

Maximising the Influence of Temporary Participants in Opinion Formation

Zhiqiang Zhuang
Tianjin University
Tianjin, China
zhuang@tju.edu.cn

Kewen Wang
Griffith University
Brisbane, Australia
k.wang@griffith.edu.au

Zhe Wang
Griffith University
Brisbane, Australia
zhe.wang@griffith.edu.au

Junhu Wang
Griffith University
Brisbane, Australia
j.wang@griffith.edu.au

Yinong Yang
Liaoning University
Shenyang, China
yangyinong@lnu.edu.cn

ABSTRACT

DeGroot-style opinion formation presumes a continuous interaction among agents of a social network. Hence, it cannot handle agents external to the social network that interact only temporarily with the permanent ones. Many real-world organisations and individuals fall into such a category. For instance, a company tries to persuade as many as possible to buy its products and, due to various constraints, can only exert its influence for a limited amount of time. We propose a variant of the DeGroot model that allows an external agent to interact with the permanent ones for a preset period of time. We obtain several insights on maximising an external agent's influence in opinion formation by analysing and simulating the variant.

KEYWORDS

Opinion Formation; DeGroot Model; Social Network

ACM Reference Format:

Zhiqiang Zhuang, Kewen Wang, Zhe Wang, Junhu Wang, and Yinong Yang. 2024. Maximising the Influence of Temporary Participants in Opinion Formation. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 7 pages.

1 INTRODUCTION

Consider a social network in which people have an opinion about the state of something in the world, such as the willingness to buy a product, the effectiveness of a public policy, or the reliability of an economic forecast. Rather than forming opinions on their own, people tend to learn about the state of the world via observation and communication with others. Mathematical models of *opinion formation* try to formalize these interactions by describing how people process the other's opinions and how their opinions evolve as a result of the interactions [18].

The *DeGroot model* [11] is a benchmark opinion formation model that has found usage in many disciplines. The model describes a

discrete-time opinion formation process in which agents within a social network have an initial opinion that they update by repeatedly taking a weighted average of their friends' opinions. Over the years, some highly influential variants of the DeGroot model have been proposed to take into account real-world situations that were neglected in the original model such as the *Friedkin and Johnson model* [15] and the *bounded confidence model* [10, 17]. A recurrent topic in DeGroot-style opinion formation is the identification of conditions for reaching a consensus and the quantification of individual influence in forming the consensus [24].

Implicit in the DeGroot model and its variants is the assumption that the agents of a social network interact continuously where no agent skips any interaction at any time. This is a reasonable assumption, given the dynamic nature of opinion formation and the research focuses on its limiting behaviour. However, it excludes external agents that do not have a permanent presence in the social network but may have a considerable influence to the permanent agents. A prominent example is an organisation trying to persuade people to for instance buy its products or vote for a particular candidate through advertising. Due to constraints like budget and timing, the organisation can advertise or exert their influence only for a limited amount of time, nevertheless, for some people, the organisation's influence is at least comparable to that of their friends in the social network.

Traditionally, agents who refused to be influenced by others are termed as *stubborn* agents (also called *zealots* [22] or *radicals* [25]) [24]. Assuming a permanent presence, eventually, they make every non-stubborn agent submissive to their opinions. The aforementioned external agents are also stubborn and have uncompromising opinions, however, a sharp difference with orthodox modelling is their temporary nature. To the best of our knowledge, we are the first to consider stubborn agents without a permanent presence. This setting not only is more realistic but also opens up new perspectives to investigate the behaviours of such agents. For instance, with a limited time frame to exert influence, these external agents are confronted with the strategic problem of how to allocate their resource to achieve maximum influence.

Our goal in this paper is twofold. Firstly, we propose a variant of the DeGroot model that allows an external (stubborn) agent to participate only temporarily. The variant enables the investigation of how combinations of the following four factors affect the external



This work is licensed under a Creative Commons Attribution International 4.0 License.

agent’s influence. We illustrate them in the context of a company promoting its products through TV commercials.

Coverage: the number of agents to which the external agent can exert its influence. This is the expected number of viewers of the TV commercial each time it is broadcast.

Duration: the number of times the external agent can exert its influence. This is the expected number of times the TV commercial is broadcast.

Intensity: the amount of influence the external agent can exert on other agents each time it does so. This reflects the TV commercial’s impact on its viewers each time they see it.

Timing: the time points at which the external agent exerts its influence. This reflects the time points at which the TV commercial is broadcast.

Secondly, we articulate insights on how to allocate the external agents’ resources to the four factors to maximise their influence through mathematical analysis and computer simulation. According to our analysis and simulations, the timing factor is irrelevant if the coverage factor is at its maximum; the coverage and the duration factor are equally important; and it is more effective to allocate resource to scale up the intensity factor than to scale up the duration factor. We also derive several other insights that deepen our understanding of opinion formation in general.

After giving some preliminaries, we present our model that incorporates the above factors into the opinion formation process. This is followed by insights obtained by analytical methods and subsequently simulations.

