Optimal Coalition Structures for Probabilistically Monotone Partition Function Games

JAAMAS Track

Shaheen Fatima
Loughborough University
Loughborough, United Kingdom
s.s.fatima@lboro.ac.uk

Michael Wooldridge
Oxford University
Oxford, United Kingdom
Michael.Wooldridge@cs.ox.ac.uk

ABSTRACT

We define *probabilistically monotone partition function games*, a subclass of the well-known *partition function games* in which we introduce uncertainty. We provide a constructive proof that an exact optimum can be found using a greedy approach, present an algorithm for finding an optimum, and analyze its time complexity.

KEYWORDS

AI; MAS; Coalition games; Externalities; Optimal coalition structure

ACM Reference Format:
Shaheen Fatima and Michael Wooldridge. 2023. Optimal Coalition Structures for Probabilistically Monotone Partition Function Games: JAAMAS Track. In *Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), London, United Kingdom, May 29 – June 2, 2023*, IFAAMAS, 3 pages.

1 INTRODUCTION

Existing solutions for the *coalition structure generation* (CSG) problem [3] for *partition function games* (PFGs) were devised either by placing constraints on externalities or else on the function that maps coalition structures to values. A common feature of existing work is that it is focussed on games whose properties are known with certainty (we will call such games *deterministic*). However, stochasticity is inherent to many multi-agent settings. Given this, the goal of our present work is to investigate how to solve the CSG problem for *stochastic environments* in which some aspects of the problem are not known with certainty.

To this end, we build on our prior work [1] in which we considered the CSG problem for PFGs with priority ordered players and a restricted class of value functions, viz., those that satisfy a certain monotonicity property, and devised a polynomial time solution. In this previous work, the notion of monotonicity was deterministic in the sense that, with probability one, the function that maps coalition structures to values satisfies monotonicity. In this paper, we relax the deterministic monotonicity assumption by allowing a certain degree of non-monotonicity. Specifically, we replace the deterministic monotonicity restriction by *probabilistic monotonicity*. Probabilistic monotonicity means that the value function obeys monotonicity with a certain probability $0 < p \leq 1$ (for the deterministic case $p = 1$). For probabilistically monotone PFGs with priority ordered players, we devise an algorithm for optimally solving the CSG problem and characterize its time complexity.

Proc. of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2023), A. Ricci, W. Yeoh, N. Agmon, B. An (eds.), May 29 – June 2, 2023, London, United Kingdom.

2 BACKGROUND

There is a finite, non-empty set of *players* $N = \{1, \ldots, n\}$. The term *coalition* refers to a non-empty subset of N. The symbol C possibly with sub/superscripts denotes a coalition and $\mathcal{C}$ denotes the set of all coalitions of N.

$$\mathcal{C} = \{C \mid C \subseteq N, C \neq \emptyset\}$$

A *coalition structure* is an exhaustive partition of a set of players into mutually disjoint coalitions. Formally:

Definition 2.1. For any coalition C, let Π^C denote the set of all coalition structures over C. Then $\{C_1, C_2, \ldots, C_k\} \in \Pi^C$ iff

$$\cup_{i=1}^{k} C_i = C, \quad \forall i\; C_i \neq \emptyset, \quad \text{and} \quad \forall i \forall j \neq i\; C_i \cap C_j = \emptyset.$$

The symbol π possibly with sub/superscripts will denote a coalition structure. An *embedded coalition* is a coalition together with a specification of how the non-members are organised into coalitions. It is formally defined as follows:

Definition 2.2. Let $\mathcal{E}$ denote the set of all embedded coalitions. Then

$$\mathcal{E} = \{(C, \pi) \mid C \in \pi \in \Pi^N\}$$

Definition 2.3. A characteristic function game (CFG) is a pair (N, v_1) where $v_1 : 2^N \to \mathbb{R}$ and 2^N denotes the set of all subsets of N. A PFG is a pair (N, v_2) where $v_2 : \mathcal{E} \to \mathbb{R}$.

Thus CFGs are a subclass of PFGs.

Definition 2.4. The value of a coalition structure over N is given by an objective function $v : \Pi^N \to \mathbb{R}$.

In the literature, the function that maps coalition structures to values, i.e., the *objective function* v, is a social welfare function. It is commonly assumed to be the sum of coalition values. In the proposed model, however, the value of a structure does not have to be the sum of the values of its coalitions but could be any function. The CSG problem then is to find an *optimal structure*, i.e., a structure $\mathbb{O}$ such that $v(\mathbb{O})$ is the highest between all coalition structures.

3 COALITION STRUCTURE GENERATION

Let $N = \{1, \cdots, n\}$ be the set of players and $\Delta \subseteq N$ be the priority ordered ones with $\delta = |\Delta|$. Let $\mathbb{P}_i$ denote the ith priority player. Any coalition that contains at least one priority player is called a priority coalition. An ordering over Δ induces an ordering over the priority coalitions: they are ordered as per the priorities of their highest priority members.

