

# The Parameterized Complexity of Welfare Guarantees in Schelling Segregation

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## ABSTRACT

Schelling’s model considers  $k$  types of agents each of whom needs to select a vertex on an undirected graph, where every agent prefers neighboring agents of the same type. We are motivated by a recent line of work that studies solutions that are optimal with respect to notions related to the welfare of the agents. We explore the parameterized complexity of computing such solutions. We focus on the well-studied notions of social welfare (WO) and Pareto optimality (PO), alongside the recently proposed notions of group-welfare optimality (GWO) and utility-vector optimality (UVO), both of which lie between WO and PO. Firstly, we focus on the fundamental case where  $k = 2$  and there are  $r$  red agents and  $b$  blue agents. We show that all solution-notions we consider are intractable even when  $b = 1$  and that they do not admit an FPT algorithm when parameterized by  $r$  and  $b$ , unless  $\text{FPT} = \text{W}[1]$ . In addition, we show that WO and GWO remain intractable even on cubic graphs. We complement these negative results with an FPT algorithm parameterized by  $r, b$  and the maximum degree of the graph. For the general case with  $k$  types of agents, we prove that for any of the notions we consider the problem remains hard when parameterized by  $k$  for a large family of graphs that includes trees. We accompany these negative results with an XP algorithm parameterized by  $k$  and the treewidth of the graph.

## KEYWORDS

Schelling; Parameterized Algorithmics; Welfare Maximization

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## 1 INTRODUCTION

Residential segregation is a phenomenon that is observed in many areas around the globe. As a result of de-facto segregation, people group together forming communities based on traits such as race, ethnicity, and socioeconomic status, and residential areas become noticeably divided into segregated neighborhoods. Half a century

ago, Schelling [26] proposed a simple agent-based model to study how residential segregation emerges from individuals’ perceptions.

At a high level, Schelling’s model works as follows. There are two types of agents, say red and blue, each of whom is placed on a unique node on a graph. Agents are aware of their neighborhood; agents of the same type are considered “friends” and those of the opposite type “enemies”. Agents are indifferent to empty vertices. An agent is happy with their location if the fraction of friends in their neighborhood is at least  $\tau$ , where  $\tau \in [0, 1]$  is a tolerance parameter. Schelling proposed a random process that starts from a random initial assignment and agents who are unhappy in their current neighborhood relocate to a different, random, empty node, whilst happy agents stay put. It is expected that when agents are not tolerant towards a diverse neighborhood,  $\tau > \frac{1}{2}$ , these dynamics will converge to a segregated assignment. However, Schelling’s experimentation on grid graphs showed that even when agents are in favour of integration, i.e.  $\tau \approx \frac{1}{3}$ , the final assignment will be segregated.

Since Schelling’s model was proposed, it has been the subject of many empirical studies in sociology [11], in economics [27, 28], and more recently in computer science. For example, [2–5, 16] analyze Schelling’s model on a grid graph with its original random dynamics, as well as many variants of this random process. They show that assignments converge to large monochromatic subgraphs with a high probability, confirming Schelling’s research.

More recently, Bullinger et al. [9] studied assignments with certain *welfare guarantees* for the agents and the computational complexity of computing them. In Schelling’s model high social welfare translates to high segregation. However, there are certain scenarios where segregation essentially captures the effectiveness of an allocation of agents over a network. As an example, think of the nodes of the graph as the resources of an organization, the edges as compatibility and interference between the resources, and the types of agents as different working groups, or skilled workers. Under this point of view, “segregation” is desirable, since we have better utilization of both the available workers and resources. For this reason, the welfare guarantees studied by [9] are the focus of this paper, albeit under the prism of parameterized complexity.

In parameterized algorithmics [12], the running-time of an algorithm is studied with respect to a parameter  $k \in \mathbb{N}_0$  and input size  $n$ . The basic idea is to find a parameter that describes the structure of the instance such that the combinatorial explosion can be confined to this parameter. In this respect, the most favorable complexity class is FPT (*fixed-parameter tractable*), which contains all problems that can be decided by an algorithm running in time  $f(k) \cdot n^{O(1)}$ , where  $f$  is a computable function. Algorithms with this running-time are called *fixed-parameter (FPT) algorithms*. A less favorable,



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but still positive, outcome is an XP algorithm, which is an algorithm running in time  $O(n^{f(k)})$ ; problems admitting such algorithms belong to the class XP. Finally, showing that a problem is W[1]-hard rules out the existence of a fixed-parameter algorithm under the well-established assumption that  $W[1] \neq FPT$ .

## 1.1 Our Contributions

We explore the parameterized complexity of computing assignments for Schelling’s model that optimizes some welfare guarantee. We study four solution notions: social-welfare optimality (WO), Pareto optimality (PO), group-welfare optimality (GWO) and utility-vector optimality (UVO). We denote the problem as  $\phi$ -SCHELLING, where  $\phi \in \{WO, PO, GWO, UVO\}$ . The task is to find an assignment that satisfies notion  $\phi$  for a given Schelling instance. While WO and PO are well-studied notions in various domains, the solution concepts of GWO and UVO were proposed by Bullinger et al. [9]. There it was proven that both UVO and GWO lie between WO and PO. At a high level, an assignment is GWO if we cannot increase the total utility of one type of agents without decreasing the utility of the other type; an assignment is UVO if it is not possible to improve the sorted utility vector of the agents. While Bullinger, Suksompong, and Voudouris showed that all four notions do not admit an polynomial-time algorithm in general, unless  $P = NP$ , their parameterized complexity remained open.

We firstly focus on the fundamental case where we have two types of agents:  $r$  red agents and  $b$  blue agents. In Theorem 3 we show that  $\phi$ -SCHELLING is NP-hard (unless  $P = NP$ ) even when  $b = 1$ , for every  $\phi \in \{WO, PO, GWO, UVO\}$ . In Theorem 5 we extend this negative result and show that deciding if there exists a *perfect* assignment, i.e. an assignment where every agent has only friends as neighbors, is W[1]-hard when parameterized by  $r + b$ . This implies Corollary 6: For every  $\phi \in \{WO, PO, GWO, UVO\}$ ,  $\phi$ -SCHELLING when parameterized by  $r + b$  does not admit a fixed parameter algorithm unless  $FPT = W[1]$ . Hence, if we want to derive a positive result, we *need* to restrict the topology of the graph. In Theorem 7, we show that restricting the maximum degree of the graph does not always suffice; we prove that both WO-SCHELLING and GWO-SCHELLING remain intractable even on cubic graphs. We complement these negative results with Theorem 8; we show that  $\phi$ -SCHELLING is in FPT, for all four optimality notions, parameterized by  $r + b + \Delta$ , where  $\Delta$  denotes the maximum degree of the graph. In fact, we show that  $\phi$ -SCHELLING admits a polynomial time preprocessing algorithm, called *kernel*, that yields an instance with at most  $O(\Delta^2 \cdot r^2 \cdot b^2)$  many vertices.

Then, we turn our attention to the general case where there are *multiple* types of agents, which we denote by SCHELLINGM. In Theorem 9, we prove that deciding existence of a perfect assignment is W[1]-hard when parameterized by the number of types of agents,  $k$ , for a large family of graphs that includes trees<sup>1</sup>. The same proof also establishes that it is NP-hard when  $k$  is part of the input and not bounded by a function of the parameter for the same family of graphs. Again, we get the corresponding intractability for  $\phi$ -SCHELLINGM as corollaries, for every  $\phi \in \{WO, PO, GWO, UVO\}$ . We complement this with three positive results. In Theorem 12 we derive an XP algorithm parameterized by the number of types

<sup>1</sup>See Thm. 9 for the exact definition of this subclass of graphs.

and the treewidth of the graph. By using the same algorithm, we get Corollary 17 that shows an FPT algorithm for  $\phi$ -SCHELLINGM parameterized by the number of agents plus the treewidth of the graph. Finally, by slightly modifying this algorithm, we get Corollary 18 that shows that if the number of types is any fixed constant, then the problem of finding a perfect assignment, if one exists, admits an FPT algorithm parameterized by the treewidth.