2 PRELIMINARIES

In this paper, we write matrices as uppercase letters in boldface such as \mathbf{T} , vectors as lowercase letters in boldface such as \mathbf{p} , and scalars as lowercase letters such as m . We denote the set of real numbers and integers as \mathbb{R} and \mathbb{Z} respectively. The entry in the i -th row and j -th column of a matrix \mathbf{T} is denoted as t_{ij} . The *transpose* of a matrix \mathbf{T} is written as \mathbf{T}^\top . A matrix is *non-negative* if all its entries are non-negative. A non-negative matrix \mathbf{T} is *stochastic* if all its rows sum to 1, that is $\sum_j t_{ij} = 1$ for all i . Vectors are considered single-column matrices unless otherwise specified. The i th component of a vector \mathbf{p} is denoted as p_i . The “zero” vector $(0, \dots, 0)^\top$ and the “one” vector $(1, \dots, 1)^\top$ are denoted as $\mathbf{0}$ and $\mathbf{1}$ respectively and are with dimensions suitable to the context they appear.

A directed graph is a pair (V, E) where V is the set of *nodes* and $E \subseteq V \times V$ the set of *edges*. In opinion formation models, a directed graph (V, E) is often identified by its adjacency matrix \mathbf{T} which is a non-negative matrix such that $(i, j) \in E$ iff $t_{ij} > 0$. *Paths*, *cycles* and their *lengths* in a directed graph are defined in the standard way. A directed graph or equivalently an adjacency matrix is *strongly connected* if there is a path from any node to any other node and it is *aperiodic* if the greatest common divisor of the lengths of its cycles is one.

3 MODELS OF OPINION FORMATION

In this section, we present our opinion formation model. It involves n permanent agents interacting continuously and an external agent interacting temporarily with the permanent ones. Outside the external agent’s period of interaction, our model reduces to the DeGroot

Model in which the pattern of interaction is represented as a $n \times n$ stochastic matrix \mathbf{T} . An entry t_{ij} of \mathbf{T} represents the weight agent i places on agent j . The weights t_{ij} for $j = 1, 2, \dots, n$ are considered as finite resources, distributed by i to itself and others. A positive t_{ij} indicates j is able to influence i at each round of interaction; and the greater t_{ij} is, the stronger the influence. We refer to \mathbf{T} as the *interaction matrix* which can be seen as the adjacency matrix that captures the social network structure. Conventionally, opinions are represented as real numbers and time is measured in rounds of interactions. We denote agent i ’s opinion and the vector of the n agents’ opinions after the t th round of interaction as $p_i^{(t)}$ and $\mathbf{p}^{(t)}$ respectively. Moreover, we refer to $p_i^{(0)}$ as the *initial opinion* of i and $\mathbf{p}^{(0)}$ the *initial opinion vector*. The agents repeatedly interact by synchronously taking weighted averages of the opinions of agents who can influence them, that is $\mathbf{p}^{(t)}$ obeys the following equation for $t \geq 1$

$$\mathbf{p}^{(t)} = \mathbf{T}^t \mathbf{p}^{(0)}. \quad (1)$$

The weight t_{ij} is therefore the contribution of j ’s opinion at each round of interaction to i ’s opinion at the next round. The weight t_{ii} agent i places on itself represents its openness to other agent’s influence: $t_{ii} = 0$ indicates an open-minded agent who totally relies on the others’ opinions whereas $t_{ii} = 1$ indicates a *stubborn* agent whose opinion remains unchanged.

An opinion formation (as described by Equation (1)) is *convergent* if

$$\mathbf{p}^\infty = \lim_{t \rightarrow \infty} \mathbf{T}^t \mathbf{p}^{(0)}$$

exists for any $\mathbf{p}^{(0)}$. A convergent opinion formation reaches a *consensus* if all components of \mathbf{p}^∞ are identical, which happens when all rows of $\lim_{t \rightarrow \infty} \mathbf{T}^t$ are identical. We refer to \mathbf{p}^∞ as the *limiting opinion vector* and p_i^∞ as the limiting opinion of i . It is shown that if the interaction matrix is strongly connected, then a convergent opinion formation always reaches a consensus. Moreover, an opinion formation with a strongly connected interaction matrix \mathbf{T} is convergent iff \mathbf{T} is aperiodic or equivalently there is a unique left eigenvector \mathbf{s} of \mathbf{T} , corresponding to eigenvalue 1 such that $\sum_{i=1}^n s_i = 1$ and

$$p_i^\infty = \left(\lim_{t \rightarrow \infty} \mathbf{T}^t \mathbf{p} \right)_i = \mathbf{s} \mathbf{p}^{(0)} \quad (2)$$

for $i = 1, 2, \dots, n$. The result establishes whether an opinion formation converges and what it converges to when it does. Also the result implies $\lim_{t \rightarrow \infty} \mathbf{T}^t$ is a matrix with identical rows each of which is the unique left eigenvector \mathbf{s} . As p_i^∞ are identical for $i = 1, 2, \dots, n$, we refer to all of them as the *limiting opinion*. See the survey [24] for the other convergence and consensus conditions and [23] for the technical details on the matrix.

According to Equation (2), the limiting opinion is a weighted average of the initial opinions where agent i ’s weight is s_i . These weights are commonly taken as the measure of an agent’s influence in a DeGroot-style opinion formation and are sometimes referred to as the agents’ *social influence* [11, 18]. We refer to \mathbf{s} as the *social influence vector*. Note that, \mathbf{s} being a left eigenvector of \mathbf{T} with an eigenvalue of 1 means $\mathbf{s}\mathbf{T} = \mathbf{s}$ which implies $s_i = \sum_{j=1}^n t_{ji}s_j$. Thus the social influence of i is a weighted sum of the social influences of the various agents who can be influenced by i . This is a very natural property of a measure of influence and entails that an influential

person is one who is trusted by other influential persons. We adopt this measure of influence to quantify the external agent's influence in opinion formation.

