# A Logic of Only-Believing over
# Arbitrary Probability Distributions

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## ABSTRACT

When it comes to robotic agents operating in an uncertain world, a major concern in knowledge representation is to better relate high-level logical accounts of beliefs and actions to the low-level probabilistic sensorimotor data. Perhaps the most general formalism for dealing with degrees of belief in formulas, and in particular, with how that should evolve in the presence of noisy sensing and acting is the first-order logical account by Bacchus, Halpern, and Levesque. The main advantage of such a logical account is that it allows a specification of beliefs that can be partial or incomplete, in keeping with whatever information is available about the domain, making it particularly attractive for general-purpose cognitive robotics. Recently, this model was extended to handle continuous probability distributions. However, it is limited to finitely many nullary fluents and defines beliefs and integration axiomatically, the latter making semantic proofs about beliefs and meta-beliefs difficult.

In this paper, we revisit the continuous model and cast it in a modal language. We will go beyond nullary fluents and allow fluents of arbitrary arity as is usual in the standard situation calculus. This necessitates a new and general treatment of probabilities on possible worlds, where we define measures on uncountably many worlds that interpret infinitely many fluents. We then show how this leads to a fairly simple definition of knowing, degrees of belief, and only-knowing. Properties thereof will also be analyzed. In this paper, we focus on the static setting and conclude with some thoughts about extending this account to actions as the next step and what challenges might arise.

## KEYWORDS

knowledge representation; epistemic logic; only-knowing; only-believing

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## 1 INTRODUCTION

When it comes to robotic agents operating in an uncertain world, a major concern in knowledge representation is to better relate high-level logical accounts of beliefs and actions to the low-level probabilistic sensorimotor data. In these and other applications, it is often not sufficient to say that a formula $\phi$ is unknown: we may need to say which of $\phi$ or $\neg\phi$ is more likely, and by how much. Motivated by such concerns, the unification of logic and probability has received much attention in recent years [34].

Perhaps the most general formalism for dealing with degrees of belief in formulas is the first-order logical account by Bacchus, Halpern, and Levesque (BHL) [1]. The main advantage of a logical account like BHL is that it allows a specification of beliefs that can be partial or incomplete, in keeping with whatever information is available about the domain, making it particularly attractive for general-purpose high-level programming [24]. It does not require specifying a prior distribution over some random variables from which posterior distributions are then calculated, as in Kalman filters, for example, [40]. Nor does it require specifying the conditional independences among random variables and how these dependencies change as the result of actions, as in the temporal extensions to Bayesian networks [40]. In the BHL model, some logical constraints are imposed on the initial state of belief. These constraints may be compatible with one or very many initial distributions and sets of independence assumptions. All the properties of belief will then follow at a corresponding level of specificity.

The BHL account is an extension to Reiter's reworking of the situation calculus [33]. The situation calculus and its counterparts, such as dynamic epistemic logic [42] and the fluent calculus [39], have enjoyed numerous extensions for time, processes, concurrency, exogenous events, reactivity, sensing, and knowledge [33]. The BHL account is a surprisingly simple extension to the epistemic situation calculus [36]: Instead of a categorical knowledge operator that says whether a formula $\phi$ is known or not, defined in terms of accessible situations satisfying said formula $\phi$, it specifies a weight on situations. We quantify the degree to which $\phi$ is believed by summing the weights of those situations where $\phi$ is true.

The simplicity of the BHL model has led to two major classes of extensions. Owing to its limitation to discrete distributions and

the lack of a solution for the projection problem (determining what holds after actions), recent results have demonstrated how it can be extended to continuous distributions (BL henceforth) [6, 9], as well as how a notion of regression and progression [10] can be defined for both the discrete and the continuous model [7, 8].

But perhaps the more major extension is owing to the fact that it is defined axiomatically, as is usual in the situation calculus [33]. Even in the non-probabilistic case of knowledge and actions, an axiomatic definition makes semantic proofs about modalities difficult [23]. The situation is far worse with probabilities: degrees of belief in BHL are defined by summing the weights of situations, but these weights themselves are provided by a successor state axiom that stitches together action executability, unobservable outcomes and likelihoods of actions in one formula. This makes it difficult to unpack, even informally, how degrees of beliefs change over actions and sensing. This motivated a new logical language, the logic $\mathcal{DS}$ [4], which casts the BHL framework in a modal language, allowing a semantical apparatus to reason about actions, beliefs, meta-beliefs (including introspection) and only knowing [25] in a single logical framework. Extensions to $\mathcal{DS}$ [27, 28] further considered adapting the regression and progression results from the BHL model.

In this work, we wish to continue and combine these two strands of work. However, one major limitation of the (continuous) BL extension [6, 9] to BHL is that it only allows finitely many nullary fluents. This is because unlike the discrete case (both BHL and $\mathcal{DS}$), where we can sum over situations (and worlds, respectively), there does not seem to be a feasible way to integrate over situations or worlds. So in the BL model, they define belief in terms of the values of fluents initially and then "project" how the weights of these situations change after actions. This makes the model only applicable to finitely many absolutely continuous fluents. Working with mixed discrete-continuous or mixing discrete and continuous needs to be done meta-linguistically by swapping the integration with sums or some other operators, which is further defined axiomatically in BL.

In this paper, we will consider the full first-order fragment (i.e., going beyond nullary fluents and allowing fluents of arbitrary arity as is usual in the standard situation calculus) but allow combinations of discrete and continuous probabilities. This necessitates a new and general treatment of probabilities on possible worlds, different from $\mathcal{DS}$ and BHL, where one used summing, and from BL, where one integrated over fluent values. Going back to early work on probabilistic logic [15, 16], but now in a first-order setting, we define measures on uncountably many worlds that interpret infinitely many fluents. We then show how this leads to a fairly simple definition of knowing, believing, and only-knowing. Properties thereof will also be analyzed. We conclude with some thoughts about how to extend this logic to actions at the next step and what challenges might arise.

## 2 THE LOGIC $\mathcal{OBL}c$

The logic $\mathcal{OBL}c$ of only-believing with continuous degree of beliefs is a second-order many-sorted epistemic modal logic. For simplicity, we only consider functions with equality (=) and omit predicates. There are two sorts: *object* and *real number* $\mathbb{R}$. Second-order quantification is only used when talking about numbers.

### 2.1 The Language

The vocabulary consists of *standard names*, *variables*, and *function symbols*. Standard names can be viewed as a fixed countable domain with the *unique names assumption*. Conventionally, we use $n$ with (sub-)superscripts for object standard names, e.g. $n_1, n_2, \ldots n', n'', \ldots$ etc. We use Arabic numbers both decimals and fractions for number standard names, e.g. 0.8, $\frac{1}{2}$, ..., etc. First-order (FO) variables are denoted by $x, y, u, v \ldots$, etc. We use $V, V', \ldots$, etc. for second-order variables. Function symbols include

- *rigid function symbols* of every arity including mathematical functions like $+, \times, e^x$.
- *fluent function symbols* of every arity, such as *distanceTo(x)*, *heightOf(y),salaryInYear(x,y), GDPInMonthYear(x,y,z)*.

Here, we use the terminology from the situation calculus [33], and by *rigid*, we mean the meaning of the function is fixed, while by *fluent*, we mean the meaning of the function might vary.

Besides, standard FO connectives $\wedge, \neg, \forall$ and modal operators $\{\boldsymbol{B}, \boldsymbol{O}\}$, are used to construct formulas.

**Terms** (for respective sort) of the language are the least set of expressions such that

(1) every standard name and first-order variable is a term;
(2) If $t_1, ..., t_k$ are terms and $f$ is a $k$-ary function symbol, then $f(t_1, ..., t_k)$ is a term of the same sort as $f$.
(3) If $t_1, ..., t_k$ are terms and $F$ is a $k$-ary second-order variable, then $F(t_1, ..., t_k)$ is a term of the same sort as $F$.