## 1.2 Further Related Work

A different line of work studies Schelling *games*, a strategic setting of Schelling’s model. There, unhappy agents will move to different positions that maximize the fraction of friends in the neighborhood. The focus is now shifted to the existence of Nash equilibria, i.e., assignments where no agent has incentives to change their position. Agarwal et al. [1] consider *jump* Schelling games with  $k \geq 2$  types, with agents that can deviate to empty nodes in the graph and *stubborn* agents which do not move regardless of their utility. They proved NP-hardness for computing a Nash equilibrium and for WO. Also, Agarwal et al. [1] study *swap* Schelling games, where agents of different types exchange their positions if at least one of them strictly increase their utility. Again, they showed that deciding whether a Nash equilibrium exists is NP-hard. Furthermore, in order to measure the diversity in assignments, they introduced the *degree of integration* that counts the number of agents exposed to agents not of their type. They showed that computing assignments that maximize this measure is hard.

More recently, Kreisel et al. [22] answered some open questions from Agarwal et al. [1] where they proved stronger NP-hardness results for the existence of swap-equilibria and jump-equilibria. In addition, they introduced and studied two measures that capture the robustness of equilibria in Schelling games.

Bilò et al. [6] investigate the existence of equilibria via finite improvement paths on different graph classes for swap Schelling games, and study a local variant wherein agents can only swap with agents in their neighborhood. Kanellopoulos et al. [19] study price of anarchy and price of stability in *modified* Schelling games, where the agent includes herself as part of the neighborhood. They prove tight bounds on the price of anarchy for general and some specific graphs with  $k \geq 2$  and  $k = 1$ . Furthermore, there are other extensions and variations of Schelling games [7, 10, 14, 15, 20].

*Statements where proofs or details are omitted due to space constraints are marked with  $\star$ . A version containing all proofs and details is provided as supplementary material.*

## 2 PRELIMINARIES

For every positive integer  $n$ , let  $[n] = \{1, 2, \dots, n\}$ . Given two vectors  $\mathbf{x}, \mathbf{y}$  of length  $n$ , we say that  $\mathbf{x}$  *weakly dominates*  $\mathbf{y}$  if  $\mathbf{x}(i) \geq \mathbf{y}(i)$  for every  $i \in [n]$ ;  $\mathbf{x}$  *strictly dominates*  $\mathbf{y}$  if at least one of the inequalities is strict.

A *Schelling instance*  $\langle G, A \rangle$ , consists of an undirected graph  $G = (V, E)$  and a set of *agents*  $A$ , where  $|A| \leq |V|$ . Every agent has a *type*, or *color*. When there are only two colors available, we assume that  $A = R \cup B$ , where  $R$  contains red agents and  $B$  contains blue agents. We denote  $r = |R|$  and  $b = |B|$ . Agents  $i$  and  $j$  are *friends* if they have the same color; otherwise, they are *enemies*. For any agent  $i \in A$  we use  $F(i)$  to declare the set of his friends.

An assignment  $\mathbf{v} = (v(1), \dots, v(|A|))$  for the Schelling instance  $\langle G, A \rangle$  maps every agent in  $A$  to a vertex  $v \in V$ , such that every vertex is occupied by at most one agent. Here,  $v(i) \in V$  is the vertex of  $G$  that agent  $i$  occupies. For any assignment  $\mathbf{v}$  and any agent  $i \in A$ ,  $N_i(\mathbf{v}) = \{j \in A : v(i)v(j) \in E\}$  denotes the set of neighbors of  $v(i) \in V$  that are occupied under  $\mathbf{v}$ . Let  $f_i(\mathbf{v}) = |N_i(\mathbf{v}) \cap F(i)|$  and let  $e_i(\mathbf{v}) = |N_i(\mathbf{v})| - f_i(\mathbf{v})$  be respectively the numbers of neighbors of agent  $i$  who are his friends and his enemies under  $\mathbf{v}$ . The *utility* of agent  $i$  under assignment  $\mathbf{v}$ , denoted  $u_i(\mathbf{v})$ , is 0 if  $|N_i(\mathbf{v})| = 0$ , and is defined as

$$u_i(\mathbf{v}) = \frac{f_i(\mathbf{v})}{|N_i(\mathbf{v})|} = \frac{f_i(\mathbf{v})}{f_i(\mathbf{v}) + e_i(\mathbf{v})}$$

if  $|N_i(\mathbf{v})| \neq 0$ . The *social welfare* of  $\mathbf{v}$  is the sum of the utilities of all agents, formally  $SW(\mathbf{v}) = \sum_{i \in A} u_i(\mathbf{v})$ . For  $X \in \{R, B\}$  we denote  $SW_X(\mathbf{v}) = \sum_{i \in X} u_i(\mathbf{v})$ .

We use  $\mathbf{u}(\mathbf{v})$  to denote the vector of length  $|A|$  that contains the utilities of the agents under  $\mathbf{v}$ , sorted in non-increasing order. Similarly, let  $\mathbf{u}_X(\mathbf{v})$  denote the corresponding vector of utilities of the agents in  $X \in \{R, B\}$ . An assignment  $\mathbf{v}$  is *utility-vector dominated* by  $\mathbf{v}'$  if  $\mathbf{u}(\mathbf{v}')$  strictly dominates  $\mathbf{u}(\mathbf{v})$ ;  $\mathbf{v}$  is *group-welfare dominated* by  $\mathbf{v}'$  if  $SW_X(\mathbf{v}') \geq SW_X(\mathbf{v})$ , where  $X \in \{R, B\}$ , and at least one of the inequalities is strict. An assignment  $\mathbf{v}$  is:

- *welfare optimal*, denoted WO, if for every other assignment  $\mathbf{v}'$  we have  $SW(\mathbf{v}) \geq SW(\mathbf{v}')$ ;
- *Pareto optimal*, denoted PO, if and only if there is no  $\mathbf{v}'$  such that  $\mathbf{u}_X(\mathbf{v}')$  weakly dominates  $\mathbf{u}_X(\mathbf{v})$  for  $X \in \{R, B\}$  and at least one of the dominations is strict;
- *utility-vector optimal*, denoted UVO, if it is not utility-vector dominated by any other assignment;
- *group-welfare optimal*, denoted GWO, if it is not group-welfare dominated by any other assignment;
- *perfect*, denoted Perfect, if every agent gets utility 1.

As previously mentioned, UVO and GWO were introduced by Bullinger et al. [9] where the following were proven.

**PROPOSITION 1.** *If an assignment  $\mathbf{v}$  is WO, then it is UVO, GWO, and PO. If  $\mathbf{v}$  is UVO or GWO, then it is PO.*

**OBSERVATION 1.** *If Schelling instance  $\langle G, A \rangle$  admits a Perfect assignment, then every PO assignment is Perfect.*

In this paper we study the complexity of  $\phi$ -SCHELLING, where  $\phi \in \{\text{WO, PO, GWO, UVO, Perfect}\}$ . In other words, given a Schelling instance  $\langle G, A \rangle$ , we want to find an assignment  $\mathbf{v}$  satisfying the given optimality notion.

#### $\phi$ -SCHELLING

Input: A Schelling instance  $\langle G, A \rangle$   
 Question: Output an assignment  $\mathbf{v}$  satisfying  $\phi$  or correctly output that no such assignment exists.

Observe that when  $\phi \in \{\text{WO, PO, GWO, UVO}\}$  there always exists an assignment that satisfies the optimality notion; this fact on its own does not imply anything for the complexity of the associated problem. On the other hand, Perfect assignments do not always exist and for this reason, we define the decision problem Perfect-SCHELLING-E.

#### Perfect-SCHELLING-E

Input: A Schelling instance  $\langle G, A \rangle$   
 Question: Decide if there exists a perfect assignment  $\mathbf{v}$ .

