan, and Paritosh Verma. 2023. Fair Chore Division under Binary Supermodular Costs. https://arxiv.org/abs/2302.11530

[7] Siddharth Barman and Paritosh Verma. 2022. Approximating Nash Social Welfare Under Binary XOS and Binary Subadditive Valuations. In *Web and Internet Economics*. 373–390.

[8] Nawal Benabbou, Mithun Chakraborty, Ayumi Igarashi, and Yair Zick. 2020. Finding fair and efficient allocations when valuations don't add up. In *International Symposium on Algorithmic Game Theory*. Springer, 32–46.

[9] Anna Bogomolnaia and Hervé Moulin. 2004. Random matching under dichotomous preferences. *Econometrica* 72, 1 (2004), 257–279.

[10] Anna Bogomolnaia, Hervé Moulin, and Richard Stong. 2005. Collective choice under dichotomous preferences. *Journal of Economic Theory* 122, 2 (2005), 165–184.

[11] Steven J Brams and Alan D Taylor. 1996. *Fair Division: From cake-cutting to dispute resolution.* Cambridge University Press.

[12] Eric Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy* 119, 6 (2011), 1061–1103.

[13] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation (TEAC)* 7, 3 (2019), 1–32.

[14] Andreas Darmann and Joachim Schauer. 2015. Maximizing Nash product social welfare in allocating indivisible goods. *European Journal of Operational Research* 247, 2 (2015), 548–559.

[15] Soroush Ebadian, Dominik Peters, and Nisarg Shah. 2022. How to Fairly Allocate Easy and Difficult Chores. In *21st International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2022, Auckland, New Zealand, May 9-13, 2022*. 372–380.

[16] Ulle Endriss. 2018. Lecture notes on fair division. *arXiv preprint arXiv:1806.04234* (2018).

[17] Uriel Feige, Ariel Sapir, and Laliv Tauber. 2021. A tight negative example for MMS fair allocations. In *International Conference on Web and Internet Economics*. Springer, 355–372.

[18] D. Foley. 1967. Resource allocation and the public sector. *Yale Econ Essays* 7, 1 (1967), 45–98.

[19] Jugal Garg, Aniket Murhekar, and John Qin. 2022. Fair and efficient allocations of chores under bivalued preferences. In *Proc. 36th Conf. Artif. Intell.(AAAI). To appear.*

[20] Daniel Halpern, Ariel D Procaccia, Alexandros Psomas, and Nisarg Shah. 2020. Fair division with binary valuations: One rule to rule them all. In *International Conference on Web and Internet Economics*. Springer, 370–383.

[21] Xin Huang and Pinyan Lu. 2021. An algorithmic framework for approximating maximin share allocation of chores. In *Proceedings of the 22nd ACM Conference on Economics and Computation*. 630–631.

[22] David Kurokawa, Ariel D Procaccia, and Nisarg Shah. 2018. Leximin allocations in the real world. *ACM Transactions on Economics and Computation (TEAC)* 6, 3-4 (2018), 1–24.

[23] David Kurokawa, Ariel D Procaccia, and Junxing Wang. 2018. Fair enough: Guaranteeing approximate maximin shares. *Journal of the ACM (JACM)* 65, 2 (2018), 1–27.

[24] Richard J Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. 2004. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM conference on Electronic commerce*. 125–131.

[25] Benjamin Plaut and Tim Roughgarden. 2020. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics* 34, 2 (2020), 1039–1068.

[26] Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. 2005. Pairwise kidney exchange. *Journal of Economic theory* 125, 2 (2005), 151–188.

[27] Paul Anthony Samuelson. 1948. Foundations of economic analysis. *Science and Society* 13, 1 (1948).

[28] W. Stromquist. 1980. How to Cut a Cake Fairly. *The American Mathematical Monthly* 87, 8 (1980), 640–644.