A Comparison of the Myerson Value and the Position Value

Extended Abstract

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ABSTRACT

In the realm of graph-restricted games, the underlying network structure plays a pivotal role, enforcing a key constraint: communication between agents is only feasible if a valid path connecting them exists within the network. This constraint significantly influences the dynamics and strategies, particularly in value allocation scenarios among connected agents.

Among various contributions to the allocation rules for such network-centric scenarios, Myerson's pioneering work stands out [4, 5]. Named after him, the Myerson value represents an adaptation of the Shapley value [7]. Another prominent concept in this domain is the position value [1, 6]. Both serve as solution concepts, offering distinct perspectives, with the Myerson value focusing on agents and the position value on links between agents.

We provide an axiomatic characterization of the Myerson value based on two fundamental axioms. Expanding our investigation, a subtle modification of the first axiom leads to a characterization of the position value. This extension enables comparing these value operators, highlighting their essential distinctions and similarities.

KEYWORDS

Game Theory; Myerson Value; Position Value; Shapley Value; Allocation Rule

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1 INTRODUCTION

Unlike previous axiomatizations of the Myerson value, we do not confine ourselves to component additive value functions; instead, the value function can take any form. Our primary axiom for characterizing the Myerson value centers on a crucial principle: given a value function, when there is an epsilon increase (or decrease) of the value function at a network, say at g, and at each network containing g, then that epsilon increase (or decrease) must be distributed equally between the agents that are linked in g.

The second axiom of our characterization is a condition only on the value function where the value of each possible network is zero. It requires that if the value function is zero at any network, then each agent must be treated uniformly, receiving zero payoff at all networks. This property, named as the null-game property, reinforces fairness in such an obvious situation. We show that the Myerson value is precisely characterized by these two axioms, which are independent of each other.

The central axiom guiding our characterization of the position value is analogous to that for the Myerson value: given a value function when there is an epsilon increase (or decrease) of the value function at a network, say at g, and at each network containing g (as in the previous case), then that epsilon increase (or decrease) must be distributed equally between the links of g. Subsequently, the decided payoff for each link is distributed uniformly between the two edges forming the link. We show that the position value is characterized by this axiom and the null-game property. These two axioms are independent of each other.

Our approach employs the 'basis' concept indirectly within the axiomatization process. By utilizing it, a significant contribution of our work emerges: the central axiom for the position value aligns closely with the key axiom for the Myerson value. This alignment highlights the fundamental difference between these two allocation rules in a straightforward and natural manner.

2 PRELIMINARIES

Let $n \geq 2$. Let $N = \{1, \dots, n\}$ be a finite set of agents who are connected in some network relationship, we take N fixed.

Let g_K stand for the complete (undirected, loop free) graph with vertex set N. Any subgraph of g_K will be referred as a *network* (in a non-standard way, all N vertices are kept). Let g_0 stand for the network that has no edges, i.e. g_0 is *the null-network*, i.e. the null-graph, edgeless graph with n nodes. Let G^N denote the set of all networks with agent set N.

The vertices of a network g correspond to the agents and the edges between the agents correspond to bilateral relationship between the agents. For any $i, j \in N$, we write $l = ij$ for the edge (link) between the agents i and j. The network obtained by adding a link l to an existing network $g \in G^N$ is denoted by $g + l$.

Let $D_i(g)$ be the set of all edges which are incident to vertex i, $d_i(g)$ be the degree of vertex i in network g, and $d(g)$ be the total number of links in g.

For each $g \in G^N$, let $N(g) = \{i \in N : \exists j \text{ s.t.} ij \in g\}$, i.e, $N(g)$ is the set of agents who has at least one link in g.

To indicate a supergraph, we write $g \subseteq \overline{g}$, if $[ij \in g \Rightarrow ij \in \overline{g}]$.

A function $v : G^N \to \mathbb{R}$ where $v(g_0) = 0$ is called a *value function* for G^N. Let V^N denote the set of all value functions for G^N.

Throughout, let $v_0 \in V^N$ denote the value function that assigns zero to each network in G^N, i.e., for each $g \in G^N$, $v_0(g) = 0$.

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For each $g \in G^N \setminus \{g_0\}$, let v_g denote the value function that satisfies

$$v_g(\overline{g}) = \begin{cases} 1 & \text{if } g \subseteq \overline{g}, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that $\mathfrak{B} = \{v_g : g \in G^N \setminus \{g_0\}\}$ forms a *basis* for V^N [2].

A rule distributing the value of a network between the agents is called a allocation rule. Formally, *an allocation rule* is a function $Y : G^N \times V^N \to \mathbb{R}^N$ that assigns a payoff vector $Y(g, v)$ to each $(g, v) \in G^N \times V^N$, such that $\sum_{i \in N} Y_i(g, v) = v(g)$. The number $Y_i(g, v)$ represents the *payoff* of agent i at (g, v).

We employ the extension of the Myerson value provided in [3] as our definition of the Myerson value. The Myerson value, denoted as Y^{MV}, is defined as follows:

$$Y_i^{MV}(g, v) = \sum_{S \subseteq N \setminus \{i\}} \left(v(g|_{S \cup \{i\}}) - v(g|_S) \right) \frac{|S|!\,(n - |S| - 1)!}{n!}.$$

As an extension of a theorem by Myerson, Jackson and Wolinsky [3] demonstrate the following:"*An allocation rule satisfying component balancedness and equal bargaining power if and only if the allocation rule is equal to the Myerson value for each $g \in G^N$ and each component additive $v \in V^N$.*" In the aforementioned characterization, it's important to note a constraint on the value function-it must be component additive.