## 2.1 Parameterized Complexity

We refer to the handbook by Diestel [13] for standard graph terminology. We also refer to the standard books for a basic overview of parameterized complexity theory [12], and assume that readers are aware of the complexity classes FPT, XP, W[1], and the notion of polynomial kernels. We denote by  $\mathbb{N}$  the set of natural numbers, by  $\mathbb{N}_0$  the set  $\mathbb{N} \cup \{0\}$ . Let  $K_{i,j}$  be the complete bipartite graph with parts of size  $i$  and  $j$ .

**Treewidth.** A *nice tree-decomposition*  $\mathcal{T}$  of a graph  $G = (V, E)$  is a pair  $(T, \chi)$ , where  $T$  is a tree (whose vertices we call *nodes*) rooted at a node  $ro$  and  $\chi$  is a function that assigns each node  $t$  a set  $\chi(t) \subseteq V$  such that the following hold:

- For every  $uv \in E$  there is a node  $t$  such that  $u, v \in \chi(t)$ .
- For every vertex  $v \in V$ , the set of nodes  $t$  satisfying  $v \in \chi(t)$  forms a subtree of  $T$ .
- $|\chi(\ell)| = 0$  for every leaf  $\ell$  of  $T$  and  $|\chi(ro)| = 0$ .
- There are only three kinds of non-leaf nodes in  $T$ :

**Introduce node:** a node  $t$  with exactly one child  $t'$  such that  $\chi(t) = \chi(t') \cup \{v\}$  for some vertex  $v \notin \chi(t')$ .

**Forget node:** a node  $t$  with exactly one child  $t'$  such that  $\chi(t) = \chi(t') \setminus \{v\}$  for some vertex  $v \in \chi(t')$ .

**Join node:** a node  $t$  with two children  $t_1, t_2$  such that  $\chi(t) = \chi(t_1) = \chi(t_2)$ .

The *width* of a nice tree-decomposition  $(T, \chi)$  is the size of a largest set  $\chi(t)$  minus 1, and the *treewidth* of the graph  $G$ , denoted  $\text{tw}(G)$ , is the minimum width of a nice tree-decomposition of  $G$ . We let  $T_t$  denote the subtree of  $T$  rooted at a node  $t$ , and use  $\chi(T_t)$  to denote the set  $\bigcup_{t' \in V(T_t)} \chi(t')$  and  $G_t$  to denote the graph  $G[\chi(T_t)]$  induced by the vertices in  $\chi(T_t)$ .

**PROPOSITION 2 ([21]).** *There exists an algorithm which, given an  $n$ -vertex graph  $G$  and an integer  $k$ , in time  $2^{O(k)} \cdot n$  either outputs a nice tree-decomposition of  $G$  of width at most  $2k + 1$  and  $O(n)$  nodes, or determines that  $\text{tw}(G) > k$ .*

## 3 PARAMETERIZING BY $r$ AND $b$

In this section, we study  $\phi$ -SCHELLING parameterized by the number of red and blue agents. Firstly, we focus on the number of blue agents,  $b$ . Observe that in this case, if  $r + b = |V|$ , there is a trivial XP algorithm since there are  $\binom{n}{b} = O(|V|^b)$  assignments in total; for any choice of the positions of the  $b$  blue agents, the remaining vertices have to be occupied by red agents. This XP algorithm is, in a sense, the best we can hope for; Bullinger et al. [9], although they do not mention it, show that WO-SCHELLING parameterized by  $b$  does not admit an FPT algorithm unless  $\text{FPT} = \text{W}[1]$ .

The above-mentioned XP algorithm works because we can trivially extend a choice for the positions of the blue agents to a complete assignment; there are no choices to be made for red agents. This is no longer possible when  $r + b < |V|$ . For this case, Agarwal et al. [1] showed that WO-SCHELLING is NP-hard even when  $b = 1$ . However, their proof was relying on the assumption that the blue

agent is “stubborn”, i.e. the blue agent had a fixed position on the graph. We strengthen their result by showing that the problem remains intractable, even when the blue agent is not stubborn.

**THEOREM 3.** *Assuming  $P \neq NP$ , there is no polynomial-time algorithm for  $\phi$ -SCHELLING, for  $\phi \in \{WO, PO, UVO, GWO\}$ , even when  $b = 1$ .*

**PROOF.** We will prove hardness via a reduction from CLIQUE, where we are given a graph  $H$  and an integer  $k$  and the goal is to decide the existence of a set  $S \subseteq V(H)$ , where  $|S| = k$ , such that  $H[S]$  induces a clique. Namely, we will construct a Schelling instance  $\langle G, A \rangle$  such that if someone gives us an assignment  $\mathbf{v}$  for  $\langle G, A \rangle$  with a guarantee that  $\mathbf{v}$  satisfies PO, then we can in polynomial time decide CLIQUE on  $\langle H, k \rangle$ . Note that by Proposition 1, if an assignment  $\mathbf{v}$  satisfies  $\phi \in \{WO, UVO, GWO\}$ , then it also satisfies PO, hence an algorithm that outputs an assignment  $\mathbf{v}$  that satisfies  $\phi \in \{WO, PO, UVO, GWO\}$ , always outputs an assignment that satisfies PO. Given an instance  $\langle H, k \rangle$  of CLIQUE, where  $H = (V', E')$  and  $|V'| = s$ , we construct an instance of PO-SCHELLING as follows.

- There are  $s^2 + k$  red agents and one blue agent.
- $G'$  is a clique of size  $s^2$ , where  $G' = (X, Y)$ .
- The topology  $G = (V, E)$  is defined so that  $V = \{x\} \cup V' \cup X$ , where  $x$  is a fresh new vertex, and  $E = E' \cup Y \cup \{xv : v \in V'\} \cup \{vw : v \in V', w \in X\}$ .

Note that since  $b = 1$ , it follows that for every assignment  $\mathbf{v}$  we get  $\mathbf{u}_B(\mathbf{v}) = 0$ . We show that  $H$  admits a clique of size  $k$  if and only if the utility-vector for the red agents of every PO assignment  $\mathbf{v}$  is equal to the vector  $\mathbf{u}_{PO} = (1, \dots, 1, \frac{s^2+k-1}{s^2+k}, \dots, \frac{s^2+k-1}{s^2+k})$ , where there are  $s^2$  red agents with utility 1 and  $k$  red agents with utility  $\frac{s^2+k-1}{s^2+k} = 1 - \frac{1}{s^2+k}$ .

First, assume  $H$  has a clique of size  $k$ . We create an assignment  $\mathbf{v}$  for  $\langle G, A \rangle$  such that  $s^2$  red agents are on vertices in  $G'$  and the remaining  $k$  red agents are assigned to the clique in  $H$ . The blue agent is assigned to vertex  $x$ . Observe that, since there is no edge between  $G'$  and  $x$ , all agents in  $G'$  are only connected to agents of their type. No agents in  $G'$  are isolated because it is clique of size  $s^2$ . Hence,  $u_i(\mathbf{v}) = 1$  for every agent  $i \in G'$ . For the remaining  $k$  red agents, observe that they are connected to every other red agent and the blue agent. Hence,  $u_i(\mathbf{v}) = \frac{s^2+k-1}{s^2+k}$  for every agent  $i \in H$ . Therefore,  $\mathbf{u}(\mathbf{v}) = \mathbf{u}_{PO}$ . Thus, if there exists a clique of size  $k$  in  $G$  then an assignment  $\mathbf{v}$  with the utility vector equal to  $\mathbf{u}_{PO}$  exists. For the other direction, we will show that if such assignment  $\mathbf{v}$  with  $\mathbf{u}_R(\mathbf{v}) = \mathbf{u}_{PO}$  exists, then it is necessarily PO and there exists a clique of size  $k$  in  $H$ . Let  $\mathbf{v}$  be an assignment in  $\langle G, A \rangle$ . We show that either  $\mathbf{u}_{PO}$  strictly dominates  $\mathbf{u}_R(\mathbf{v})$ , or  $\mathbf{u}_R(\mathbf{v}) = \mathbf{u}_{PO}$  and the graph  $H$  admits a clique of size  $k$ . Observe that if the blue agent is not at vertex  $x$ , but instead at a vertex in  $V(H) \cup X$ , then minimum  $s^2 + k - (s - 1)$  of the red agents lose utility. Each of these red agents lose minimum  $\frac{1}{s^2+k}$  of their utility due to the blue agent becoming part of their neighborhood and the fact that there are  $s^2 + k + 1$  agents in total. It follows that in this case  $\mathbf{u}_{PO}$  strictly dominates  $\mathbf{u}_R(\mathbf{v})$ . Else, the blue agent is assigned the vertex  $x$  by  $\mathbf{v}$ . Now, every red agent at a vertex in  $H$  loses at least  $\frac{1}{s^2+k}$  of their utility. It follows that exactly  $k$  red agents are assigned a vertex in  $H$ , otherwise  $\mathbf{u}_{PO}$  strictly dominates  $\mathbf{u}_R(\mathbf{v})$ . Finally, if a red agent  $i$  at a vertex in  $H$  is not adjacent to all the red agents, then the loss