The novelty of our model lies in the treatment of the external agent that interacts with the permanent ones for a finite number of rounds. We reserve the letter k for this finite number which indicates the *duration* of the external agent's influence. To represent the pattern of interaction involving the external agent, we extend the interaction matrix \mathbf{T} to form a $(n+1) \times (n+1)$ interaction matrix \mathbf{A} in which the external agent is identified as the $(n+1)$ th agent whose corresponding weights occupy the $(n+1)$ th row and column. We refer to \mathbf{A} as the *extended interaction matrix*. Since the external agent acts as an organisation or individual with the solitary goal of persuading others of its opinion, the external agent is, in our terminology, a stubborn agent, hence $a_{(n+1)(n+1)} = 1$. For the rest of the entries in the $(n+1)$ th column, $a_{i(n+1)} > 0$ means the external agent is able to influence agent i . We reserve the letter m for the number of such positive entries which indicates the *coverage* of the external agent's influence. A simplifying assumption we make is that the $m > 0$ agents place an identical weight of λ to the external agent, that is $a_{i(n+1)} = \lambda$ whenever $a_{i(n+1)} > 0$ and $i \leq n$. The weight λ indicates the *intensity* of the external agent's influence. We also assume, without loss of generality, that the m agents occupy the first m rows and columns of \mathbf{A} . We reserve the letter Λ for the vector that occupies the first n entries of the $(n+1)$ th column. Lastly, for each entry a_{ij} with $1 \leq i, j \leq n$, if $a_{i(n+1)} = 0$, then it inherits the corresponding entry t_{ij} in \mathbf{T} , otherwise it is shrunk from t_{ij} by a factor of $(1 - \lambda)$ to make \mathbf{A} stochastic. Putting these together, we have

$$\mathbf{A} = \begin{bmatrix} (1-\lambda)t_{11} & \cdots & (1-\lambda)t_{1n} & \lambda \\ \vdots & \ddots & \vdots & \vdots \\ (1-\lambda)t_{m1} & \cdots & (1-\lambda)t_{mn} & \lambda \\ t_{(m+1)1} & \cdots & t_{(m+1)n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_{n1} & \cdots & t_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

where m is the number of agents the external agent can influence, t_{ij} are entries of \mathbf{T} , and λ is the weight placed on the external agent.

The $n+1$ agents interact exactly as in the DeGroot model only that it is now governed by both \mathbf{T} and \mathbf{A} . The opinion vector $\mathbf{p}^{(t)}$ obeys Equation (1) when the external agent does not participate and the following one when it does.

$$\mathbf{p}^{(t)} = \left(\mathbf{A} \begin{pmatrix} \mathbf{p}^{(t-1)} \\ a \end{pmatrix} \right)_{1, \dots, n} \quad (3)$$

where a is the external agent's unchanged opinion. For a matrix \mathbf{W} , we denote the matrix formed by the first n rows of \mathbf{W} as $(\mathbf{W})_{1, \dots, n}$. For example, $(\begin{smallmatrix} a & b \\ c & d \\ e & f \end{smallmatrix})_{1,2} = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$. To illustrate this interaction, suppose the external agent participates in the third and fourth rounds of interaction, then

$$\mathbf{p}^{(4)} = \left(\mathbf{A}^2 \begin{pmatrix} \mathbf{T}^2 \mathbf{p}^{(0)} \\ a \end{pmatrix} \right)_{1, \dots, n}$$

and $\mathbf{p}^\infty = \lim_{t \rightarrow \infty} \mathbf{T}^t \mathbf{p}^{(4)}$.

Following Equation (2), after expressing the limiting opinion as the weighted average of the $n+1$ agents' initial opinions, that is

$$p_i^\infty = w_1 p_1^{(0)} + w_2 p_2^{(0)} + \cdots + w_n p_n^{(0)} + w_{(n+1)} a \quad (4)$$

we take weight $w_{(n+1)}$ as the external agent's social influence. Due to the external agent's intervention, the social influence vector \mathbf{s} no longer gives accurate social influence for the permanent agents. It does so if the external agent did not participate at all in which case our model reduces to the DeGroot model.

For the rest of this paper, we assume all interaction matrices are strongly connected and aperiodic to ensure an opinion formation always reaches a consensus. Also, we assume, without loss of generality, that the external agent's fixed opinion is 1 and the permanent agents have an initial opinion of 0. Note that agents' opinions are irrelevant to their influence and we are only concerned with the external agent's influence. By this assumption, it follows from Equation (4) that the limiting opinion is precisely the external agent's social influence.

4 ANALYTICAL RESULTS

We have argued that an organisation's influencing effort depends on the coverage, duration, intensity, and timing of its influence. Our model captures these factors respectively as the number m of agents the external agent can influence; the number k of rounds of interactions the external agent participates; the weight λ that is placed on the external agent; and the time points in which the k rounds of interaction take place. Of those factors, the first three reflect the amount of resource available for the influencing effort, the more resources there are, the larger these factors' values. As organisations usually, if not always, have a finite resource, a vital question is how and when to allocate it to achieve maximum influence.