A term is said to be *rigid* if and only if it does not contain fluents. *Ground terms* are terms without variables.

The epistemic expression $\boldsymbol{B}(\alpha : r)$ should be read as "$\alpha$ is believed with a degree $r$". $\boldsymbol{K}\alpha$ means "$\alpha$ is known" and is an abbreviation for $\boldsymbol{B}(\alpha : 1)$. $\boldsymbol{O}(\alpha_1 : r_1, \ldots \alpha_k : r_k)$ may be read as "all that believed are conjunctively $\alpha_i$ with degree $r_i$". Similarly, $\boldsymbol{O}\alpha$ means "$\alpha$ is only known" and is an abbreviation for $\boldsymbol{O}(\alpha : 1)$.

As usual, we treat $\alpha \vee \beta, \alpha \supset \beta, \alpha \equiv \beta$, and $\exists v.\alpha$ as abbreviations. A *sentence* is a formula without free variables. We use TRUE as an abbreviation for $\forall x(x = x)$, and FALSE for its negation. A formula without $\boldsymbol{B}$ and $\boldsymbol{O}$ is called *objective*. A formula with no fluent outside $\boldsymbol{B}$ or $\boldsymbol{O}$ is called *subjective*. An objective formula without fluent functions is called a *rigid formula*.

### 2.2 The Semantics

The semantics is given in terms of possible worlds, where a *world* is a Tarski-like structure (Recall that we only consider static case in this paper). Formally, we assume a fixed domain of discourse $\mathcal{D} = \mathcal{D}_{obj} \cup \mathbb{R}$ where $\mathcal{D}_{obj}$ is a countable infinite set of objects, $\mathbb{R}$ the real numbers. The set of standard names $\mathcal{N} = \mathcal{N}_{obj} \cup \mathcal{N}_{num}$ is a countable subset of $\mathcal{D}$ and $\mathcal{N}_{obj} = \mathcal{D}_{obj}$.[1] Lastly, we fix $\mathcal{N}_{num}$ to the set of *computable numbers* [41] which is a countable subset of $\mathbb{R}$ but still includes important irrational numbers such as $\pi, e$.

*2.2.1 Objective Formulas.* A **world** is a mapping from all function symbols to functions of the corresponding sorts.

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[1]Even if the domain is uncountable, we can only assign standard names to a countable subset of it.

Formally, a world $w$ maps every $k$-ary object function symbol $f_{obj}$ and number function symbol $f_{num}$ to a function of the corresponding sort, i.e. $w[f_{obj}]\colon (\mathcal{D})^k \to \mathcal{D}_{obj}$ and $w[f_{num}]\colon (\mathcal{D})^k \to \mathcal{D}_{num}$ satisfying the following constraints:

(1) **Rigidity:** if $f$ is a rigid function symbol, then for all $w, w'$, $w[f] = w'[f]$;

(2) **Arithmetical Correctness:** arithmetical function symbol (e.g. $+, \times, e^x$) are interpreted in the usual sense. For example, $w[+](1,1) = 2$ for any $w$. [2]

Let $W$ be the set of all such worlds. We denote the set of all first-order variables and second-order variables as $\mathcal{V}_{FO}$ and $\mathcal{V}_{SO}$ respectively. A **variable map** $\lambda$ maps each element in $\mathcal{V}_{FO}$ to $\mathcal{D}$ of the right sort and maps each element in $\mathcal{V}_{SO}$ to a function of the corresponding sorts. We write $\lambda \sim_v \lambda'$ to mean $\lambda$ and $\lambda'$ agree excepts perhaps on variable $v$ and $\lambda \sim_V \lambda'$ to mean $\lambda$ and $\lambda'$ agree excepts perhaps on SO variable $V$. The *denotation* of terms is defined recursively:

*Definition 2.1.* the denotation of a term $t$ under a pair of world and variable map $\langle w, \lambda \rangle$ is defined as: (assuming $t_i$ are terms)

- $\|t\|_{w,\lambda} = t$ if $t \in \mathcal{N}$;
- $\|t\|_{w,\lambda} = \lambda(t)$ if $t \in \mathcal{V}_{FO}$;
- $\|t\|_{w,\lambda} = w[f](\|t_1\|_{w,\lambda}, ..., \|t_k\|_{w,\lambda})$ if $t$ is of the form $f(t_1, ..., t_k)$ where $f$ is a function symbol.
- $\|t\|_{w,\lambda} = \lambda(V)(\|t_1\|_{w,\lambda}, ..., \|t_k\|_{w,\lambda})$ if $t$ is of the form $V(t_1, ..., t_k)$ where $V$ is a SO variable.

For simplicity, we write $\|t\|_\lambda$ when $t$ is rigid, $\|t\|_w$ when $t$ does not contain variables, and $\|t\|$ when $t$ is both rigid and ground. By a *model* we mean a pair $\langle w, \lambda \rangle$. Truth of objective formulas is then given as:

- $w, \lambda \models t_1 = t_2$ iff $\|t_1\|_{w,\lambda}$ and $\|t_2\|_{w,\lambda}$ are identical;
- $w, \lambda \models \neg\alpha$ iff $w, \lambda \not\models \alpha$;
- $w, \lambda \models \alpha \wedge \beta$ iff $w, \lambda \models \alpha$ and $w, \lambda \models \beta$;
- $w, \lambda \models \forall v.\alpha$ iff $w, \lambda' \models \alpha$ for all $\lambda' \sim_v \lambda$;
- $w, \lambda \models \forall V.\alpha$ iff $w, \lambda' \models \alpha$ for all $\lambda' \sim_V \lambda$;

*2.2.2 Beliefs.* To give the semantics of $\boldsymbol{B}$ and $\boldsymbol{O}$, we need the notion of **epistemic state**. We begin with a brief recap of some key concepts of probability theory. A measure space is a tuple $\langle X, \mathcal{X}, \mu \rangle$ where $X$ is a set, $\mathcal{X}$ is a $\sigma$-algebra on the set $X$ (i.e. a non-empty collection of subsets closed under complements and countable unions), and $\mu : \mathcal{X} \mapsto [0, +\infty]$ is a measure. More concretely, $\mathcal{X}$ is a non-empty subset of $2^X$ such that:

- for all $X_1, X_2, \ldots \in \mathcal{X}$, it holds $\bigcup_i X_i \in \mathcal{X}$ (closed under countable unions); and
- for all $X' \in \mathcal{X}$, $X'_c = X \backslash X' \in \mathcal{X}$ (closed under complements).

Moreover, $\mu$ is a mapping $\mu : \mathcal{X} \mapsto [0, +\infty]$ that satisfies:

- $\mu(\emptyset) = 0$;
- for all pairwise disjoint $X_1, X_2, \ldots \in \mathcal{X}$, it holds $\mu(\bigcup_i X_i) = \sum_i \mu(X_i)$.

Typical measure spaces include the Lebesgue measure spaces $\langle \mathbb{R}^k, \mathcal{M}, m \rangle$ where $\mathbb{R}^k$ is a $k$-dimensional Euclidean space, $\mathcal{M}$ is a $\sigma$-algebra on $\mathbb{R}^k$, and $m$ is the Lebesgue measure. A probability space is a special measure space whose measure is normalized, i.e. $\mu(X) = 1$. For probability spaces, usually $X$ is called the *sample space*, $\mathcal{X}$ the *event set*, $\mu$ the *probability measure*. A probability space $\langle X, \mathcal{X}, \mu \rangle$ is said to be *complete* if for all $B \in \mathcal{X}$ with $\mu(B) = 0$ and all $A \subseteq B$, one has $A \in \mathcal{X}$ and $\mu(A) = 0$. Intuitively, completeness means that if an event has zero probability, any subset of it is also an event and has zero probability; likewise, if an event has probability 1, all its supersets are events and have probability 1. We restrict ourselves to complete probability spaces since each probability space can be uniquely extended to a complete probability space.