of utility of agent  $i$  is at least  $\frac{1}{s^2+k-1} > \frac{1}{s^2+k}$ . Therefore, if the  $k$  red agents at the vertices of  $H$  do not form a clique, then  $\mathbf{u}_{PO}$  strictly dominates  $\mathbf{u}_R(\mathbf{v})$ . It follows that if  $\mathbf{u}_{PO}$  does not strictly dominates  $\mathbf{u}_R(\mathbf{v})$ , then the  $k$  agents that are assigned the vertices of  $H$  by the assignment  $\mathbf{v}$  form a clique and  $\mathbf{u}_R(\mathbf{v}) = \mathbf{u}_{PO}$ .  $\square$

Recall, there is always an assignment  $\mathbf{v}$  for  $\phi$ -SCHELLING when  $\phi \in \{WO, PO, UVO, GWO\}$ . However, as our theorem shows, it is still intractable to find such an assignment (assuming  $P \neq NP$ ). Someone could wonder if the associated problems are complete for some of the complexity classes that belong to NP and capture problems that are guaranteed to admit a solution, like PPAD, PPA, or PLS [18, 25]. However, this is unlikely to be the case. The proof of Theorem 3 can be easily modified (for example by planting a clique minus one edge in the original clique instance) to show that, assuming  $P \neq NP$ , there is no polynomial-time algorithm to verify whether an assignment  $\mathbf{v}$  satisfies an optimality condition. Hence, it is unlikely that  $\phi$ -SCHELLING for  $\phi \in \{WO, PO, UVO, GWO\}$  belongs to any of the above mentioned classes and in TFNP more generally [24].

On the positive side, we can easily get an XP algorithm for Perfect-SCHELLING parameterized by  $b$ .

**THEOREM 4.** *For Perfect-SCHELLING there is an XP-algorithm parameterized by  $b$ .*

**PROOF.** Observe that when  $b = 1$ , there is no Perfect assignment since the unique blue agent cannot get utility 1. Now, for  $b > 1$  we proceed as follows. We guess the  $b$  vertices, denoted  $B$ , that blue agents occupy under the constraint that every connected component induced by  $B$  has size at least 2. There are  $O(|V|^b)$  such many guesses. Then, we consider the graph induced by  $S := V - B - N(B)$ , where  $N(B)$  contains all the vertices adjacent to at least one vertex in  $B$ . We focus on the connected components induced by  $S$  that have size at least 2; let us denote this graph  $G'$ . If  $G'$  contains less than  $r$  vertices, then we reject the guess. Otherwise, we start assigning red agents to the vertices of  $G'$  in the following manner. We order the connected components of  $G'$  in decreasing size. Then, greedily we start assigning red agents to the vertices of each component under the constraint that every new agent we assign to the current connected component is either the first agent assigned to a vertex of this component, or he is adjacent to a vertex already occupied by an agent. When we will consider the last agent, there are two cases.

- The last agent is assigned to a vertex adjacent to a vertex with a red agent. In this case, every red agent has at least one red neighbor and no blue neighbors and every blue agent has a blue neighbor and no red neighbors. Thus, the assignment is Perfect.
- The last agent  $a$  is the first agent assigned to a vertex of a connected component of  $G'$ . Then, we check the largest connected component of  $G'$ . If there is a red agent that can be reassigned to a vertex adjacent to  $a$ , while the remaining red agents of his previous component still have utility 1, then we make the reassignment and we have a Perfect assignment. Otherwise, we can conclude that the original guess of  $B$  cannot be extended to a Perfect assignment and we proceed to the next guess.

Hence, in time  $O(|V|^b)$  we can decide if a Perfect assignment exists and if it does compute one in the same time.  $\square$

Next, we show that the XP-algorithm from Theorem 4 is actually the best we can hope for. In fact we show that the problem is hard even if we parameterize by  $r + b$ .

**THEOREM 5.** *Perfect-SCHELLING-E is W[1]-hard when parameterized by  $r + b$ .*

**PROOF.** We will prove the theorem via a reduction from BiCLIQUE. The input for an instance of BiCLIQUE is a bipartite graph  $H = (P \cup Q, Y)$  and an integer  $k$ . The task is to decide whether  $H$  contains a complete bipartite subgraph with  $k$  vertices in each side of  $H$ . It is known that BiCLIQUE is W[1]-hard when parameterized by  $k$  [23]. Given  $H = (P \cup Q, Y)$  and  $k$ , we construct an instance  $\langle G, A \rangle$  of Perfect-SCHELLING as follows. The graph  $G = (V, E)$  will be the complement of  $H$ . This means that  $V = P \cup Q$  and  $uv \in E$  if and only if  $uv \notin Y$ . Furthermore, we create  $k$  red and  $k$  blue agents, i.e.  $r = b = k$ . We will ask if there is a perfect assignment  $\mathbf{v}$ .

Firstly, assume that  $\{p_1, \dots, p_k\} \in P$  and  $\{q_1, \dots, q_k\} \in Q$  form a complete bipartite subgraph of  $H$ . We create an assignment  $\mathbf{v}$  for  $\langle G, A \rangle$ , by assigning a blue agent to vertex  $p_i$  and a red agent to vertex  $q_i$  for every  $i \in [k]$ . Observe that since  $p_i$  is adjacent to all  $q_1, \dots, q_k$  in  $H$ , it follows that in  $G$ , which recall is the complement of  $H$ , the vertex  $p_i$  is not adjacent to any of the vertices  $q_1, \dots, q_k$ . Hence, there is no edge  $uv \in E$  such that  $u$  is occupied by a red agent and  $v$  is occupied by a blue agent. In addition, observe that in  $G$  the vertices  $p_1, \dots, p_k$  induce a clique and the vertices  $q_1, \dots, q_k$  induce a clique as well. Hence, under  $\mathbf{v}$  every blue agent is adjacent to all remaining blue agents and no red agent, and every red agent is a neighbor to all remaining red agents and no blue agent. This means that  $\mathbf{v}$  is perfect.

For the other direction, assume that there exists a perfect assignment  $\mathbf{v}$  in  $\langle G, A \rangle$ . Hence, no red agent has a blue agent as a neighbor in  $\mathbf{v}$ ; if this was the case, then both agents would get utility strictly less than 1. So, assume that under  $\mathbf{v}$ ,  $v \in P$  is occupied by a blue agent. Then there is no vertex  $u \in P$  occupied by a red agent under  $\mathbf{v}$ ; this is because  $P$  forms a clique in  $G$ . Thus, all red agents occupy vertices of  $Q$  and no blue agent occupies a vertex in  $Q$ ; this is because  $Q$  forms a clique in  $G$ . Hence, we can conclude that in  $\mathbf{v}$ :

- the blue agents occupy the vertices  $p_1, p_2, \dots, p_k$  in  $P$ ;
- the red agents occupy the vertices  $q_1, q_2, \dots, q_k$  in  $Q$ ;
- there is no edge  $p_i q_j$ , where  $i \in [k]$  and  $j \in [k]$ .