To measure the distance between any two structures π^1 and π^2 over N, we define a metric d in terms of the positions of the priority players, i.e., in terms of the restriction of π^1 and π^2 to Δ.

$$d(\pi^1_{|\delta}, \pi^2_{|\delta}) \quad = \quad \delta - \max_{1 \le i \le \delta} \{\pi^1_{|i} = \pi^2_{|i}\} \tag{1}$$

Let $\Pi_{\mathbb{O}}$ denote the set of all optimal coalition structures and $\mathbb{S}$ be the set of all ordered pairs of coalition structures over N. For a game of n players, $|\mathbb{S}| = Bell(n) \times (Bell(n) - 1)$. Then, for a player ordering, deterministic monotonicity and probabilistic monotonicity of an objective function v are defined as follows.

Definition 3.1. **Deterministic monotonicity**: For any two structures π^1 and π^2, and a unique optimum $\mathbb{O}$:

$$d(\mathbb{O}, \pi^1) < d(\mathbb{O}, \pi^2) \quad \Rightarrow \quad v(\pi^1) > v(\pi^2) \tag{2}$$

with **probability one**. □

Definition 3.2. **Probabilistic monotonicity**: For a random pair $(\pi^1, \pi^2) \in \mathbb{S}$ such that $\pi^1 \notin \Pi_{\mathbb{O}}$ and $\pi^2 \notin \Pi_{\mathbb{O}}$, and any optimal structure $\mathbb{O} \in \Pi_{\mathbb{O}}$

$$d(\mathbb{O}_{|\delta}, \pi^1_{|\delta}) < d(\mathbb{O}_{|\delta}, \pi^2_{|\delta}) \quad \Rightarrow \quad v(\pi^1) > v(\pi^2) \tag{3}$$

with a certain probability. Specifically, for a random pair $(\pi^1, \pi^2) \in \mathbb{S}$ such that neither π^1 nor π^2 is an optimum, the probability that $v(\pi^1) > v(\pi^2)$ conditional on $d(\mathbb{O}_{|\delta}, \pi^1_{|\delta}) < d(\mathbb{O}_{|\delta}, \pi^2_{|\delta})$ is

$$p\Big(v(\pi^1) > v(\pi^2) \ \mid \ d(\mathbb{O}_{|\delta}, \pi^1_{|\delta}) < d(\mathbb{O}_{|\delta}, \pi^2_{|\delta})\Big) \quad \le \quad 1$$

For a random pair $(\pi^1, \pi^2) \in \mathbb{S}$ such that any one element of the pair, say π^1, is an optimum, $v(\pi^1) > v(\pi^2)$ with probability 1. □

Definition 3.3. A *partition function game* (N, v_2, v) is probabilstically monotonic if v is probabilistically monotone for some player ordering. □

Let the set $\mathbb{A}$ be defined as follows:

$$\mathbb{A} = \{(u, v) \mid u \in \{<, =, >\} \text{ and } v \in \{<, =, >\}\}.$$

Let the functions $f : \mathbb{S} \rightarrow \mathbb{A}$, $r^d : \mathbb{S} \rightarrow \{<, =, >\}$, and $r^v : \mathbb{S} \rightarrow \{<, =, >\}$ be defined as follows. For any $(x, y) \in \mathbb{S}$, $f(x, y) = (r^d(x, y), r^v(x, y))$ where

$$r^d(x, y) = \begin{cases} < & \text{if } \ d(\mathbb{O}_{|\delta}, x_{|\delta}) < d(\mathbb{O}_{|\delta}, y_{|\delta}) \\ = & \text{if } \ d(\mathbb{O}_{|\delta}, x_{|\delta}) = d(\mathbb{O}_{|\delta}, y_{|\delta}) \\ > & \text{if } \ d(\mathbb{O}_{|\delta}, x_{|\delta}) > d(\mathbb{O}_{|\delta}, y_{|\delta}) \end{cases}$$

$$r^v(x, y) = \begin{cases} < & \text{if } \ v(x) < v(y) \\ = & \text{if } \ v(x) = v(y) \\ > & \text{if } \ v(x) > v(y) \end{cases}$$

The set $\mathbb{S}$ can be partitioned into nine pairwise disjoint subsets as follows:

(1) $\mathbb{S}_{ee} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (=, =)\}$
(2) $\mathbb{S}_{eg} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (=, >)\}$.
(3) $\mathbb{S}_{el} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (=, <)\}$
(4) $\mathbb{S}_{ge} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (>, =)\}$
(5) $\mathbb{S}_{gg} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (>, >)\}$
(6) $\mathbb{S}_{gl} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (>, <)\}$
(7) $\mathbb{S}_{le} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (<, =)\}$
(8) $\mathbb{S}_{lg} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (<, >)\}$