An **epistemic state** $e$ is then defined as a set of $\langle \mathcal{W}, \mu \rangle$ pairs s.t. $\langle W, \mathcal{W}, \mu \rangle$ forms a complete probability space (henceforth, we call such $\langle \mathcal{W}, \mu \rangle$ pairs probability spaces directly). [3] We expand the model with the epistemic state. Namely, a model is now a triple $\langle e, w, \lambda \rangle$. For objective formulas, truth is given the same as before since the epistemic state $e$ plays no role.

Let $W_\alpha^{e,\lambda} = \{w' \mid e, w', \lambda \models \alpha\}$. Specifically, when $e$ contains only one element, i.e. $e = \{\langle \mathcal{W}, \mu \rangle\}$, we write $W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda}$ instead of $W_\alpha^{\{\langle \mathcal{W}, \mu \rangle\}, \lambda}$. In case $\alpha$ has no free variables, we write $W_\alpha^e$. Let $r, r_i$ denote rigid terms. Truth for $\boldsymbol{B}$ and $\boldsymbol{O}$ is given as:

- $e, w, \lambda \models \boldsymbol{B}(\alpha : r)$ iff $\forall \langle \mathcal{W}, \mu \rangle$, $\langle \mathcal{W}, \mu \rangle \in e$ *implies* $W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda} \in \mathcal{W}$ and $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda}) = \|r\|_\lambda$;
- $e, w, \lambda \models \boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$ iff $\forall \langle \mathcal{W}, \mu \rangle$, $\langle \mathcal{W}, \mu \rangle \in e$ iff $W_{\alpha_i}^{\langle \mathcal{W}, \mu \rangle, \lambda} \in \mathcal{W}$ and $\mu(W_{\alpha_i}^{\langle \mathcal{W}, \mu \rangle, \lambda}) = \|r_i\|_\lambda$ for $i \in \{1, \ldots, k\}$.

Intuitively, $e, w, \lambda \models \boldsymbol{B}(\alpha : r)$ if for all $\langle \mathcal{W}, \mu \rangle \in e$, the set of worlds that satisfy $\alpha$ under $\langle \mathcal{W}, \mu \rangle$, i.e. $W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda}$, has probability measure $\|r\|_\lambda$. Likewise, $e, w, \lambda \models \boldsymbol{O}(\alpha : r)$ iff $e$ is the maximal set of such probability spaces. Essentially, beliefs $\boldsymbol{B}(\alpha : r)$ defined in this way are indeed probabilities over possible worlds.

For a sentence $\alpha$, we write $e, w \models \alpha$ to mean $e, w, \lambda \models \alpha$ for all variable maps $\lambda$. When $\Sigma$ is a set of sentences and $\alpha$ is a sentence, we write $\Sigma \models \alpha$ (read: $\Sigma$ logically entails $\alpha$) to mean that for every $e$ and $w$, if $e, w \models \alpha'$ for every $\alpha' \in \Sigma$, then $e, w \models \alpha$. We say that $\alpha$ is valid ($\models \alpha$) if $\{\} \models \alpha$. Satisfiability is then defined in the standard way. If $\alpha$ is an objective sentence, we write $w \models \alpha$ instead of $e, w \models \alpha$. Similarly, we write $e \models \alpha$ instead of $e, w \models \alpha$ if $\alpha$ is subjective.

# 3 PROPERTIES OF BELIEF

In this section, we show that our logic has many reasonable properties. As we shall see, many properties in the logic $\mathcal{DS}$ and its static predecessor, the logic $\mathcal{OBL}$ [5], are retained.

To begin with, we have the following properties in terms of validity and satisfiability:

- $\boldsymbol{B}(\textsc{True}: 1)$ is valid

  PROOF. This is straightforward: 1) $w \models \textsc{True}$ for all world $w \in W$; 2) for all epistemic state $e$ and all probability space $\langle \mathcal{W}, \mu \rangle \in e$, we have $W \in \mathcal{W}$ and $\mu(W) = 1$. Hence $e \models \boldsymbol{B}(\textsc{True}: 1)$. $\qquad\square$

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[2]Sometime, one might also wish to use the predicate '$<$' (similarly for '$\leq$') in formulas, this can be done by assuming a rigid function *lessthan* which takes values from two reserved standard names $\{\top, \bot\}$, additionally, for all worlds $w$ and real number $x, y$, $w[lessthan](x, y) = \top$ iff $x < y$.

[3]As observed in [18], using a single probability space as an epistemic state would result in that agents necessarily have *de re* knowledge about their degrees of belief, i.e. for all $\phi$, $\exists x.\boldsymbol{KB}(\phi: x)$ is valid, which is counter-intuitive.

- for negative $r$, $\boldsymbol{B}(\alpha : r)$ is satisfiable only by the empty epistemic state $\emptyset$.

  PROOF. Supposing $e \neq \emptyset$ and $e \models \boldsymbol{B}(\alpha : r)$ with $\|r\| < 0$, then for all $\langle \mathcal{W}, \mu \rangle \in e$, $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = \|r\| < 0$. This contradicts with the definition of $\mu$. □

### 3.1 Additivity and Equivalence

Besides, $\boldsymbol{B}$ entertains the properties of probability as in [15]:

- if $\models \alpha \equiv \beta$, then $\models \boldsymbol{B}(\alpha : r) \equiv \boldsymbol{B}(\beta : r)$ for all $r$.

  PROOF. For any $e$, and every $\langle \mathcal{W}, \mu \rangle \in e$, since $W_\alpha^{\langle \mathcal{W}, \mu \rangle} = W_\beta^{\langle \mathcal{W}, \mu \rangle}$, if $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = \|r\|$, then $\mu(W_\beta^{\langle \mathcal{W}, \mu \rangle}) = \|r\|$ and vice versa. Thus $e \models \boldsymbol{B}(\alpha : r) = \boldsymbol{B}(\beta : r)$. □

- $\models \boldsymbol{B}(\alpha : r) \supset \boldsymbol{B}(\neg\alpha : 1 - r)$
  It means $\boldsymbol{B}$ satisfies the complement law of probability.

  PROOF. Suppose $e \models \boldsymbol{B}(\alpha : r)$, then for all $\langle \mathcal{W}, \mu \rangle \in e$, $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = \|r\|$. Then $\mu(W_{\neg\alpha}^{\langle \mathcal{W}, \mu \rangle}) = \mu(W \setminus W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = \mu(W) - \mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = 1 - \|r\|$. □

- $\models \boldsymbol{B}(\alpha \wedge \beta : r) \wedge \boldsymbol{B}(\alpha \wedge \neg\beta : r') \supset \boldsymbol{B}(\alpha : r + r')$