Hence, in the dual of  $G$ , which is  $H$ , for every  $i \in [k]$  and every  $j \in [k]$  the edge  $p_i q_j$  exists. Thus,  $\{p_1, \dots, p_k\}$  and  $\{q_1, \dots, q_k\}$  form a solution of BiCLIQUE.  $\square$

The combination of Theorem 5, Proposition 1, and Observation 1, gives us the following corollary.

**COROLLARY 6.** *There is no FPT algorithm for  $\phi$ -SCHELLING when parameterized by  $r + b$ , for  $\phi \in \{\text{Perfect}, \text{WO}, \text{PO}, \text{UVO}, \text{GWO}\}$ , unless  $\text{FPT} = \text{W}[1]$ .*

## 4 BOUNDED DEGREE GRAPHS

In light of the negative results from the previous section, we turn our attention on instances where the structure of  $G$  is restricted. In

this section, we focus on the maximum degree  $\Delta$  of  $G$ . We prove that, unless  $\text{P} = \text{NP}$ , no polynomial-time algorithm that finds a WO, resp. GWO, assignment exists even when  $G = (V, E)$  is cubic, i.e. every vertex has degree 3.

**THEOREM 7.** *Assuming  $\text{P} \neq \text{NP}$ , there is no polynomial-time algorithm solving WO-SCHELLING and GWO-SCHELLING on cubic graphs, even if  $r + b = |V|$ .*

**PROOF.** It follows from Proposition 1 that it suffices to show that, unless  $\text{P} = \text{NP}$ , no polynomial-time algorithm finds a GWO assignment on cubic graphs. The proof is via a reduction from MINBISECTION on cubic graphs [8]. An instance of MINBISECTION consists of a graph  $G = (V, E)$  and an integer  $k$ . We have to decide if there exists a partition of  $V$  into  $P$  and  $Q$  such that  $|P| = |Q| = \frac{|V|}{2}$  where the number of edges  $uv \in E$  with  $u \in P$  and  $v \in Q$  is at most  $k$ . The constructed instance  $\langle G, A \rangle$  of GWO-SCHELLING is on the same graph  $G$  and set of agents  $A$  which has  $\frac{|V|}{2}$  red agents and  $\frac{|V|}{2}$  blue agents. We will ask if there is an assignment  $\mathbf{v}$  such that the welfare of each group is at least  $\frac{|V|}{2} - \frac{k}{3}$ .

So, assume that there is a partition  $V$  into  $P$  and  $Q$  with  $|P| = |Q| = \frac{|V|}{2}$  such that there are exactly  $\ell \leq k$  edges between  $P$  and  $Q$ . We create an assignment  $\mathbf{v}$  by placing all blue agents on  $P$  and all red agents on  $Q$ . Observe that since  $|A| = |V|$  and since the graph is cubic, for every  $i$  we have that  $f_i(\mathbf{v}) = 3 - e_i(\mathbf{v})$  and that  $|N_i(\mathbf{v})| = 3$ . Thus, for  $X \in \{R, B\}$

$$\begin{aligned} \text{SW}_X(\mathbf{v}) &= \sum_{i \in X} \frac{f_i(\mathbf{v})}{|N_i(\mathbf{v})|} = \sum_{i \in X} \frac{3 - e_i(\mathbf{v})}{3} \\ &= \frac{|V|}{2} - \sum_{i \in X} \frac{e_i(\mathbf{v})}{3}. \end{aligned}$$

Furthermore, observe that if  $i$  is a blue agent, i.e. occupies a vertex in  $P$ , all his enemies occupy vertices in  $Q$  and vice versa. Hence,  $\sum_{i \in B} e_i(\mathbf{v}) = \sum_{i \in R} e_i(\mathbf{v}) = \ell \leq k$ . Thus, for  $X \in \{R, B\}$ ,  $\text{SW}_X(\mathbf{v}) = \frac{|V|}{2} - \frac{\ell}{3} \geq \frac{|V|}{2} - \frac{k}{3}$ .

For the other direction, assume that we have an assignment  $\mathbf{v}$  such that for  $X \in \{R, B\}$  we have  $\text{SW}_X(\mathbf{v}) \geq \frac{|V|}{2} - \frac{k}{3}$ . From the arguments above, we know that  $\text{SW}_R(\mathbf{v}) = \text{SW}_B(\mathbf{v}) = \frac{|V|}{2} - \frac{\ell}{3}$ , where  $\ell$  is the number of edges between vertices assigned to blue agents and the vertices assigned to red agents. Thus,  $\ell \leq k$  and there exists at most  $k$  edges in  $G$  where one of the endpoints is occupied by a red agent and the other endpoint is occupied by a blue agent. The proof is completed by creating a partition of  $V$  by setting  $P$  to contain all the vertices occupied by blue agents and  $Q$  to contain all the vertices occupied by red agents. It follows that there are at most  $k$  edges between  $P$  and  $Q$ .  $\square$

Theorems 5 and 7 show that we cannot hope for an efficient algorithm, at least for WO and GWO, just by parameterizing only by  $r + b$  or only by the maximum degree  $\Delta$ . We complement this by providing an FPT algorithm for  $\phi$ -SCHELLING when parameterized by  $r + b + \Delta$ .

**THEOREM 8 (★).** *For every  $\phi \in \{\text{Perfect}, \text{WO}, \text{PO}, \text{UVO}, \text{GWO}\}$ , there is an FPT-algorithm for  $\phi$ -SCHELLING parameterized by  $r + b + \Delta$ .*

Moreover,  $\phi$ -SCHELLING admits a kernel with at most  $O(\Delta^2 \cdot r^2 \cdot b^2)$  many vertices.

**PROOF SKETCH.** Let  $\langle G, A \rangle$  be a SCHELLING instance. Let  $C_1, \dots, C_k$  be the connected components of  $G$  such that  $|C_1| \geq |C_2| \geq \dots \geq |C_k|$ . We will prove the theorem in two steps. First we show that if  $|C_1| \geq (\Delta + 1) \cdot r \cdot (1 + \Delta \cdot b)$ , then we can construct an assignment  $\mathbf{v}$  such that  $\text{SW}(\mathbf{v}) = r + b$  in polynomial time. Afterwards, we show that there always exists a solution that maximizes social welfare that does not intersect any of the components  $C_{r+b+1}, \dots, C_k$  and if  $f(r + b + \Delta) \geq |C_1| \geq |C_2| \geq \dots \geq |C_{r+b}|$ , we can find a solution that maximizes social welfare in FPT time; for example by trying all possible assignments that assign all agents to the components  $C_1, C_2, \dots, C_{r+b}$ .

Assume that  $|C_1| \geq (\Delta + 1) \cdot r \cdot \Delta \cdot (1 + b)$  and let us pick an arbitrary set  $X \subseteq C_1$  such that  $G[X]$  is connected and  $|X| = r$ . We assign all red agents to the vertices in  $X$ . Now  $|N[X]| \leq (\Delta + 1)r$ , where  $N[X]$  is the closed neighborhood of  $X$ , and  $|N(N[X])| \leq \Delta(\Delta + 1)r$ . However, every connected component of  $G[C_1 \setminus N[X]]$  contains a vertex with a neighbor in  $N[X]$ . Hence,  $G[C_1 \setminus N[X]]$  has at most  $\Delta(\Delta + 1)r$  many connected components and if  $|C_1| \geq (\Delta + 1)r\Delta(1 + b)$  it follows that  $|C_1 \setminus N[N[X]]| \geq (\Delta + 1) \cdot r \cdot \Delta \cdot b$  and by the pigeonhole principle at least one of the connected components of  $G[C_1 \setminus N[X]]$  has  $b$  vertices. We can assign all blue agents to a connected subgraph of such a component. Let  $\mathbf{v}$  be the assignment we obtained above. Since no blue agent is assigned to  $N(X)$  and  $X$  is connected, we get  $\text{SW}_R(\mathbf{v}) = r$  whenever  $r \geq 2$  and, since all blue agents are also assigned to a connected subgraph of  $G$ , we get  $\text{SW}_B(\mathbf{v}) = b$  whenever  $b \geq 2$ . Hence from now on we can assume that all connected components of  $G$  have size at most  $(\Delta + 1) \cdot r \cdot \Delta \cdot (1 + b)$ .