Meessen [1, 6] introduces an alternative allocation rule for network games, known as the position value, denoted asY^{PV}, defined as follows:

$$Y_i^{PV}(g, v) = \sum_{l \in D_i(g)} \frac{1}{2} \sum_{\overline{g} \subseteq g - l} \left(v(\overline{g} + l) - v(\overline{g}) \right) \frac{d(\overline{g})!(d(g) - d(\overline{g}) - 1)!}{d(g)!}.$$

3 MAIN AXIOMS

For establishing the main axioms, we initially introduce a new value function obtained from v, denoted as $v_{(g,\epsilon)}$ as follows.

Definition 3.1. For each $g \in G^N \setminus \{g_0\}$, each $v \in V^N$ and each $\epsilon \in \mathbb{R} \setminus \{0\}$, we define the value function $v_{(g,\epsilon)}$ as follows:

$$v_{(g,\epsilon)}(\overline{g}) = \begin{cases} v(\overline{g}) + \epsilon & \text{if } g \subseteq \overline{g}, \\ v(\overline{g}) & \text{otherwise,} \end{cases}$$

In other words, $v_{(g,\epsilon)} = v + \epsilon v_g$ where $v_g \in \mathfrak{B}$.

Next, we present our fundamental axioms.

Definition 3.2. An allocation rule Y satisfies *equal division between the source agents at a monotonic increment of the value function* (for short, we call *EDBA*) if for each $g \in G^N \setminus \{g_0\}$, $v \in V^N$ and $\epsilon \in \mathbb{R} \setminus \{0\}$, the following conditions hold:

(a) for each $\overline{g} \in G^N$ such that $\overline{g} \supseteq g$,

$$Y_i(\overline{g}, v_{(g,\epsilon)}) = \begin{cases} Y_i(\overline{g}, v) + \frac{\epsilon}{|N(g)|} & \text{if } i \in N(g), \\ Y_i(\overline{g}, v) & \text{otherwise,} \end{cases}$$

(b) for each $\overline{g} \in G^N$ such that $\overline{g} \not\supseteq g$,

$$Y_i(\overline{g}, v_{(g,\epsilon)}) = Y_i(\overline{g}, v),$$

for each $i \in N$.

As per EDBA, if a change in the value function occurs at g and at each supergraph of g, the source of the change is g, thus, this change must be be equally distributed among all agents having at least one link in g.

Definition 3.3. An allocation rule Y satisfies *equal division between the source links at a monotonic increment of the value function* (for short, we call *EDBL*) if for each $g \in G^N \setminus \{g_0\}$, $v \in V^N$ and $\epsilon \in \mathbb{R} \setminus \{0\}$, the following conditions hold:

(a) for each $\overline{g} \in G^N$ such that $\overline{g} \supseteq g$,

$$Y_i(\overline{g}, v_{(g,\epsilon)}) = \begin{cases} Y_i(\overline{g}, v) + \frac{\epsilon d_i(g)}{2d(g)} & \text{if } i \in N(g), \\ Y_i(\overline{g}, v) & \text{otherwise.} \end{cases}$$

(b) for each $\overline{g} \in G^N$ such that $\overline{g} \not\supseteq g$,

$$Y_i(\overline{g}, v_{(g,\epsilon)}) = Y_i(\overline{g}, v),$$

for each $i \in N$.

As per EDBL, in a situation where there is a change of the value function at g and at each supergraph of g, the network g is the source of the change, thus, this change must be distributed equally between the links of g, and then equally distributed among the agents who own these links.

Definition 3.4. We say that an allocation rule Y satisfies *null-game property* if for each $g \in G^N$ and each $i \in N$, $Y_i(g, v_0) = 0$.

4 CHARACTERIZATIONS

First, we show that Myerson value satisfies EDBA. For that, we first show the following lemma.

LEMMA 4.1. *For any $n, m \in \mathbb{Z}$, $n \geq 2$ and $0 \leq m \leq n - 2$,*

$$\sum_{k=0}^{m} \binom{n - m + k}{k} \frac{1}{n - m + k} = \binom{n}{n - m} \frac{1}{n - m}.$$

The proof of Lemma 4.1 follows by induction on n, or alternatively, employing binomial coefficients and Pascal's rule.

THEOREM 4.2. *There is a unique allocation rule that satisfies EDBA and null-game property, namely Y^{MV}.*

Sketch of the proof: It is straightforward to verify that Y^{MV} satisfies null-game property. By employing a proof by cases argument and using Lemma 4.1, we show that Y^{MV} satisfies EDBA.

Conversely, we show that there is a unique allocation rule that satisfies EDBA and null-game property. The proof utilizes the fact that $\mathfrak{B}$ serves as a basis for V^N, employing a recursive argument to establish the uniqueness of Y. Given that Y^{MV} satisfies EDBA and null-game property, we conclude that Y is indeed equal to Y^{MV}, which completes the proof.

In a similar way, we give a characterization of the position value.

THEOREM 4.3. *There is a unique allocation rule that satisfies EDBL and null-game property, namely Y^{PV}.*

The proof follows a similar structure to that of Theorem 4.2. It is straightforward to verify that Y^{PV} satisfies null-game property. By employing a proof by cases argument and using Lemma 4.1, we show that Y^{PV} satisfies EDBL.

For the converse, the proof technique is exactly as in the proof of Theorem 4.2. Assuming Y is an allocation rule satisfying EDBL and null-game property, as in the proof of Theorem 4.2, Y is uniquely determined. Given that Y^{PV} satisfies EDBL and null-game property, we conclude that Y is indeed equal to Y^{PV}, completing the proof.

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