  PROOF. Suppose $e \models \boldsymbol{B}(\alpha \wedge \beta : r)$ and $e \models \boldsymbol{B}(\alpha \wedge \neg\beta : r')$, this means for all $\langle \mathcal{W}, \mu \rangle \in e$, $\mu(W_{\alpha \wedge \beta}^{\langle \mathcal{W}, \mu \rangle}) = \|r\|$ and $\mu(W_{\alpha \wedge \neg\beta}^{\langle \mathcal{W}, \mu \rangle}) = \|r'\|$. Since $W_{\alpha \wedge \beta}^{\langle \mathcal{W}, \mu \rangle} \cap W_{\alpha \wedge \neg\beta}^{\langle \mathcal{W}, \mu \rangle} = \emptyset$ and $W_{\alpha \wedge \beta}^{\langle \mathcal{W}, \mu \rangle} \cup W_{\alpha \wedge \neg\beta}^{\langle \mathcal{W}, \mu \rangle} = W_\alpha^{\langle \mathcal{W}, \mu \rangle}$, we have $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = \mu(W_{\alpha \wedge \beta}^{\langle \mathcal{W}, \mu \rangle} \cup W_{\alpha \wedge \neg\beta}^{\langle \mathcal{W}, \mu \rangle}) = \mu(W_{\alpha \wedge \beta}^{\langle \mathcal{W}, \mu \rangle}) + \mu(W_{\alpha \wedge \neg\beta}^{\langle \mathcal{W}, \mu \rangle}) = \|r\| + \|r'\|$. □

- $\models \boldsymbol{B}(\alpha : r) \wedge \boldsymbol{B}(\beta : r') \wedge \boldsymbol{B}(\alpha \wedge \beta : r'') \supset \boldsymbol{B}(\alpha \vee \beta : r + r' - r'')$
  That is $\boldsymbol{B}$ satisfies the addition law of probability. The proof is rather similar to the above one, hence we skip it here.

All the above properties follow from the fact that $\boldsymbol{B}$ is essentially a probability over possible worlds.

### 3.2 Knowledge

Recall that $\boldsymbol{K}\alpha$ is an abbreviation for $\boldsymbol{B}(\alpha : 1)$. Our modal of knowledge $\boldsymbol{K}$ also satisfies many properties in the epistemic logic $\mathcal{KL}$ by Levesque and Lakemeyer [26], including universal and existential versions of the Barcan formula:

- $\models \boldsymbol{K}\alpha \supset \boldsymbol{K}(\alpha \vee \beta)$

  PROOF. Suppose that $e \models \boldsymbol{K}\alpha$, then for all $\langle \mathcal{W}, \mu \rangle \in e$, $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle}) = 1$. Since $W_\alpha^{\langle \mathcal{W}, \mu \rangle} \subseteq W_{\alpha \vee \beta}^{\langle \mathcal{W}, \mu \rangle}$, the completeness of the probability space $\langle W, \mathcal{W}, \mu \rangle$ guarantees that $\mu(W_{\alpha \vee \beta}^{\langle \mathcal{W}, \mu \rangle}) = 1$. □

- $\models \boldsymbol{K}\alpha \wedge \boldsymbol{K}\beta \supset \boldsymbol{K}(\alpha \wedge \beta)$

  PROOF. Suppose that $\boldsymbol{K}\alpha \wedge \boldsymbol{K}\beta$ holds. By the complement law we have $\boldsymbol{B}(\neg\alpha : 0) \wedge \boldsymbol{B}(\neg\beta : 0)$. Further, with the first property of knowledge and the complement law we have $\boldsymbol{B}(\neg\alpha \wedge \neg\beta : 0)$. According to the addition law of probability,

$\boldsymbol{B}(\neg\alpha : 0) \wedge \boldsymbol{B}(\neg\beta : 0) \wedge \boldsymbol{B}(\neg\alpha \wedge \neg\beta : 0) \supset \boldsymbol{B}(\neg\alpha \vee \neg\beta : 0)$. Thus $\boldsymbol{K}(\alpha \wedge \beta)$ holds. □

- $\models \boldsymbol{K}\alpha \wedge \boldsymbol{K}(\alpha \supset \beta) \supset \boldsymbol{K}\beta$

  PROOF. Suppose that $e \models \boldsymbol{K}\alpha \wedge \boldsymbol{K}(\alpha \supset \beta)$. To prove $e \models \boldsymbol{K}\beta$ it suffice to prove that $e \models \boldsymbol{K}((\alpha \vee \beta) \wedge (\neg\alpha \vee \beta))$. $\boldsymbol{K}(\alpha \vee \beta)$ can be proved by the first property of knowledge. $\boldsymbol{K}(\neg\alpha \vee \beta)$ is equivalent to $\boldsymbol{K}(\alpha \supset \beta)$. Based on the second property of knowledge $e \models \boldsymbol{K}\beta$. □

- $\models \exists x.\boldsymbol{K}\alpha \supset \boldsymbol{K}\exists x.\alpha$

  PROOF. Suppose $e, \lambda \models \exists x.\boldsymbol{K}\alpha$, then there exists $\lambda'$ with $\lambda \sim_x \lambda'$ such that $e, \lambda' \models \boldsymbol{K}\alpha$. Namely, for all $\langle \mathcal{W}, \mu \rangle \in e$, $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'}) = 1$ where by definition $W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'} = \{w | \langle \mathcal{W}, \mu \rangle, w, \lambda' \models \alpha\}$. On the other hand,

$$W_{\exists x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda} = \{w | \langle \mathcal{W}, \mu \rangle, w, \lambda \models \exists x.\alpha\}$$
$$= \bigcup_{\{\lambda'' | \lambda \sim_x \lambda''\}} \{w | \langle \mathcal{W}, \mu \rangle, w, \lambda'' \models \alpha\}$$

Thus $W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'} \subseteq W_{\exists x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda}$. Since $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'}) = 1$, we have $\mu(W_{\exists x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda}) = 1$ □

The converse of the above formula does not hold: knowing that $\alpha$ holds for someone does not imply knowing that individual. For universal quantification, we have

- $\models \boldsymbol{K}\forall x.\alpha \supset \forall x.\boldsymbol{K}\alpha$

  PROOF. Suppose that $e, \lambda \models \boldsymbol{K}\forall x.\alpha$. For any $\langle \mathcal{W}, \mu \rangle \in e$,

$$W_{\forall x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda} = \{w | \langle \mathcal{W}, \mu \rangle, w, \lambda \models \forall x.\alpha\}$$
$$= \{w | \forall \lambda'. \lambda' \sim_x \lambda, \langle \mathcal{W}, \mu \rangle, w, \lambda' \models \alpha\}$$
$$= \bigcap_{\{\lambda' | \lambda \sim_x \lambda'\}} W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'}$$

For any $\lambda'$ with $\lambda \sim_x \lambda'$, $W_{\forall x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda} \subseteq W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'}$. Since $\mu(W_{\forall x.\alpha}^{\langle \mathcal{W}, \mu \rangle, \lambda}) = 1$, for all $\lambda'$ with $\lambda' \sim_x \lambda$, $\mu(W_\alpha^{\langle \mathcal{W}, \mu \rangle, \lambda'}) = 1$. Thus $e, \lambda' \models \boldsymbol{K}\alpha$ for all $\lambda' \sim_x \lambda$, i.e. $e, \lambda \models \forall x.\boldsymbol{K}\alpha$ □

Unfortunately, in general the converse of the above formula does not hold as well:

- $\not\models \forall x.\boldsymbol{K}\alpha \supset \boldsymbol{K}\forall x.\alpha$
  The reason is: in a probability space with uncountably many samples, there could be uncountably many distinct events where each of them has probability 1, the countable intersection of any of these events has probability 1, yet an uncountable intersection does not have probability 1.
  More concretely, we provide a counter-example as follows, assuming $h$ is a nullary fluent taking values from $\mathbb{R}$, then $\not\models \forall x.\boldsymbol{K}(h \geq 0 \vee h \neq x) \supset \boldsymbol{K}\forall x.(h \geq 0 \vee h \neq x)$.
  Informally, given an epistemic state $e = \langle W, \mu \rangle$, the fluent $h$ can be viewed as a random variables over $\mathbb{R}$. We further assume that the distribution of $h$ is standard Gaussian in $e$. For this particular $e$, $e \models \forall x.\boldsymbol{K}(h \geq 0 \vee h \neq x)$ yet $e \not\models \boldsymbol{K}\forall x.(h \geq 0 \vee h \neq x)$.