It remains to show that we can focus on the  $r + b$  largest components. We can observe that if agents of some type are assigned to some component  $C_q$ ,  $q > r + b$ , then we can reassign all of them to some component  $C_p$ ,  $p \leq r + b$ , such that no vertex of  $C_p$  is occupied by any agent so far. Moreover, this operation does not decrease the social welfare. Hence, we can remove all connected components  $C_q$ ,  $q > r + b$ , from the instance and we are left with at most  $r + b$  components, each with at most  $O(\Delta^2 \cdot r \cdot b)$  many vertices. We can solve this in time  $O((\Delta^2 \cdot r \cdot b \cdot (r + b))^{r+b})$  by enumerating all possible assignments of the agents.  $\square$

## 5 MULTIPLE TYPES

Next, we depart from the standard model and study Schelling instances with multiple types, denoted  $\phi$ -SCHELLINGM and Perfect-SCHELLINGM-E, respectively. We show that Perfect-SCHELLINGM-E is  $W[1]$ -hard when parameterized by (the number of) agent-types. Our reduction shows that the intractability holds for a variety of graphs. In fact, it reveals that the sizes of the connected components of the graph are sufficient for proving hardness, without depending on the internal structure of every component. We will prove our result in two steps. In the first step, we will prove that the problem is hard even when  $G$  is a collection of connected components with arbitrary structure. Then, we will show how to get hardness even when  $G$  is a tree.

**THEOREM 9 (★).** *Let  $\mathcal{G}$  be an arbitrary class of connected graphs that contains at least one graph of size  $s$  for every  $s \in \mathbb{N}$ . Perfect-SCHELLINGM-E is NP-hard and  $W[1]$ -hard when parameterized by agent-types, even when every connected component of  $G$  is in  $\mathcal{G}$ .*

**PROOF SKETCH.** We will prove our result via a reduction from UNBINPACKING. An instance of UNBINPACKING consists of a set  $I$  of items, where every item  $i \in I$  has a positive integer size  $s_i > 1$  given in unary, and  $k$  bins of size  $B$  each<sup>2</sup>. The task is to decide if there is a partition of the items into  $k$  subsets  $I_1, I_2, \dots, I_k$  such that the size of each subset is exactly  $B$ . UNBINPACKING is  $W[1]$ -hard parameterized by the number of bins  $k$  [17].

Given an instance of UNBINPACKING, we will create an instance of SCHELLINGM with  $k$  types of agents, where for each type there are  $B$  agents, hence there are  $k \cdot B$  agents in total. The graph  $G$  will be the union of the graphs  $G_1, G_2, \dots, G_{|I|}$ , where  $G_i$  is isomorphic to a connected graph in  $\mathcal{G}$  with  $s_i$  vertices. We will ask whether there is a perfect assignment for  $\langle G, A \rangle$ .

The correctness is proven by observing that in any perfect partition, the agents are neighbors only to agents of the same type, and hence any connected component contains only agents of a specific type.  $\square$

As a corollary of Theorem 9, we can prove that the problem remains  $W[1]$ -hard even when  $G$  is a tree.

**COROLLARY 10 (★).** *Perfect-SCHELLINGM-E is NP-hard and  $W[1]$ -hard when parameterized by agent-types, even if  $G$  is a tree.*

Again, using Proposition 1 and Observation 1, we can get the following corollary.

**COROLLARY 11.** *Assuming  $P \neq \text{NP}$ , for every  $\phi \in \{\text{WO}, \text{PO}, \text{UVO}, \text{GWO}, \text{Perfect}\}$ , there is no polynomial-time algorithm for  $\phi$ -SCHELLINGM even when  $G$  is a tree. Moreover, assuming  $\text{FPT} \neq W[1]$ , there is no FPT algorithm for  $\phi$ -SCHELLINGM parameterized by agent-types, even when  $G$  is a tree.*

In the rest of this section, we give an algorithm for  $\phi$ -SCHELLINGM that matches the lower bound from Corollary 11. Namely, we give an XP algorithm for the problem parameterized by the number of agent-types and the treewidth of the graph. Recall that trees have treewidth 1 and hence SCHELLINGM does not admit an FPT algorithm parameterized by agent-types and treewidth unless  $\text{FPT} = W[1]$ . The rest of the section is devoted to the proof of the following theorem.

**THEOREM 12.** *There is an  $|A|^{O(k \cdot \text{tw}(G))} \cdot |V(G)|$  time algorithm for  $\phi$ -SCHELLINGM,  $\phi \in \{\text{Perfect}, \text{WO}, \text{PO}, \text{GWO}, \text{UVO}\}$ , where  $k$  is the number of agent-types.*

Let  $\langle G, A \rangle$  be an instance of  $\phi$ -SCHELLINGM with  $k$  agent-types, where  $G$  has treewidth at most  $w$ . For a type  $i \in [k]$ , let  $A_i$  denote the set of all agents of type  $i$ . Moreover, let  $\mathcal{T} = (T, \chi)$  be a tree-decomposition of  $G$  of width at most  $w$ . Note that every assignment that is WO is also  $\phi$  by Proposition 1, hence regardless of  $\phi$  we can compute a WO assignment. The algorithm is a standard bottom-up dynamic programming along a nice tree-decomposition. As always, the main challenge is to decide what records we should keep for

<sup>2</sup>To get  $s_i > 1$ , we can multiply  $B$  and every  $s_i$  without changing the answer to the decision question.

each node  $t$  of  $T$ . Here each record models an equivalence class of partial assignments for the sub-instance induced by the vertices in  $G_t$ , i.e., the graph  $G$  induced by all vertices contained in bags in the subtree rooted at  $t$ .

Consider some node  $t \in V(T)$ . We would like to compute a table  $\Gamma_t$ , where each entry of the table corresponds to some equivalence class of partial assignments and the value of that entry is the "best" partial assignment in the equivalence class.

For the algorithm to be efficient, we need the number of equivalence classes to be small and we should be able to compute each entry in  $\Gamma_t$  efficiently from the tables for the children of  $t$ . First, let us formally define a *partial assignment over the subset of agents*  $A' \subseteq A$  to be an assignment  $\mathbf{v}_{A'} = (v(a_1), v(a_2), \dots, v(a_{|A'|}))$ , where  $A' = \{a_1, a_2, \dots, a_{|A'|}\}$ . A *partial assignment* is then an assignment over some subset of agents. In node  $t$ , we are interested in the partial assignment that we can obtain by taking a (full) assignment and restricting it to the agents that are assigned some vertex in  $G_t$ . For a partial assignment  $\mathbf{v}$  we let  $\text{SW}^t(\mathbf{v}) = \sum_{i \in A'} u_i(\mathbf{v})$ , where  $A' \subseteq A$  is the set of agents assigned to vertices in  $G_t - \chi(t)$  by  $\mathbf{v}$ . Note that  $\text{SW}^t(\mathbf{v})$  does not contain the utilities of vertices in the bag  $\chi(t)$ . This is because the utilities of these vertices still depend on the vertices in  $V(G) \setminus V(G_t)$  and might be different in an assignment  $\mathbf{v}'$  whose restriction to  $G_t$  results in  $\mathbf{v}$ .

