Maximin Share Allocations for Assignment Valuations

Extended Abstract

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ABSTRACT

In this paper, we initiate the study of fairly dividing a set of indivisible resources under the fairness notion of Maximin share (MMS), for the setting where the agents have assignment or OXS valuation functions. These are a popular subclass of functions that lie between the well-studied submodular and additive function classes.

KEYWORDS

Fair Division; OXS Valuations; Maximin share

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Typically in much of economics and game theory, when investigating resource allocation among agents, the valuation functions of agents are assumed to be *beyond-additive*, and have a decreasing marginal returns property; this is essentially a *complement-free-kind* property, where the value of a set of resources is less than the sum of the values of any subsets that cover this set. [16] described a hierarchy of five valuation function classes that have this property. OXS is the first beyond-additive class in this hierarchy, and holds particular importance within microeconomic theory, for instance [3, 6, 9, 16, 19, 22]. Furthermore, OXS functions have a rich structure, and can be syntactically defined in various ways: for instance, as depth 2 trees of ORs of XORs of additive functions [16], or using bipartite matchings [10], or matrices [20]. We use the definition via bipartite matchings, which intuitively is as follows. Every agent is associated with a weighted bipartite graph where all the goods form one part of vertices. The value of a set of goods is the value of the maximum weight matching in the graph induced by this set. It is somewhat surprising that no work, to the best of our knowledge, has explored the fundamental problem of finding *fair* resource allocations, which is a central focus in both economics and game theory, under OXS valuations. We initiate this study under the fairness notion of Maximin share (MMS) [7].

The best known algorithmic and non-existence results for the MMS problem under OXS valuations are due to the results for submodular and additive cases, which relate to the OXS class as: Additive ⊂ OXS ⊂ Submodular. A PTAS to find a 1/3-MMS allocation for submodular functions [12], and the non-existence of 39/40-MMS allocations for the case with additive valuations [11] is known. We note that, even when the number of agents is two, no better than a 1/3-MMS algorithm is known. Given the strong connection of OXS functions with bipartite matching, and the richness of the mathematical structure this allows, it is natural to ask the following questions.

Q: For an efficient computation, can the barrier of 1/3 *be broken for* OXS *functions?*

Q: Can the 39/40 *factor non-existence result be improved when valuations are beyond-additive?*

We note that a negative result for OXS would extend to its super classes as well. Similarly, consider the task of computing the MMS value of any agent. This problem is equivalent to finding a 1-MMS allocation when the agents are identical. But solving this problem, even approximately up to any constant factor better than 1/3 remains open for valuations that are beyond additive. This is in sharp contrast to the additive valuations where a PTAS is known [23], motivating the third question,

Q: Does a PTAS *exist for computing the* MMS *value for* OXS *valuations? If not, is there an algorithm for a factor better than* 1/3*?*

We analyze all the three questions in this paper, and as a result provide efficient algorithms, non-existence result, as well as hardness results, summarised below. As a corollary, we provide additional guarantees of EF1, PO, and *max social welfare* for the special case with agents with identical valuations.

Efficient algorithm. We show the existence of $\frac{1}{3}(1+\frac{2/3}{(n-2/3)})$-MMS allocation, breaking the barrier of 1/3 for the OXS valuations, and design a PTAS to compute one. As a corollary, this yields improved factors for small number of agents, *e.g.,* 1/2 with 2 agents, 3/7 with 3 agents, 2/5 with 4 agents and so on. Note that the study of α-MMS allocations for small number of agents is a well-studied problem by itself. For instance, under additive valuations, for three agents [1] showed that a 7/8-MMS allocation always exists. This factor was later improved to 8/9 in [13]. For four agents, [12] showed that a 4/5-MMS allocation always exists. [15] gave an algorithm that, given an instance with constantly many agents, and any $\epsilon > 0$, computes an $(\alpha - \epsilon)$-MMS allocation, for the highest $\alpha \in (0, 1]$ for which an α-MMS allocation exists for the instance.

To break the barrier of 1/3, we uncover important properties of OXS functions (w.r.t. the MMS problem). Importantly, we show

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that, every agent can assign one *representative value* to every good and as a result there is an ordering of the goods. Using this, we analyze the round-robin procedure to obtain a factor better than 1/3. The challenge in the analysis is to bound the loss in value when a good is matched to an agent, as $O(n)$ goods get discarded due to this matching, due to being connected to the same right side vertex as the matched good in the OXS graph. These insights, together with a simple algorithm for beyond-additive valuations, may be of independent interest to analyze other fairness notions for OXS and other special classes of submodular functions, like gross-substitutes.

Non-existence of better than 2/3-MMS**.** We show a simple example with 2 agents and 4 goods where an allocation strictly better than 2/3-MMS does not exist. This gives an improved non-existence result for valuations that subsume OXS, namely *gross-substitutes*, *Rado*, and *submodular* valuations, as the previous best known result was for the submodular functions [12], with a non-existence of 3/4-MMS allocations.

Computing MMS **value.** We show that the problem of computing MMS values of agents with OXS functions is strongly NP-hard, negating the possibility of a PTAS. To counter the impossibility result, we show an efficient algorithm to compute the MMS value of an agent within a factor of 1/2.

EF1 +PO **and** EF1+MSW **allocations with identical agents.** A key subroutine of our algorithm with identical agents resolves another popular fair and efficient notion, namely the EF1+PO allocation. Introduced by [7], EF1 allocations are those where every agent values their bundle more than any other agent's bundle upon removing some good from the other bundle. EF1 allocations can be found efficiently using envy cycle removal procedure introduced by [17]. An allocation is called PO if there is no other allocation where every agent receives a bundle of equal or higher value, and at least one agent gets a strictly better bundle. Finding an EF1+PO allocation is a widely studied problem, with little success. [4] showed a pseudo-polynomial time algorithm to obtain an EF1 + PO allocation on a goods manna. A series of works [2, 8, 18, 21, 24] studied special cases of the problem. [14] and [5] showed the existence and efficient computation for the case of matroid rank valuations. The existence of an EF1+PO allocation is open, even with identical agents, in the beyond-additive valuations setting.

We show such an allocation exists in the OXS valuations setting with identical agents. In fact, we show the existence of an allocation that is not only PO but also has the maximum social welfare (MSW), meaning the sum of values of all the bundles is maximized.

Our work opens several directions for future work. First, apart from closing the gap of existence and non-existence results for the problem under OXS functions, a natural direction is to study MMS approximations for other subclasses of submodular functions, like weighted matroid rank functions. Second, our result on identical agents could be improved to show Pareto efficiency as well. Note that we start with an MSW allocation. One needs to bound or remove the loss in welfare while achieving the 1/2-MMS guarantee. Finally, our analysis with non-identical agents, specifically the notion to obtain a preference relation of the items corresponding to an OXS function, using a fairness notion, may be applicable to analyze other fairness notions for OXS.

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