Algorithm 1 Computes optimal coalition structure
Input: $N = \{1, \ldots, n\}$, r^v, δ
Output: $\mathbb{P}$ and $\mathbb{O}$

```
Initialize Z (as described in [2])
while |Z| > 2 do
    Consider any two elements of Z and choose a test Ta (where a ∈ {1, 2, 3}) to do
    Generate 3𝔹 arbitrary elements of U_Ta (as described in [2])
    Compute MSR_Ta (as described in [2])
    Perform eliminations from Z (as described in [2])
end while
ℙ1 ← x, ℙ2 ← y, 𝕆[1] ← 1, 𝕆[2] ← z        ▹ where Z = {(x, y, 1, z)}
for k ← 3, δ do
    Initialize V(k)
    while |V(k)| > 1 do
        Consider any two elements of V(k) and do the test T4
        Generate 3𝔹 arbitrary elements of U_T4
        Compute MSR_T4 (as described in [2])
        Perform eliminations from V(k) (as described in [2])
    end while
    ℙk ← a, 𝕆[k] ← b,        ▹ where (a, b) ∈ V(k) is what remains in V(k) after eliminations
end for
𝕆 ← ExhaustiveSearch(𝕆^E_|δ)
return (ℙ1, ..., ℙδ, 𝕆)
```

(9) $\mathbb{S}_{ll} = \{(x, y) \ \mid \ (x, y) \in \mathbb{S} \text{ and } f(x, y) \text{ is } (<, <)\}$

Then $\mathbb{S}$ is the union of these nine subsets:

$$\mathbb{S} = \mathbb{S}_{ee} \cup \mathbb{S}_{eg} \cup \mathbb{S}_{el} \cup \mathbb{S}_{ge} \cup \mathbb{S}_{gg} \cup \mathbb{S}_{gl} \cup \mathbb{S}_{le} \cup \mathbb{S}_{lg} \cup \mathbb{S}_{ll}. \tag{4}$$

Between all these nine subsets, only $\mathbb{S}_{ee}$, $\mathbb{S}_{eg}$, $\mathbb{S}_{el}$, $\mathbb{S}_{gl}$, and $\mathbb{S}_{lg}$ satisfy monotonicity. The union of these is denoted $\mathbb{S}_{\text{MON}}$:

$$\mathbb{S}_{\text{MON}} = \mathbb{S}_{ee} \cup \mathbb{S}_{eg} \cup \mathbb{S}_{el} \cup \mathbb{S}_{gl} \cup \mathbb{S}_{lg}. \tag{5}$$

Definition 3.4. Each pair in $\mathbb{S}_{\text{MON}}$ is called a *monotonicity-satisfying pair.*

Definition 3.5. **Degree of non-monotonicity**: The degree of non-monotonicity D is the sum of the cardinalities of $\mathbb{S}_{ge}$, $\mathbb{S}_{gg}$, $\mathbb{S}_{le}$, and $\mathbb{S}_{ll}$.

$$D = |\mathbb{S}_{ge}| + |\mathbb{S}_{gg}| + |\mathbb{S}_{le}| + |\mathbb{S}_{ll}| \qquad \square$$

CSG problem: For a probabilistically monotone PFG (N, v_2, v) with a bounded degree of non-monotonicity, find the identities of $\mathbb{P}_1, \ldots, \mathbb{P}_\delta$ and an optimal structure $\mathbb{O}$ where

$$\mathbb{O} \in \{\pi \ \mid \ \arg\max_{\pi \in \Pi^N} \ v(\pi)\}$$

given as input N and the function r^v.

The complete CSG method is summarised as Algorithm 1. Here

$$Z = \{(x, y, 1, z) \ \mid \ x \in N, \ y \in N - \{x\}, \ z \in \{1, 2\}\}$$

$$V(k) = \{(x, y) \mid x \in Q(k), \ y \in \{1, \ldots, \alpha(k) + 1\}\}$$

$\alpha(k)$ is the number of coalitions in $\mathbb{O}_{|k}$, $Q(k) = N - \{\mathbb{P}_1 \cdots \mathbb{P}_k\}$, and $\mathbb{O}^E_{|\delta}$ is the set of structures whose k element prefix is $\mathbb{O}_{|k}$.

REFERENCES

[1] S. Fatima and M. Wooldridge. 2019. Computing optimal coalition structures in polynomial time. *Journal of Autonomous Agents and Multiagent Systems* 33 (2019), 35–83. Issue 1-2.

[2] Shaheen Fatima and Michael Wooldridge. 2022. Optimal coalition structures for probabilistically monotone partition function games. *Autonomous Agents and Multi-Agent Systems* 36, 2 (2022). https://doi.org/10.1007/s10458-022-09555-9.

[3] T. Rahwan, T. Michalak, M. Wooldridge, and N. Jennings. 2015. Coalition structure generation: A survey. *AI Journal* 229 (2015), 139–174.