A (description of) the equivalence class  $C$  for the node  $t$  is a tuple  $\langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$ , where

- **Sizes:**  $[k] \rightarrow \mathbb{N}$  such that  $0 \leq \text{Sizes}(i) \leq |A_i|$  for all  $i \in [k]$ ,
- **$\chi$ -Types:**  $\chi(t) \rightarrow \{0, \dots, k\}$ , and
- **$\chi$ -Neighbors:**  $\chi(t) \rightarrow \mathbb{N}^k$  such that for every vertex  $v \in \chi(t)$ , if  $\chi\text{-Neighbors}(v) = \langle n_1, n_2, \dots, n_k \rangle$ , then  $0 \leq n_i \leq |A_i|$  for all  $i \in [k]$ .

We say that a partial assignment  $\mathbf{v}$  belongs to the equivalence class  $C = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$  for the node  $t$  if and only if:

- $\mathbf{v}$  assigns agents only to vertices in  $G_t$ ,
- for all  $i \in [k]$ , the number of agents of type  $i$  assigned a vertex in  $G_t$  by  $\mathbf{v}$  is  $\text{Sizes}(i)$ ,
- for all  $v \in \chi(t)$ , if  $\chi\text{-Types}(v) = 0$ , then  $\mathbf{v}$  does not assign any agent to the vertex  $v$ , else  $\mathbf{v}$  assigns some agent of type  $\chi\text{-Types}(v)$  to  $v$ , and
- for all  $v \in \chi(t)$  and all  $i \in [k]$ , the number of neighbors of  $v$  in  $G_t$  assigned an agent of type  $i$  is equal to  $\chi\text{-Neighbors}(v)[i]$ .

Observe, every partial assignment belongs to some equivalence class.

We say that an equivalence class is *valid* if there exists a partial assignment that belongs to the equivalence class. For a valid equivalence class  $C$ , the table entry  $\Gamma_t[C]$  should contain some partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $\text{SW}^t(\mathbf{v})$ .

Moreover, recall that for the root node  $\text{ro}$  of  $T$ , we have  $G_{\text{ro}} = G$  and  $\chi(\text{ro}) = \emptyset$ , hence  $\text{SW}^{\text{ro}}(\mathbf{v}) = \text{SW}(\mathbf{v})$  and the table entry  $\Gamma_{\text{ro}}[C_{\text{ro}}]$  for the class  $C_{\text{ro}} = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$ , where for all  $i \in [k]$  we have  $\text{Sizes}(i) = |A_i|$  and  $\chi\text{-Types}$  and  $\chi\text{-Neighbors}$  are empty functions, contains an assignment that maximizes the social welfare. First, let us observe that the number of entries in each node is bounded.

**OBSERVATION 2 (★).** *The number of equivalence classes for node  $t$ ,  $t \in V(T)$ , is at most  $(|A| + 1)^{k(1+\chi(t))} \cdot (k + 1)^{|\chi(t)|}$ .*

It follows from Observation 2 that it suffices to show that for each node  $t$  and each equivalence class  $C$  for the node  $t$ , we can decide in time  $|A|^{O(k \cdot \text{tw}(G))}$  whether  $C$  is valid and if so find the partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $\text{SW}^t(\mathbf{v})$ .

We will distinguish four cases depending on the type of the node  $t$ . Moreover, when computing the entries for the node  $t$  we always assume that we computed all entries for the children of  $t$  in  $T$ .

**LEMMA 13 (LEAF NODE (★)).** *Let  $t \in V(T)$  be a leaf node and  $C = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$  an equivalence class for  $t$ . Then we can in  $O(k)$  time decide whether  $C$  is valid and if so compute a partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $\text{SW}^t(\mathbf{v})$ .*

**LEMMA 14 (INTRODUCE NODE (★)).** *Let  $t \in V(T)$  be an introduce node with child  $t'$  such that  $\chi(t) \setminus \chi(t') = \{v\}$  and  $C$  an equivalence class for  $t$ . Then we can in polynomial time decide whether  $C$  is valid and if so compute a partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $\text{SW}^t(\mathbf{v})$ .*

**PROOF SKETCH.** First note that from the properties of a tree-decomposition it follows that all neighbors of  $v$  are in  $\chi(v)$ . Hence, if for some  $i \in [k]$  we have that  $\chi\text{-Neighbors}(v)[i]$  does not equal to the number of neighbors  $w$  of  $v$  such that  $\chi\text{-Types}(w) = i$ , then there cannot exist a partial assignment that belongs to  $C$  and  $C$  is not valid. Else, let  $C' = \langle \text{Sizes}', \chi\text{-Types}', \chi\text{-Neighbors}' \rangle$  be an equivalence class for  $t'$  such that:

- for all  $i \in [k]$ ,  $\text{Sizes}'(i) = \text{Sizes}(i) - 1$  if  $\chi\text{-Types}(v) = i$  and  $\text{Sizes}'(i) = \text{Sizes}(i)$  otherwise;
- for all  $w \in \chi(t') = \chi(t) \setminus \{v\}$ ,  $\chi\text{-Types}'(w) = \chi\text{-Types}(w)$ ;
- for all  $i \in [k]$  and  $w \in \chi(t')$ ,
  - $\chi\text{-Neighbors}'(w)[i] = \chi\text{-Neighbors}(w)[i] - 1$  if  $vw \in E(G)$  and  $\chi\text{-Types}(v) = i$ ;
  - $\chi\text{-Neighbors}'(w)[i] = \chi\text{-Neighbors}(w)[i]$ , otherwise.

Observe that a partial assignment  $\mathbf{v}$  belongs to  $C$  if and only if the partial assignment  $\mathbf{v}'$  obtained from  $\mathbf{v}$  by restriction to  $G_{t'}$  belongs to  $C'$  and it is not difficult to see that we can also construct  $\mathbf{v}$  given  $\mathbf{v}'$  that belongs to  $C'$ . Finally,  $\text{SW}^t(\mathbf{v}) = \text{SW}^{t'}(\mathbf{v}')$  and we can get  $\Gamma_t[C]$  from  $\Gamma_{t'}[C']$ .  $\square$

**LEMMA 15 (FORGET NODE (★)).** *Let  $t \in V(T)$  be a forget node with child  $t'$  such that  $\chi(t') \setminus \chi(t) = \{v\}$  and an equivalence class for  $t$   $C = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$ . Then we can in  $O((k + 1)|A|^k)$  time decide whether  $C$  is valid and if so compute a partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $\text{SW}^t(\mathbf{v})$ .*

**PROOF SKETCH.** Let  $C$  be the set of all valid equivalence classes for  $t'$  such that for all  $C' = \langle \text{Sizes}', \chi\text{-Types}', \chi\text{-Neighbors}' \rangle \in C$  it holds that:

- $\text{Sizes} = \text{Sizes}'$ , and
- for all  $w \in \chi(t)$ 
  - $\chi\text{-Types}(w) = \chi\text{-Types}'(w)$  and
  - $\chi\text{-Neighbors}(w) = \chi\text{-Neighbors}'(w)$ .