In this section, we express the external agent's influence as a function of the dependent factors and obtain influence-maximising insights by analysing the function. For the ease of presentation, we decompose the multiplication with the matrix \mathbf{A} in Equation (3) and rewrite it as

$$\mathbf{p}^{(t)} = (\mathbf{T} - \lambda(\mathbf{T})_m) \mathbf{p}^{(t-1)} + \Lambda. \quad (5)$$

Recall that we assume $a = 1$ and Λ is the n -dimensional vector of which the first m components are λ and the rest are 0. For a matrix \mathbf{T} , $(\mathbf{T})_m$ is the matrix formed by replacing all entries of \mathbf{T} with zero except those of its first m rows. For example $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})_1 = (\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix})$.

One might notice that the timing factor is intrinsically different from the other three. Apart from being unaffected by the scarcity of resources, the timing factor is not a single factor, but a collection of k factors, one for each of the k rounds of participation. It is impractical and pointless to consider all variations of the k factors, instead, we focus on three timing options that echo real-world situations: (1) the external agent participates in k rounds of interaction and it only participates after the other agents have reached a consensus; (2) the external agent participates in the first k rounds of interaction; and (3) the external agent randomly participates in k rounds of interaction according to a uniform distribution over a range of time points. We refer to them as *consensus*, *start*, and *uniform* respectively. For now, we focus on the consensus timing option. Later on, we will explain how the other two cause the explosion in the number of

variables that makes function analysis infeasible. In fact, “consensus” is the most important of the three as simulations show that it gives rise to the largest social influence.

The consensus timing option resembles the real-world situation in which an organisation always allow sufficient time for its target to thoroughly digest its influence from the previous intervention before intervening and exerting its influence again. From a function analysis perspective, it keeps the number of variables small which results in a succinct expression for the limiting opinion.

THEOREM 1. *In an opinion formation, if the external agent can influence $m \leq n$ agents and it participates in $k \geq 1$ rounds of interaction, each of which is at a time point the other agents have reached a consensus, then*

$$p_i^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$$

for all i , where $s = \sum_{i=1}^m s_i$ for s the social influence vector.

PROOF. Let $\mathbf{S} = \lim_{t \rightarrow \infty} \mathbf{T}^t$, so each row of \mathbf{S} is the social influence vector \mathbf{s} . We prove by induction on k that $\mathbf{p}^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j (s\lambda, \dots, s\lambda)^\top$ where $s = \sum_{i=1}^m s_i$. For the base case of $k = 1$. It follows from Equation (1) and (5) that

$$\begin{aligned} \mathbf{p}^\infty &= \mathbf{S}((\mathbf{T} - \lambda(\mathbf{T})_m)\mathbf{0} + \Lambda) \\ &= \mathbf{S}(\lambda, \dots, \lambda, 0, \dots, 0)^\top \\ &= (\sum_{i=1}^m s_i \lambda, \dots, \sum_{i=1}^m s_i \lambda)^\top \\ &= (s\lambda, \dots, s\lambda)^\top \end{aligned}$$

For the induction step, suppose $\mathbf{p}^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j (s\lambda, \dots, s\lambda)^\top$ for $k = l$. We need to show the equality also holds for $k = l + 1$. Let \mathbf{a} be the limiting opinion vector for when the external agent participates for l rounds of interaction. Due to the induction hypothesis $\mathbf{a} = \sum_{j=0}^{l-1} (1 - s\lambda)^j (s\lambda, \dots, s\lambda)^\top$. Then for $k = l + 1$ we have

$$\begin{aligned} \mathbf{p}^\infty &= \mathbf{S}((\mathbf{T} - \lambda(\mathbf{T})_m)\mathbf{a} + \Lambda) \\ &= \mathbf{S}\mathbf{T}\mathbf{a} - \lambda\mathbf{S}(\mathbf{T})_m\mathbf{a} + \mathbf{S}\Lambda \\ &= \mathbf{a} - \lambda\mathbf{S}(\mathbf{T}\mathbf{a})_m + \mathbf{S}\Lambda \\ &= \mathbf{a} - \lambda\mathbf{S}(\mathbf{a})_m + (s\lambda, \dots, s\lambda)^\top \\ &= \mathbf{a} - \lambda(\sum_{i=1}^m s_i)\mathbf{a} + (s\lambda, \dots, s\lambda)^\top \\ &= (1 - s\lambda)\mathbf{a} + (s\lambda, \dots, s\lambda)^\top \\ &= \sum_{j=0}^l (1 - s\lambda)^j (s\lambda, \dots, s\lambda)^\top \end{aligned}$$

This complete the induction step. Thus we have $p_i^\infty = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$ for all i . \square

Essentially, the limiting opinion is the sum of the first k terms of a geometric series with start term $s\lambda$ and constant ratio $1 - s\lambda$ where the scalar s is the sum of the first m components of the social influence vector \mathbf{s} . Since our model does not specify the identities of the m agents that can be influenced by the external agent, the same value of m can lead to different values of s . So ultimately it is the value of s rather than m that matters in determining the limiting opinion.

We can represent the limiting opinion, which is the external agent’s social influence, as a function $I : X \times Y \times Y \rightarrow \mathbb{R}$ such that

$$I(k, \lambda, s) = \sum_{j=0}^{k-1} (1 - s\lambda)^j s\lambda$$

where $X = \{x \in \mathbb{Z} \mid x \geq 1\}$ and $Y = \{y \in \mathbb{R} \mid 0 < y < 1\}$. As explained, the variable m should not appear in the function and s is the combined social influence of the m agents for when the external agent did not participate. Substituting $I(k, \lambda, s)$ with the formula for the sum of a geometric series, we have

$$I(k, \lambda, s) = 1 - (1 - s\lambda)^k \quad (6)$$

It is easy to see that $I(k, \lambda, s)$ increases as s gets larger. Since a few influential agents can have the same combined social influence as that of many less influential ones, a large value of m does not necessarily give a large value of s . This leads to our first insight through function analysis:

(A1) if the timing option is consensus, then the external agent should aim for the more influential ones to maximise its social influence.