For the former part: for all real $r < 0$, $h \geq 0 \vee h \neq r$ is equivalent to $h \neq r$. The probability of this event would be 1: it leaves out a single point ($h = r$) from the whole sample space of a Gaussian distribution. The remaining samples have a probability of 1. Additionally, for all real $r \geq 0$, $h \geq 0 \vee h \neq r$ is equivalent to TRUE, hence the event $h \geq 0 \vee h \neq r$ has probability 1 trivially. Together, we have $e \models \forall x.K(h \geq 0 \vee h \neq x)$.

For the later part: $\forall x.(h \geq 0 \vee h \neq x)$ is equivalent to $h \geq 0$, hence $e \models B(\forall x.(h \geq 0 \vee h \neq x) : \frac{1}{2})$ (recall $h$ is distributed as a standard Gaussian in $e$). Hence $e \not\models K\forall x.(h \geq 0 \vee h \neq x)$.

Nevertheless, if we only consider the countable object domain, we still have:

- $\models \forall x.K\alpha \supset K\forall x.\alpha$ if $x$ is a variable of sort object.

PROOF. Suppose that $e, \lambda \models \forall x.K\alpha$. By the semantics we have $e, \lambda' \models K\alpha$ for all $\lambda' \sim_x \lambda$. For all $\langle W, \mu \rangle \in e$ and $\lambda' \sim_x \lambda$, $\mu(W_\alpha^{\langle W, \mu \rangle, \lambda'}) = 1$. Since $x$ is of sort object, the set $\{\lambda' | \lambda' \sim_x \lambda\}$ is countable.

$$\mu(W \backslash W_{\forall x.\alpha}^{\langle W, \mu \rangle, \lambda}) = \mu(W \backslash \bigcap_{\{\lambda' | \lambda' \sim_x \lambda\}} W_\alpha^{\langle W, \mu \rangle, \lambda'})$$
$$= \mu(\bigcup_{\{\lambda' | \lambda' \sim_x \lambda\}} (W \backslash W_\alpha^{\langle W, \mu \rangle, \lambda'}))$$
$$\leq \sum_{\{\lambda' | \lambda' \sim_x \lambda\}} \mu(W \backslash W_\alpha^{\langle W, \mu \rangle, \lambda'}) = 0$$

The inequality above is derived from the property of measure and based on the fact that $\{\lambda' | \lambda' \sim_x \lambda\}$ is countable. Thus $\mu(W_{\forall x.\alpha}^{\langle W, \mu \rangle, \lambda}) = 1$ for all $\langle W, \mu \rangle \in e$, i.e. $e, \lambda \models K\forall x.\alpha$. □

E.g. $\models \forall x.K(father(x) \neq joe) \supset K\forall x.father(x) \neq joe$. Namely, if for all persons, it's known that the person's father is not $joe$, as a consequence, it's known that $joe$ is not anyone's father.

### 3.3 Introspection

Lastly, let us turn to introspection. Formally, for positive introspection, we have:

- $\models B(\alpha : r) \supset KB(\alpha : r)$

PROOF. Supposing $e \models B(\alpha : r)$, by semantics for all $\langle W, \mu \rangle \in e$, $\mu(W_\alpha^{\langle W, \mu \rangle}) = r$. On the other hand, $W_{B(\alpha : r)}^{\langle W, \mu \rangle} = W$ if $\langle W, \mu \rangle \models B(\alpha : r)$. Hence for all $\langle W, \mu \rangle \in e$, $\mu(W_{B(\alpha : r)}^{\langle W, \mu \rangle}) = \mu(W) = 1$. By semantics, $e \models KB(\alpha : r)$ □

We comment that the converse of the above formula holds as well, namely, $\models KB(\alpha : r) \supset B(\alpha : r)$. The proof is in the same spirit as above, hence we skip it. As a special case, we have the usual $K$ properties [17]:

- $\models K\alpha \supset KK\alpha$

Unfortunately, negative introspection does not hold:

- $\not\models \neg B(\alpha : r) \supset K\neg B(\alpha : r)$

PROOF. Supposing $e_1 \models B(\alpha : r)$ and $e_2 \not\models B(\alpha : r)$, then let $e = e_1 \cup e_2$, clearly, $e \not\models B(\alpha : r)$ since there is at least one $\langle W, \mu \rangle \in e_2$ for which $\langle W, \mu \rangle \not\models B(\alpha : r)$. Now we show, $e \not\models K\neg B(\alpha : r)$.

This amounts to show $\exists \langle W', \mu' \rangle \in e$ s.t. $\mu'(W_{\neg B(\alpha : r)}^{\langle W', \mu' \rangle}) \neq 1$. It is not hard to see this hold by just picking $\langle W', \mu' \rangle$ from $e_1$. Since $e_1 \models B(\alpha : r)$, we have $\mu'(W_\alpha^{\langle W', \mu' \rangle}) = r$. Then $W_{\neg B(\alpha : r)}^{\langle W', \mu' \rangle} = \{w | \langle W', \mu' \rangle, w \models \neg B(\alpha : r)\} = \emptyset$, hence, $\mu'(W_{\neg B(\alpha : r)}^{\langle W', \mu' \rangle}) = \mu'(\emptyset) = 0 \neq 1$. □

This is because in the truth of $B$, i.e. $e \models B(\alpha : r)$, only individuals $\langle W, \mu \rangle$ in $e$ are used in $W_\alpha$, namely $W_\alpha^{\langle W, \mu \rangle, \lambda}$, instead of the whole epistemic state $W_\alpha^{e, \lambda}$. The latter approach [4, 5] might result in negative introspection. However, it also complicates the epistemic states which satisfy $O(\alpha : r)$. Our logic uses the former idea. Consequently, we have the unique model theorem for only-believing as shown below.

## 4  ONLY-BELIEVING

Another important part of our formalism lies in the notion of only-knowing (or only-believing) [26]. Only-knowing captures the intuition that the beliefs and non-beliefs of an agent are precisely those that follow from its knowledge base. Hence it is useful to characterize a knowledge base. Here, we show that our modality $O$ faithfully captures the notion of only-believing by examining its properties.

To begin with, the unique model theorem holds for only-believing as in the work [28].

THEOREM 4.1 (UNIQUE MODEL THEOREM). *For any sentence $\alpha$ and rigid ground term $r$, there is an unique epistemic state $e$ such that $e \models O(\alpha : r)$*

PROOF. By the semantics of $O$, we have $e \models O(\alpha : r)$ iff $e = \{\langle W, \mu \rangle | \mu(W_\alpha^{\{\langle W, \mu \rangle\}}) = \|r\|\}$. Clearly, there is only one such $e$. □

While only-believing is always uniquely satisfiable, it may be the case that $e$ is empty. For example $\emptyset \models O(\text{FALSE} : 1)$. Likewise, the maximal epistemic state $e_{\text{TRUE}} \models O(\text{TRUE} : 1)$ where $e_{\text{TRUE}} = \{\langle W, \mu \rangle | \langle W, W, \mu \rangle$ forms a probability space$\}$.