We observe that for every partial assignment  $\mathbf{v}$  that belongs to  $C$ , there exists an equivalence class  $C' \in C$  such that  $\mathbf{v}$  belongs to  $C'$ . Hence, if  $C$  is empty, we can return that  $C$  is not valid. Moreover, it is easy to see that a partial assignment that belongs to some class in  $C$  also belongs to  $C$ . Finally, we can show that the difference  $\text{SW}^t(\mathbf{v}) - \text{SW}^{t'}(\mathbf{v}')$  depends only on the class  $C'$  and is

easy to compute. Hence, to get  $\Gamma_t[C]$  we simply iterate over all equivalence classes in  $C$  and output  $\Gamma_{t'}[C']$  in the class  $C'$  that maximizes  $SW^t(\Gamma_{t'}[C'])$ . The running time then follows from the observation that  $|C| \leq (k+1)|A|^k$ .  $\square$

**LEMMA 16 (JOIN NODE  $(\star)$ ).** *Let  $t \in V(T)$  be a join node with children  $t_1$  and  $t_2$ , where  $\chi(t) = \chi(t_1) = \chi(t_2)$  and let  $C$  be an equivalence class for  $t$ . Then we can in  $(O(|A|+1)^{k(1+|\chi(t)|)})$  time decide whether  $C$  is valid and if so compute a partial assignment  $\mathbf{v}$  that belongs to  $C$  and maximizes  $SW^t(\mathbf{v})$ .*

**PROOF SKETCH.** We can show that for any partial assignment  $\mathbf{v}$  that belongs to  $C$  there are two valid equivalence classes

- $C_1 = \langle \text{Sizes}_1, \chi\text{-Types}_1, \chi\text{-Neighbors}_1 \rangle$  for  $t_1$ , and
- $C_2 = \langle \text{Sizes}_2, \chi\text{-Types}_2, \chi\text{-Neighbors}_2 \rangle$  for  $t_2$ ,

such that:

- (A)  $\chi\text{-Types} = \chi\text{-Types}_1 = \chi\text{-Types}_2$ ;
- (B) for all  $i \in [k]$ ,  $\text{Sizes}(i) = \text{Sizes}_1(i) + \text{Sizes}_2(i) - |S_i|$ ;
- (C) for all  $v \in \chi(t)$  and all  $i \in [k]$  it holds that  $\chi\text{-Neighbors}(v)[i] = \chi\text{-Neighbors}_1(v)[i] + \chi\text{-Neighbors}_2(v)[i] - |N_{\chi(t)}^i(v)|$ .

Now, let  $\mathbf{v}_1$  be a partial assignment that belongs to  $C_1$  and  $\mathbf{v}_2$  be a partial assignment that belongs to  $C_2$ . We can construct a partial assignment  $\mathbf{v}$  that belongs to  $C$  by assigning each vertex  $v \in \chi(t)$  to an agent of type  $\chi\text{-Types}(v)$ , every vertex  $w \in G_{t_1} \setminus \chi(t)$  an agent of the same type as the agent assigned the vertex  $w$  by  $\mathbf{v}_1$ , and every vertex  $u \in G_{t_2} \setminus \chi(t)$  an agent of the same type as the agent assigned the vertex  $u$  by  $\mathbf{v}_2$ . It is straightforward to verify that  $\mathbf{v}$  belongs to  $C$  and that  $SW^t(\mathbf{v}) = SW^{t_1}(\mathbf{v}_1) + SW^{t_2}(\mathbf{v}_2)$ . Hence, the lemma follows by trying all pairs  $C_1$  and  $C_2$  satisfying (A)–(C).  $\square$

**PROOF OF THEOREM 12.** Let  $\langle G, A \rangle$  be an instance of  $\phi$ -SCHELLINGM, with  $\phi \in \{\text{WO}, \text{PO}, \text{GWO}, \text{UVO}\}$ . We will compute a WO assignment, which by Proposition 1 is PO, GWO, and UVO. The algorithm first computes a nice tree decomposition  $\mathcal{T} = (T, \chi)$  of  $G$  of width  $w \leq 2 \text{tw}(G) + 1$  in FPT-time [21]. Afterwards, we use the algorithms of Lemmas 13, 14, 15, and 16 to compute for every node  $t$  and every valid equivalence class  $C$  for  $t$  a partial assignment that belongs to  $C$  and maximizes  $SW^t(\mathbf{v})$  among all the partial assignments that belong to the equivalence class  $C$ . An assignment that maximizes the social welfare is then the partial assignment that we computed for the root node of  $T$  for the equivalence class  $C_{\text{ro}} = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$ , where for all  $i \in [k]$  we have  $\text{Sizes}(i) = |A_i|$  and  $\chi\text{-Types}$  and  $\chi\text{-Neighbors}$  are empty functions. The correctness follows from the correctness of Lemmas 13, 14, 15, and 16. The running time of the algorithm is at most the number of nodes of  $T$ , i.e., at most  $w^2|V(G)|$ , times the maximum number of equivalence classes for a node  $t$ , i.e.,  $(|A|+1)^{k(2+w)} \cdot (k+1)^{w+1}$  by Observation 2, times the maximum time required to compute a partial assignment for a node  $t$  and an equivalence class  $C$  for any of the four node types of a nice tree-decomposition which, because of Lemmas 13, 14, 15, and 16, is at most  $(O(|A|+1)^{k(2+w)})$ . Therefore,  $|A|^{O(kw)} \cdot |V(G)| = |A|^{O(k \cdot \text{tw}(G))} \cdot |V(G)|$  is the total running time of the algorithm.  $\square$

Observe that the number of agent-types is always at most the number of agents. It then follows from the running time of the

algorithm that SCHELLINGM is actually FPT when parameterized by treewidth plus the number of agents.

**COROLLARY 17.**  *$\phi$ -SCHELLINGM is in FPT when parameterized by treewidth and the number of agents, for every  $\phi \in \{\text{WO}, \text{PO}, \text{GWO}, \text{UVO}\}$ .*

Finally, while Corollary 11 implies that we cannot obtain an FPT algorithm if the number of agent-types is part of the parameter, unless  $\text{FPT} = \text{W}[1]$ , it remains an interesting open question whether this is possible for a constant number of agent types. While our algorithm cannot resolve this in general, we would like to point out a specific case that can be solved in FPT-time with a very minor modification of our algorithm. Assume that we wish to maximize the social welfare under the additional constraint that every agent is allowed to have only neighbors of the same type. Note that this is the only way to get social welfare equal to the number of agents. In this case, we would reject any equivalence class  $C = \langle \text{Sizes}, \chi\text{-Types}, \chi\text{-Neighbors} \rangle$ , where  $\chi\text{-Neighbors}[v][i] > 0$  such that  $\chi\text{-Types}[v] \neq i$ . Moreover, if  $\chi\text{-Types}[v] = i$ , then we only care whether  $\chi\text{-Neighbors}[v][i] > 0$  and the utility of the agent that is assigned vertex  $v$  is 1 or  $\chi\text{-Neighbors}[v][i] = 0$  and the utility of this agent is 0. Hence, we can replace  $\chi\text{-Neighbors}$  by a function from  $\chi(t) \rightarrow \{0, 1\}$  with meaning:

- if  $\chi\text{-Neighbors}(v) = 0$ , then no neighbor of  $v$  is assigned to any agent;
- if  $\chi\text{-Neighbors}(v) = 1$ , then at least one of the neighbors of  $v$  is assigned some agent of  $\chi\text{-Types}(v)$  and no neighbor of  $v$  is assigned an agent of any other type.

The algorithm then follows more or less analogously the proof of Theorem 12. It is easy to see that the number of these "modified" equivalence classes is at most  $|A|^k \cdot 2^{|\chi(t)|} \cdot (k+1)^{|\chi(t)|}$ , which is FPT by treewidth if the number of agent-types  $k$  is a fixed constant.

**COROLLARY 18.** *When the number of types is constant, Perfect-SCHELLINGM admits an FPT algorithm parameterized by treewidth.*

## 6 CONCLUSIONS

In this paper, we studied  $\phi$ -SCHELLING for several notions,  $\phi \in \{\text{WO}, \text{PO}, \text{UVO}, \text{GWO}, \text{Perfect}\}$ . We presented both strong negative results and accompanying algorithms. Our results show that tractability of  $\phi$ -SCHELLING for every optimality notion considered *requires* the underlying graph to be constrained. We highlight some immediate open questions that deserve further study and we believe that novel algorithmic techniques are required to answer them.

Settle the parameterized complexity for  $\phi$ -SCHELLING when parameterized by  $b+\Delta$  only. Recall, Theorem 8 shows that the problem is fixed parameter tractable when parameterized by  $r+b+\Delta$ . Can we strengthen this result by removing parameter  $r$ , or does the problem become intractable?

What is the parameterized complexity of  $\phi$ -SCHELLING under the vertex cover parameter? It is not too hard to show that when the number of agents equals the number of vertices of the graph the problem is fixed parameter tractable when parameterized by vertex cover. However, the problem remains challenging if the number of agents is less than the number of vertices of the graph.

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