What this analysis also tells us is that the number of agents that can be influenced by the external agent is not the most accurate measure of “coverage.” A better choice is the combined social influence of such agents.

With the function $I(k, \lambda, s)$, we note that its three variables have nothing to do with the structure of the social network, which means the latter has no impact on the external agent’s social influence. This leads to our second insight:

(A2) if the timing option is consensus, then the structure of the social network is irrelevant to the external agent’s social influence.

The insight might seem trivial, nevertheless, it is of great significance in practice. Often a social network’s structure is unknown to an external organisation, so knowing that it is irrelevant brings certainty and assurance to an organisation’s influencing effort. Hence, we consider this irrelevance as an advantage of the “consensus” timing option over the others with which the structure does matter.

With $I(k, \lambda, s)$, we also note that the extra influence accumulated by participating one more round of interaction is $I(k + 1, \lambda, s) - I(k, \lambda, s)$. Substituting $I(k + 1, \lambda, s)$ and $I(k, \lambda, s)$ with their full expression, we have

$$I(k + 1, \lambda, s) - I(k, \lambda, s) = (1 - s\lambda)^k s\lambda.$$

Since $0 < 1 - s\lambda < 1$, $I(k + 1, \lambda, s) - I(k, \lambda, s)$ decreases as k gets larger. This leads to our third insight:

(A3) if the timing option is consensus, then the external agent’s influencing effort becomes less and less effective as it participates in more rounds of interaction.

Next, we will analyse $I(k, \lambda, s)$ to decide which of the three variables s , k , and λ has a more profound effect on the external agent’s social influence. As an immediate consequence of Equation (6), the following lemma shows that $I(k, \lambda, s)$ gains the same increase by scaling up s as it does by scaling up λ .

LEMMA 1. *Let $r \in \mathbb{R}$, $r \geq 1$, $rs < 1$ and $r\lambda < 1$. Then*

$$I(k, r\lambda, s) = I(k, \lambda, rs).$$

Note that both the weight the other agents place on the external agent and the combined social influence of the agents that can be influenced by the external one are percentages, thus the precondition in Lemma 1. The lemma leads to our four insights:

(A4) if the timing option is consensus, then it is equally effective to scale up the coverage or intensity factor to maximise the external agent's social influence.

Since λ and s are equally important in determining the value of $I(k, \lambda, s)$, it remains to compare either one of them with k . The following lemma shows that $I(k, \lambda, s)$ increases more by scaling up s than it does by scaling up k .

LEMMA 2. Let $r \in \mathbb{Z}$, $r \geq 2$, $rs < 1$ and $r\lambda < 1$. Then

$$I(k, \lambda, rs) > I(rk, \lambda, s).$$

PROOF. Since, according to Equation (6), $I(k, \lambda, rs) = 1 - (1 - rs\lambda)^k$ and $I(rk, \lambda, s) = 1 - (1 - s\lambda)^{rk}$, it suffices to show $(1 - rs\lambda) < (1 - s\lambda)^r$. We will prove by induction on r . For the base case, we have $r = 2$. Then $(1 - 2s\lambda) < (1 - s\lambda)^2$ follows from $(1 - s\lambda)^2 = 1 - 2s\lambda + (s\lambda)^2$ and $s\lambda > 0$.

For the induction step, suppose $(1 - rs\lambda) < (1 - s\lambda)^r$ holds for $r = l$, we need to show it also holds for $r = l + 1$.

$$\begin{aligned} 1 - rs\lambda &= 1 - (l + 1)s\lambda \\ &= (1 - ls\lambda) - s\lambda \end{aligned}$$

and

$$\begin{aligned} (1 - s\lambda)^r &= (1 - s\lambda)^{l+1} \\ &= (1 - s\lambda)^l (1 - s\lambda) \\ &= (1 - s\lambda)^l - s\lambda(1 - s\lambda)^l. \end{aligned}$$

We have by the induction hypothesis that $(1 - ls\lambda) < (1 - s\lambda)^l$. Also since $(1 - s\lambda)^l < 1$, we have $s\lambda > s\lambda(1 - s\lambda)^l$. It follows from $(1 - ls\lambda) < (1 - s\lambda)^l$ and $s\lambda > s\lambda(1 - s\lambda)^l$ that $(1 - ls\lambda) - s\lambda < (1 - s\lambda)^l - s\lambda(1 - s\lambda)^l$ which implies $1 - rs\lambda < (1 - s\lambda)^r$. This completes the proof. \square

Note that it is meaningless to scale up the number of participation rounds by a non-integer or by the integer 1, thus the precondition of Lemma 2. The lemma leads to our fifth insight:

(A5) if the timing option is "consensus," then it is more effective to scale up the coverage or intensity factor than the duration factor to maximise the external agent's social influence.

Finally, it is not uncommon that an organisation can influence everyone in its target group. For example, this may happen if the group is relatively small with respect to the organisation's resources. In the remaining of this section, we deal with this special but realistic "full coverage" case.