Besides, as in the logic $O\mathcal{L}$ [26], only-knowing implies knowing and not knowing about what is not entailed by the knowledge base. Below, let $\alpha$ be an arbitrary sentence and $\phi, \psi$ objective sentences:

- $\models O\alpha \supset K\alpha$
- $\models O\phi \supset K\psi$ iff $\models \phi \supset \psi$

PROOF. Since $\phi$ is objective, for all epistemic state $e$ and $\langle W, \mu \rangle \in e$, the set $W_\phi^{\langle W, \mu \rangle}$ is irrelevant to $\langle W, \mu \rangle$ and hence we write it as $W_\phi$ (respectively we write $W_\psi$).
($\Rightarrow$) Suppose that $\models O\phi \supset K\psi$, i.e. for all $\langle W, \mu \rangle$, if $\mu(W_\phi) = 1$, then $\mu(W_\psi) = 1$.
Assuming that $\not\models \phi \supset \psi$, then there exists $w$ such that $w \models \phi$ but $w \not\models \psi$. Let $W'$ be any $\sigma$-algebra which contains $\{w\}, W_\phi$ and $W_\psi$ as elements. Let $\mu'$ be a probability measure such that $\mu'(\{w\}) = 0.1$ and $\mu'(W_\phi \backslash \{w\}) = 0.9$. Then $\mu'(W_\phi) = 1$ but $\mu'(W_\psi) \leq \mu'(W \backslash \{w\}) = 0.9$ which leads to contradiction. Thus $\models \phi \supset \psi$.
($\Leftarrow$) Suppose that $\models \phi \supset \psi$, for all $w \in W$, $w \models \phi$ implies $w \models \psi$. Thus $W_\phi \subseteq W_\psi$. For all $\langle W, \mu \rangle$, if $\mu(W_\phi) = 1$, then $\mu(W_\psi) = 1$. Hence $\models O\phi \supset K\psi$ □

- $\models \boldsymbol{O}\phi \supset \neg\boldsymbol{K}\psi$ iff $\phi \nvDash \psi$

  This is an easy consequence of the above property and the unique model theorem.

  PROOF. As the negation of $\models \boldsymbol{O}\phi \supset \boldsymbol{K}\psi$ iff $\models \phi \supset \psi$, we have $\boldsymbol{O}\phi \nvDash \boldsymbol{K}\psi$ iff $\phi \nvDash \psi$. Due to the unique model theorem, there exists a unique model $e$ s.t. $e \models \boldsymbol{O}\phi$, hence $\boldsymbol{O}\phi \nvDash \boldsymbol{K}\psi$ iff $\boldsymbol{O}\phi \models \neg\boldsymbol{K}\psi$. □

We note that the above statement does not hold if the $\boldsymbol{O}$ modality is replaced with $\boldsymbol{K}$.

More generally, we have properties for only-believing as follows:

- $\models \boldsymbol{O}(\alpha : r) \supset \boldsymbol{B}(\alpha : r)$
- $\models \boldsymbol{O}(\alpha : r) \supset \neg\boldsymbol{B}(h(\vec{n}) = m : r')$ for all $r, r'$, where $\vec{n}, m$ are standard names and $h$ is a fluent not in $\alpha$.

E.g. $\boldsymbol{O}(1.7 \leq heightOf(A) : 0.9) \models \boldsymbol{B}(1.7 \leq heightOf(A) : 0.9)$ and $\boldsymbol{O}(1.7 \leq heightOf(A) : 0.9) \models \forall x. \neg\boldsymbol{B}(weightOf(A) \leq 50 : x)$ That is, only-believing the person $A$'s height is greater than 1.7m with degree 0.9 entails believing the person's height is greater than 1.7m with degree 0.9, and also, no believing $A$'s weight is no more than 50kg with any degree.

## 5 BELIEF DISTRIBUTIONS

Lastly, we demonstrate how to represent probability distributions and how to reason about beliefs in our logic.

### 5.1 The Nullary Fluents Fragment

In the works [27, 28] (extensions to the logic $\mathcal{DS}$), a formula of the form $\forall \vec{x}.\boldsymbol{B}(\vec{h} = \vec{x} : f(\vec{x}))$ is used to express a joint *discrete* distribution, i.e. *belief distribution*, over a finite set of random variables, where $\vec{h}$ is a set of *nullary fluents* (of sort number) $\vec{h} = \{h_1, h_2, \ldots h_k\}$ each corresponding to a random variable, $\vec{h} = \vec{x}$ denotes $\bigwedge_i h_i = x_i$ and $f$ is a rigid mathematics function s.t. $\sum_{\vec{x}} f(\vec{x}) = 1$ describing the joint distribution.[4] In our logic, we can use a similar formula to express such belief distribution: a (potentially continuous) belief distribution over a finite set of nullary fluents $\vec{h} = \{h_1, h_2, \ldots h_k\}$ is a formula of the form $\forall \vec{x}.\boldsymbol{B}(\bigwedge_i h_i \leq x_i : f(\vec{x}))$ where $f$ is a rigid mathematical function (which satisfies $f(-\vec{\infty}) = 0$ and $f(+\vec{\infty}) = 1$). [5] We use $\boldsymbol{B}^f$ for short. Intuitively, $f(\vec{x})$ represents the joint cumulative distribution function of random variables $\vec{h}$.

PROPOSITION 5.1. *Given a belief distribution $\boldsymbol{B}^f$ over a finite set of nullary fluents $\vec{h} = \{h_1, h_2, \ldots h_k\}$, there exists a non-empty epistemic state $e$ such that $e \models \boldsymbol{B}^f$.*

The proof is based on the fact that there exists a subset $W_{\mathbb{R}^k}$ of $W$ such that $W_{\mathbb{R}^k}$ is isomorphic to $\mathbb{R}^k$ in the sense that for all $\vec{x} \in \mathbb{R}^k$ there is a unique $w_{\vec{x}} \in W_{\mathbb{R}^k}$ s.t. $w_{\vec{x}} \models \bigwedge_i h_i = x_i$. Likewise, one can select a $\sigma$-algebra $\mathcal{W}_{\mathbb{R}^k}$ of $W_{\mathbb{R}^k}$ such that $\mathcal{W}_{\mathbb{R}^k}$ is isomorphic to the Borel $\sigma$-algebra of $\mathbb{R}^k$. Based on the Caratheodory's Extension

Theorem [38], there exists a unique probability measure $\tilde{\mu}$ on $\mathcal{W}_{\mathbb{R}^k}$ such that $\tilde{\mu}(\{w \in W_{\mathbb{R}^k} \mid \bigwedge_i w[h_i] \leq x_i\}) = f(\vec{x})$ for any $\vec{x} \in \mathbb{R}^k$. Lastly, $\langle \mathcal{W}_{\mathbb{R}^k}, \tilde{\mu} \rangle$ can be extended to $\sigma$-algebra and measure on $W$ $\langle \mathcal{W}, \mu \rangle$ by assigning zero-measure on any subset of $W \setminus W_{\mathbb{R}^k}$, and $\{\langle \mathcal{W}, \mu \rangle\}$ is the desired epistemic state.

Our notion of belief distribution can express both discrete and continuous distributions.

*Example 5.2.*
- The belief distribution $\forall x.\boldsymbol{B}(h \leq x : f_d(x))$ of the nullary fluent $h$ with

$$f_d(x) = \begin{cases} 0 & x < 1 \\ 0.5 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

  expresses a discrete uniform distribution over two points $\{1,2\}$ and it is equivalent to $\boldsymbol{B}(h = 1 : 0.5) \wedge \boldsymbol{B}(h = 2 : 0.5)$.
- The belief distribution $\forall x.\boldsymbol{B}(h \leq x : f_c(x))$ expresses a continuous distribution where $f_c(x) = \int_{-\infty}^{x} \mathcal{N}(y; 0, 1) dy$ and $\mathcal{N}(y; u, \sigma)$ represents the Gaussian distribution with mean $u$ and variance $\sigma$ as [6] $\mathcal{N}(y; u, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-u)^2}{2\sigma^2}}$

We emphasize that our notion of belief or belief distribution is more general than the classical notion of probability or probability distribution in that it is possible to express uncertainty of beliefs. For example, the following formulas $\boldsymbol{B}(\alpha : \frac{1}{2}) \vee \boldsymbol{B}(\alpha : \frac{1}{3})$, $\boldsymbol{B}^{f_d} \vee \boldsymbol{B}^{f_c}$, $\forall x \exists y \exists z.\boldsymbol{B}(h \leq x : \int_{-\infty}^{x} \mathcal{N}(x'; y, z) dx')$ express respectively that $\alpha$ is believed with a degree either $\frac{1}{2}$ or $\frac{1}{3}$, $h$ is believed to distribute either discretely as $f_d$ or continuously as $f_c$, and $h$ is believed to be distributed as Gaussian but with unknown mean and variance. All of these are impossible in classical probability theory.