In our model, full coverage means $m = n$ with which Equation (5) reduces to

$$\mathbf{p}^{(t)} = (1 - \lambda)\mathbf{T}\mathbf{p}^{(t-1)} + \Lambda \quad (7)$$

where Λ is the n -dimensional vector $(\lambda, \dots, \lambda)^\top$.

The distinguishing property of the full coverage case is that the timing factor does not play a part in determining the external agent's social influence. The key to appreciating this property is the following easily verifiable equality.

$$(1 - \lambda)\mathbf{T}(\mathbf{T}\mathbf{p}^{(t)}) + \Lambda = \mathbf{T}((1 - \lambda)\mathbf{T}\mathbf{p}^{(t)} + \Lambda)$$

By Equation (1) and (7), if the current opinion vector is $\mathbf{p}^{(t)}$ then the LHS of the equality is the opinion vector after two rounds of interaction where the external agent participates in the second round and the RHS is the opinion vector after two rounds of interaction where the external agent participates in the first round. Since this "one-step" change does not affect the opinion vector after two rounds of interaction neither does it to the limiting opinion. By realising that we can repeat this "one-step" change for any number of times to have the k rounds of participation at any k time points without affecting the limiting opinion, the property must hold. This leads to our final insight obtained through function analysis.

(A6) If the external agent can influence all agents, then the timing factor is irrelevant to its social influence.

5 SIMULATION RESULTS

Function analysis has its limits, in this section, we resort to simulations to obtain insights tied to the timing options. We think simulation might be our last resort for the start and uniform timing options. Suppose the external agent participates in the first and r th rounds where r is 2 for "start" and an arbitrary number for "uniform." Then the limiting opinion and thus the external agent's social influence can be derived as follows

$$\begin{aligned} \mathbf{p}^\infty &= \lim_{t \rightarrow \infty} \mathbf{T}^t \mathbf{p}^{(r)} \\ &= \lim_{t \rightarrow \infty} \mathbf{T}^t ((\mathbf{T} - \lambda(\mathbf{T})_m) \mathbf{p}^{(r-1)} + \Lambda) \\ &= \lim_{t \rightarrow \infty} \mathbf{T}^t ((\mathbf{T} - \lambda(\mathbf{T})_m) \mathbf{T}^{r-2} \mathbf{p}^{(1)} + \Lambda) \\ &= \lim_{t \rightarrow \infty} \mathbf{T}^t ((\mathbf{T} - \lambda(\mathbf{T})_m) \mathbf{T}^{r-2} \Lambda + \Lambda) \\ &= \lim_{t \rightarrow \infty} \mathbf{T}^t (\mathbf{T}^{r-1} \Lambda - \lambda(\mathbf{T}^{r-1})_m \Lambda + \Lambda) \\ &= 2 \lim_{t \rightarrow \infty} \mathbf{T}^t \Lambda - \lambda \lim_{t \rightarrow \infty} \mathbf{T}^t (\mathbf{T}^{r-1})_m \Lambda. \end{aligned}$$

Obviously, the matrix $(\mathbf{T}^{r-1})_m$ plays a part in determining \mathbf{p}^∞ , meaning that any variation of the social network can result in a very different social influence of the external agent. Thus, the function that expresses the social influence must consider all entries of \mathbf{T} (as well as the k timing factors). This is an overwhelmingly large number of variables for function analysis to be feasible. For "consensus" however $(\mathbf{T}^{r-1})_m$ is cancelled out in the above derivation, making it irrelevant to determining the social influence (see the proof of Theorem 1).

In this paper, all simulations are conducted with 100 agents and with 3000 rounds of interactions. Additionally, 1000 simulations are conducted for each combination of values for the intensity, duration, coverage and timing factor. Experimenting with various simulation settings shows that a larger number of agents and simulations makes no difference to the exhibited patterns, which are already evident with as little as 10 agents and 10 simulations. Moreover, the patterns do not rely on any specific factor values, though some values make them easy to visualise.

Our first set of simulations intends to disentangle the varying effects of the timing options as the duration value grows. While holding the intensity and coverage factor constant, for each timing option, we simulate our model for duration values ranging from 0 to 45 (with an increment value of 1). In Figure 1, we plot the external

agent’s average social influence (in 1000 simulations) induced by each timing option against the duration values.

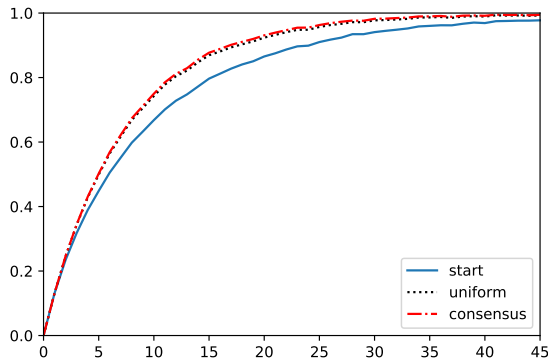


Figure 1: Comparing the timing options with respect to a growing duration factor.

Immediately we observe that the plot for “consensus” is virtually always above those of “start” and “uniform,” with a noticeable gap over the former and a tiny one over the latter. More specifically, the gap between the plots is closing towards both ends of the horizontal axis and becomes negligible at the very ends. The plots suggest “consensus” and “start” respectively gives the largest and smallest social influence while “uniform” is almost identical to “consensus.” Closing of the gap, however, cannot be interpreted as suggesting the less relevance of the timing options with small and large duration values, rather the pattern is enforced by the bounded nature of social influence values. Social influence is bounded from below by 0 and from above by 1, thus if the duration value is sufficiently small or large, then the induced social influence must be close to 0 or 1 regardless of the timing option.

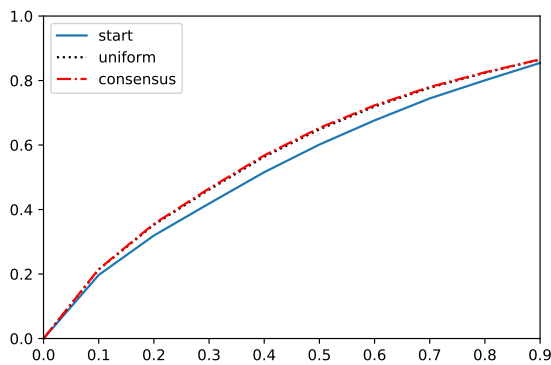


Figure 2: Comparing the timing options with respect to a growing coverage factor

In a similar fashion, while holding the duration and intensity factors constant, for each timing option, we simulate our model

for coverage values ranging from 0 to 0.9 (with an increment value of 0.1). In Figure 2, the average social influence induced by each timing option is plotted against the coverage values. Once again, we observe that “consensus” is the clear winner against “start” but not so much against “uniform”. Also, the gap is closing towards both ends of the horizontal axis. Likewise, the bounded nature of social influence plays a part in the closing of the gap. But this time, as the coverage value grows, the gap closes so drastically that the plots converge at a social influence (i.e., 0.8) well below 1. This means the growing coverage value also plays a part. So, unlike the duration value, it is righteous to interpret the closing of the gap as suggesting the larger the coverage value the less relevant the timing options. In fact, we have already proved a limiting case of this pattern in (A6) which concludes the irrelevance of the timing factor when the coverage value is at its largest. Hence, we obtain the following insight.

(S1) Once the coverage value passes certain threshold,¹ then the larger the coverage value, the less relevant the timing options in determining the external agent’s social influence.

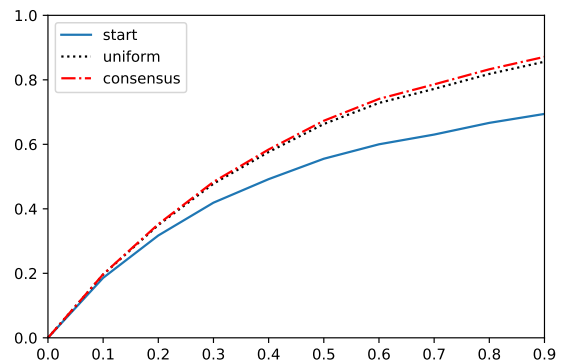


Figure 3: Comparing the timing options with respect to a growing intensity factor

Finally, we hold constant the duration and coverage factors and simulate our model for intensity values ranging from 0 to 0.9 (with an increment value of 0.1). The corresponding plots are given in Figure 3. Yet another time, we observe the superiority of “consensus” over “start” and “uniform,” but only slightly over the latter. This unequivocal pattern exhibited in all three sets of simulations leads to the following insight.

(S2) Among the three timing options, “consensus” gives rise to the largest social influence of the external agent and “start” the least.

Furthermore, the fact that “consensus” and “uniform” are almost indistinguishable indicates that the key is to spread out the participation times and avoid having them in a cluster. Last but not least, we observe that, contrary to the previous simulations, the gap between the plots enlarges towards the end of the horizontal

¹The threshold depends on the intensity and duration values.

axis with larger intensity values. Noticeably, whatever mechanism that drives this enlargement is very effective as it manages to do so even though the bounded nature of social influence acts in an opposite direction. This leads to our final insight.

(S3) *The larger the intensity factor, the more relevant the timing options in determining the external agent’s social influence.*

6 RELATED WORK

Influence maximisation is a recurring topic in studies of opinion formation and social networks. The term is often referred to as the algorithmic problem of selecting a predefined number of agents in a social network to maximise the spread of a binary opinion in an opinion diffusion process [21]. [12] is the first to pose the algorithmic problem. Later on, [19] proposed the so-called independent cascade model and gave a greedy algorithm based on submodular maximisation [19, 20]. These very influential papers have since then promoted a large number of follow-up works improving and extending various aspects of the selection techniques [6]. Although the diffusion process and representation of opinions are different from ours, these works are, in our terminology, concerning the coverage factor. Rather than focusing on this single factor, we also considered the intensity, duration and timing factors and attempt to understand how the interplay between the four factors elevates and restrains the overall influencing effort.

Apart from the above works, recent years have seen a surge in the popularity of various forms of opinion diffusion in artificial intelligence [1–5, 7–9, 13, 14, 16, 26]. Some tackled influence maximisation problems from the perspective of how the sequence of opinion updates affects the influencing effort [1, 4]. [1] considered the case of three alternative opinions in an asynchronous mode of opinion update and investigated the sequence of updates that maximises the spread of an opinion. [4] continued the effort and provided upper and lower bounds on the length of such sequences among other results.

7 CONCLUSION

In this paper, we generalised the DeGroot model of opinion formation to allow a temporary participant. We articulated four factors namely, duration, intensity, coverage and timing that dominate the temporary participant’s influencing effort and incorporated them into the opinion formation process. Through function analysis and simulation, we revealed the degree of importance and interplay between the factors which leads to crucial insights into influence maximisation. In summary, the temporary participant ought to adopt the consensus timing option and focus its resources on the intensity and coverage factor. The insights may aid organisations and individuals in making better strategic choices when facing limited resources to maximise their influence.