Besides, it is also possible to express mixture distributions via our notion of belief distribution.

*Example 5.3.*
- The belief distribution $\forall x.\boldsymbol{B}(h \leq x : f_m(x))$ of the nullary fluent $h$ with

$$f_m(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \leq x \leq 1 \\ 0.1 + 0.1x & 1 < x \leq 9 \\ 1 & x > 9 \end{cases}$$

  express a mixture distribution, where the point $h = 0$ receives probability mass 0.2 and the rest 0.8 probability mass is distributed uniformly in the interval $[1, 9]$.

This is different from the formula $\boldsymbol{B}^{f_d} \vee \boldsymbol{B}^{f_c}$, which mixes a pure discrete with a pure continuous one. Formulas like $\boldsymbol{B}^{f_d} \vee \boldsymbol{B}^{f_c}$ are possible in BL's formalism [9]; however, they cannot handle the real mixture distribution as our example above. The reason is that their notion of belief extends the BHL formalism, and belief is defined in terms of normalized sums or integral of weights of situations. Since it is unclear how to integrate over situations, they map situations to fluent values which are purely continuous, then belief is given in terms of integral over the continuous values of fluents. When the distribution is not purely continuous as Example 5.3 where the fluent $h$ has discrete probability at the point 0 and has continuous

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[4]It is also possible to use axioms to define such functions [27], for example, $\forall x \forall y.f_{=1}(x) = y \equiv x = 1 \wedge y = 1 \vee x \neq 1 \wedge y = 0$ defines the indicator function $f_{=1}(x)$ which checks if $x = 1$.

[5]Here $f(+\infty) = 1$ should be understand as $\lim_{x \to \infty} f(x) = 1$ and $\lim_{x \to \infty} f(x) = y$ can be defined as $\lim_{x \to \infty} f(x) = y := \forall u.(u > 0 \supset \exists m.\forall v.(v > m) \supset |y - f(v)| < u)$.

[6]Integral is second-order definable see [9] for details. Here we simply assume it as a rigid mathematical function just as $+, \times, e^x$, etc.

probability in the range $[1, 9]$, the integral over situations is not well-defined. Our formalism does not have such problems as we do not have explicit integrals over possible worlds.

## 5.2 Beyond Nullary Fluents

As mentioned earlier, another main advantage of our account is that the notion of belief (only-believing as well) does not restrict possible worlds to only having finitely many nullary fluents. This, among other things, allows us to express probabilities involving potentially infinitely many random variables. For example:

*Example 5.4.* Let $\Sigma$ be $\forall x.x \neq A \supset \forall y.\boldsymbol{B}(salary(x) \leq y: f^u(y))$ where $f^u(y)$ expresses a uniform distribution between $[1000, 2000]$

$$f^u(y) = \begin{cases} 0 & y < 1000 \\ \frac{y}{1000} - 1 & 1000 \leq y \leq 2000 \\ 1 & 2000 < y \end{cases}$$

then, we have

- $\Sigma$ is satisfiable
  Intuitively, $\Sigma$ says that the salary of every person except $A$ is distributed uniformly in $[1000, 2000]$. Clearly, there exists a probability space over possible worlds under which $salary(n)$ is distributed uniformly in $[1000, 2000]$ where $n \neq A$. In the set of all such probability spaces, consider one $\langle W, \mu \rangle$, which assigns non-zero probability only to worlds which satisfy $\forall x, y.x \neq A \land y \neq A \supset salary(x) = salary(y)$, then $\langle W, \mu \rangle$ is the desired epistemic state.
  One might notice that the function $f^u(y)$ is not a joint distribution here. Hence, even if there are infinitely many $salary(n_i)$ and each of them can be viewed as a random variable, the probability space $\langle W, \mu \rangle$ might not be an infinite dimensional one. In fact, the probability space $\langle W, \mu \rangle$ above is two dimensional: one dimension for $salary(A)$ and another for $salary(n)$ where $n \neq A$ (since their values are identical).
- $\boldsymbol{O}(\Sigma) \models \boldsymbol{B}(1200 \leq salary(B) \leq 1300: 0.1) \land \boldsymbol{B}(1800 \leq salary(C) \leq 1900: 0.1)$
  Only-knowing the uniform distribution of all person's salary except $A$ entails believing $B$'s salary in $[1200, 1300]$ and $C$'s salary in $[1800, 1900]$ with the same degree 0.1;
- $\boldsymbol{O}(\Sigma) \models \neg\exists x.\boldsymbol{B}(salary(A) \leq 2000: x)$
  The agent has no knowledge of the distribution of $A$'s salary.

Besides the above uniform distribution example, it is possible in our logic to express Gaussian where the underlying models could be indeed infinite-dimensional. For example, if we use second-order quantification, we have

*Example 5.5.* let $\Sigma$ and $\Sigma'$ be two sentences as

$$\Sigma := \forall x_t \forall y.\boldsymbol{B}(positionAt(x_t) \leq y: \int_{-\infty}^{y} \mathcal{N}(z; x_t, 1)dz)$$

$$\Sigma' := \forall x_n.Natural(x_n) = \top \supset \forall V \forall V' \exists u \exists \sigma \forall x.$$

$$\boldsymbol{B}(\sum_{m=1}^{x_n} V(m) \times positionAt(V'(m)) \leq x: \int_{-\infty}^{x} \mathcal{N}(y; u, \sigma)dy)$$

where $positionAt(x_t)$ returns the horizontal position of an particle at time $x_t$ and $x_n$ is a natural number, then it holds that [7]

- $\Sigma$ and $\Sigma'$ are satisfiable
  Intuitively, $\Sigma$ specifies that the agent believes the position of a particle at time $x_t$ is always distributed as Gaussian with mean $x_t$ and variance 1, while $\Sigma'$ says the agent believes selecting any linear combination $(V_m)$ of any $x_n$ time-points position of the particle $positionAt(V'(m))$ is distributed as Gaussian. In fact, $\Sigma'$ describes that the belief of $positionAt(x)$ forms a *Gaussian process* [13] and $\Sigma$ is a special case. Additionally, a Gaussian process is an infinite set of random variables defined over an infinite-dimensional probability space.

It is worth reflecting at this point that the proposal seems to be the most general framework for reasoning about first-order formulas and continuous distributions. Indeed, we emphasise that these examples are impossible in the logics of BHL [1], BL [9], $\mathcal{DS}$ [4] and its variants [27, 28]. In their works, the nullary fluent assumption is essential to ensure belief distributions are indeed probability distributions over finite-dimensional Euclidean space $\mathbb{R}^k$. Belief distribution that corresponds to probability over infinite dimensional space is not well-defined in their works. Most significantly, to the best of our knowledge, we are not aware of any work in the literature that handles this level of generality.

## 6 CHALLENGES IN EXTENDING TO ACTION

In this section, we examine problems that might arise when extending the proposed formalism to account for actions.