ACKNOWLEDGMENTS

This work was partially supported by the National Natural Science Foundation of China (NSFC) (61976153).

REFERENCES

- [1] Vincenzo Auletta, Diodato Ferraioli, Valeria Fionda, and Gianluigi Greco. 2019. Maximizing the Spread of an Opinion When Tertium Datur Est. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*. 1207–1215.
- [2] Vincenzo Auletta, Diodato Ferraioli, and Gianluigi Greco. 2020. On the complexity of reasoning about opinion diffusion under majority dynamics. *Artificial Intelligence* 284 (2020), 103–288.
- [3] Sirin Botan, Umberto Grandi, and Laurent Perrussel. 2019. Multi-Issue Opinion Diffusion under Constraints. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*. 828–836.
- [4] Robert Bredereck, Lilian Jacobs, and Leon Kellerhals. 2020. Maximizing the Spread of an Opinion in Few Steps: Opinion Diffusion in Non-Binary Networks. In *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence*. 1622–1628.
- [5] Markus Brill, Edith Elkind, Ulle Endriss, and Umberto Grandi. 2016. Pairwise Diffusion of Preference Rankings in Social Networks. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence*. 130–136.
- [6] Wei Chen, Carlos Castillo, and Lokes V. S. Lakshmanan. 2013. *Information and Influence Propagation in Social Networks*. Morgan & Claypool.
- [7] Dmitry Chistikov, Grzegorz Lisowski, Michael S. Paterson, and Paolo Turrini. 2020. Convergence of opinion diffusion is PSPACE-complete. In *Proceedings of the 34th AAAI Conference (AAAI-2020)*.
- [8] Laurence Cholvy. 2018. Opinion diffusion and influence: A logical approach. *International Journal of Approximate Reasoning* 93 (2018), 24–39.
- [9] Zoé Christoff and Davide Grossi. 2017. Binary Voting with Delegable Proxy: An Analysis of Liquid Democracy. In *Proceedings 16th Conference on Theoretical Aspects of Rationality and Knowledge*. 134–150.
- [10] Guillaume Deffuant, David Neau, Frederic Amblard, and Gérard Weisbuch. 2000. Mixing beliefs among interacting agents. *Advances in Complex Systems* 03, 01n04 (2000), 87–98.
- [11] Morris H. DeGroot. 1974. Reaching a Consensus. *J. Amer. Statist. Assoc.* 69, 345 (1974), 118–121.
- [12] Pedro Domingos and Matt Richardson. 2001. Mining the Network Value of Customers. In *Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. 57–66.
- [13] Piotr Faliszewski, Rica Gonen, Martin Koutecký, and Nimrod Talmon. 2018. Opinion Diffusion and Campaigning on Society Graphs. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence*. 219–225.
- [14] Diodato Ferraioli and Carmine Ventre. 2019. Social pressure in opinion dynamics. *Theoretical Computer Science* 795 (2019), 345–361.
- [15] Noah E. Friedkin and Eugene C. Johnsen. 1990. Social influence and opinions. *The Journal of Mathematical Sociology* 15, 3-4 (1990), 193–206.
- [16] Umberto Grandi, Emiliano Lorini, Arianna Novaro, and Laurent Perrussel. 2017. Strategic Disclosure of Opinions on a Social Network. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems*. 1196–1204.
- [17] Rainer Hegselmann and Ulrich Krause. 2002. Opinion dynamics and bounded confidence: models, analysis and simulation. *Journal of Artificial Societies and Social Simulation* 5, 3 (2002).
- [18] Matthew O. Jackson. 2008. *Social and Economic Networks*. Princeton University Press.
- [19] David Kempe, Jon Kleinberg, and Éva Tardos. 2003. Maximizing the Spread of Influence through a Social Network. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. 137–146.
- [20] David Kempe, Jon Kleinberg, and Éva Tardos. 2005. Influential Nodes in a Diffusion Model for Social Networks. In *Proceedings of the 32nd International Conference on Automata, Languages and Programming*. 1127–1138.
- [21] Yuchen Li, Ju Fan, Yanhao Wang, and Kian-Lee Tan. 2018. Influence Maximization on Social Graphs: A Survey. *IEEE Transactions on Knowledge and Data Engineering* 30, 10 (2018), 1852–1872.
- [22] Naoki Masuda. 2015. Opinion control in complex networks. *New Journal of Physics* 17, 3 (mar 2015), 033–031.
- [23] Carl D. Meyer. 2000. *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics.
- [24] Anton V. Proskurnikov and Roberto Tempo. 2017. A tutorial on modeling and analysis of dynamic social networks. Part I. *Annual Reviews in Control* 43 (2017), 65–79.
- [25] Ulrich Krause Rainer Hegselmann. 2015. Opinion dynamics under the influence of radical groups, charismatic leaders, and other constant signals: A simple unifying model. *Networks and Heterogeneous Media* 10, 3 (2015), 477–509.
- [26] Zhiqiang Zhuang, Kewen Wang, Junhu Wang, Heng Zhang, Zhe Wang, and Zhiguo Gong. 2020. Lifting Majority to Unanimity in Opinion Diffusion. In *In proceedings of the 24th European Conference on Artificial Intelligence*. 259–266.