The very first problem is how to include actions in the domain. This task seems relatively easy at first glance. Just like object and number, one could assume another sort for action. Since actions might take numbers as parameters, there would be uncountably many actions. A related problem is what action constants look like. A possible solution is to assume action constants are tuples of the form $\langle act, d_1, d_2, \ldots d_k \rangle$ and worlds give denotations for action terms $act(d_1, d_2, \ldots d_k)$ to $\langle act, d_1, d_2, \ldots d_k \rangle$. Besides, worlds have to be extended to include action sequences as parameters just as the modal variants of the situation calculus [4, 23, 27, 28]. So they behaved as tree-like structures to interpret formulas both initially as well as after any sequence of actions.

Another problem in formalizing beliefs under uncertainty is how to specify the non-deterministic effects of stochastic actions. Such problem also exists when modeling discrete degree of beliefs [1, 4, 27, 28]. An existing solution is to view the stochastic action as a set of ground actions (seen as mutual *alternatives*) where each of them has a deterministic effect, and they are *observationally-indistinguishable* to the agent. Formally, a special fluent *alt* is used to characterize such alternatives relationship among actions. When reasoning about actions, it is the user's task to use an axiom with *alt* to specify such alternatives in the basic action theories.

The third and most important problem is how to incorporate the low-level likelihood of stochastic actions or sensing into high-level beliefs. In fact, two subtle problems arise here: First, how to express actions' likelihood. In discrete formalisms such as the BHL logic

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[7]The set of natural numbers is second-order definable. Namely, $Natural(x) = \top$ can be defined as $\forall V.(V(0) = \top \land \forall y.V(y) = \top \supset V(y + 1) = \top) \supset V(x) = \top$.

and the $\mathcal{DS}$ logic, actions' likelihood is specified by a special unary fluent $l(a)$. Additionally, to ensure $l(a)$ is indeed a probability distribution over actions' alternatives, they often require that the sum of the likelihood of all alternative actions equals 1. In continuous domains, however, using a fluent $l(a)$ to give a value for each action $a$ does not guarantee $l(a)$ is indeed a probability as the alternatives of a given action could be uncountably many. It is unclear what constraints can be imposed to fulfill the requirement. Secondly, how can we incorporate actions' likelihood into beliefs? As belief is most likely to be modeled as a probability over possible worlds like in our logic, the probability over possible worlds would shift as a consequence of actions. Therefore, the belief change after actions has to reflect such shifting over possible worlds correctly. Inevitable problems such as how to perform integration over possible worlds and how such integral changes after actions need to be addressed.

## 7 RELATED WORK

We review related work on probabilistic formalisms. Many works study probabilistic reasoning such as [21, 32, 40], but our contribution is more on the representation side.

The most relevant work is the modal logic $\mathcal{OBL}$ by Belle, Lakemeyer, and Levesque [5], which investigates only-believing with discrete degrees of belief. $\mathcal{OBL}$ is rather similar to $\mathcal{OBL}c$'s first-order fragment in syntax, except that $\mathcal{OBL}c$ has variables that range over the set of real. The major difference lies in the semantics. Since the domain of $\mathcal{OBL}$ is countable, $\mathcal{OBL}$ treats quantifiers substitutionally. Hence no variable maps are needed. Besides, epistemic states in $\mathcal{OBL}$ are sets of weight functions over possible worlds. Special logical devices are imposed to ensure such weight functions are indeed discrete distributions over possible worlds. In contrast, in our account, since the domain is uncountable, possible worlds are Tarski-like structures. Additionally, we encapsulate all subsets of well-defined probability spaces over possible worlds as the epistemic state, and no additional restrictions are required. This leads to fairly succinct semantics for beliefs and only-believing. Consequently, the two formalisms have distinct properties: the universal quantified version of the Barcan formula does not hold in our logic, while it does in $\mathcal{OBL}$. Such a property arises from differences between continuous and discrete phenomena. Also, we do not have negative introspection. $\mathcal{OBL}$ is inspired by the model epistemic logic $\mathcal{OL}$ [26] where the notion of only-knowing is proposed and examined. In 2007, Gabaldon and Lakemeyer [18], proposed the logic $\mathcal{ESP}$, a probabilistic extension of the modal situation calculus [23], where the BHL formalism [1] is re-cast to include the notion of only-knowing. However, degrees of beliefs are confined to finite domains. $\mathcal{OBL}$ has been extended to probabilistic actions and changes [9, 27, 28]. We remark that all of the works above either do not include the notion of only-believing (only-knowing) or solely consider the discrete degree of beliefs.

There are also works that axiomatize the degree of beliefs. Perhaps the most well-known work is the BHL logic [1] which extends the epistemic situation calculus by Scherl and Levesque [36]. The idea is to use a special fluent $p(s, s')$ to denote the weighted accessibility relation among situations $s, s'$. Thereafter the degree belief of a formula at a situation $s$ among to the normalized sum of weights of all the situations that are considered to be accessible from the situation $s$, i.e. $bel(\phi, s) := \frac{1}{\eta} \sum_{\{s'|\phi[s']\}} p(s, s')$. The summation there is defined axiomatically by using second-order quantification. The main advantage of a logical account like BHL is that it allows a specification of beliefs that can be partial or incomplete, in keeping with whatever information is available about the domain, making it particularly attractive for general-purpose high-level programming [24]. Note that our logic also allows a partial specification of beliefs, though we only consider the static cases. Following the BHL logic, Belle and Levesque [9] proposed a variant that deals with continuous degrees of beliefs. But they impose the nullary fluent assumption and use axioms on situations to ensure that the integral over situations can be shifted to an integral over $\mathbb{R}^k$. Compared with the modal approaches (including our $\mathcal{OBL}c$ logic), such axiomatic proposal suffers when proving properties about modalities [23]. Another benefit of our modal logic $\mathcal{OBL}c$ is that we do not have to perform explicit summation or integral over possible worlds. Hence the nullary fluent assumption is unnecessary, making our logic significantly more general.

Given the interest in unifying logic and probability, there is an extensive list of related work (see [9] for a recent review) but very few of them are closely related to ours, apart from the works discussed above. Probabilistic models, Kalman filters, decision-theoretic and probabilistic planning languages are not logics of beliefs (in allowing for arbitrary connectives, nested modalities, and quantifiers) [35, 43, 44]. Relational probabilistic models [12] offer some logical features (such as clausal reasoning) but are not embedded in a general first-order modal setting, allowing one to reason about arbitrary sequences of quantifies and modalities. The closest ones, therefore, are from the knowledge representation literature. Early proposals such as Nielsen Bacchus and halpern [2, 15, 31] are either propositional or did not consider meta beliefs. Likewise works such as probabilistic epistemic dynamic logic [22] are propositional and discrete but they consider actions that we will investigate in future work. Lastly, there is some interesting related work from the machine learning community [11, 14, 20, 29, 30, 37] particularly on defining countably infinite random variables to handle incoming unseen data. However, these methods focus on providing a careful set of conditions under which distributions can be defined and they're also not a logic for reasoning about mental beliefs.

## 8 CONCLUSION

We propose a modal logic of only-believing with continuous probability. Drawing ideas from existing works on discrete degrees of beliefs [4, 5, 9], our logic has fairly succinct semantics that does not impose any restriction on possible worlds going beyond existing works like [9, 27, 28]. We show that our logic has reasonable properties. Lastly, we conclude with some thoughts about how to extend this logic to account for actions and what challenges might arise.

In terms of future work, as mentioned earlier, an extension to account for stochastic actions and noisy sensing is desirable. In this regard, the work [4, 27, 28] is relevant. Also, extending the formalism to multi-agent scenarios is a promising venue [3, 19]. Another direction is on the reasoning side. Although reasoning in our logic is generally undecidable, it would be desirable to find out fragments of the logic where reasoning is tractable.

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