Facility Location Games with Fractional Preferences and Limited Resources

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ABSTRACT

In this paper, we study the heterogeneous facility location game with fractional preferences under resource constraints. In this model, a group of agents are positioned along the interval $[0, 1]$, where each agent has position information and fractional preferences indicated as support weights for facilities. Our main focus is to design mechanisms that choose and locate one facility out of two facilities while motivating agents to truthfully report their information, aiming to approximately maximize the social utility, defined as the sum of utilities of all agents.

Based on the types of private information held by agents, we consider three different settings. For the known-preferences setting, we provide a deterministic group strategy-proof mechanism with 2-approximation and a randomized group strategy-proof mechanism with $\frac{4}{3}$-approximation. We also provide lower bounds of 2 on the approximation ratio for any deterministic strategy-proof mechanism and 1.043 for any randomized strategy-proof mechanism. For the known-positions setting and the general setting, we present a deterministic group strategy-proof mechanism with 6-approximation and a randomized strategy-proof mechanism with 4-approximation, respectively. Furthermore, we give lower bounds of 1.554 for any deterministic strategy-proof mechanism and 1.2 for any randomized strategy-proof mechanism in the known-positions setting. Finally, we extend the model to the scenario of choosing k facilities out of m facilities. For the known-preferences setting, we provide a 2-approximate deterministic group strategy-proof mechanism, which is also the best deterministic strategy-proof mechanism. For the known-positions setting, when $k \geq 2$, we give a lower bound of $2 - \frac{1}{k}$ for any deterministic strategy-proof mechanism.

KEYWORDS

Facility location game; Fractional preferences; Algorithmic mechanism design; Approximation

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1 INTRODUCTION

The facility location game is a cross-disciplinary topic in economics, operations research, and computer science, and it is a significant problem in algorithmic mechanism design. The classical facility location game captures the scenario where a government plans to build a public facility (e.g., a stadium) near a road, and a group of self-interested agents with private positions are distributed along this road. The agents are required to report their private positions, and the government uses a mechanism to determine the location of the facility based on the information provided by the agents. Since the mechanism is common knowledge, each self-interested and rational agent may strategically misreport their private position to manipulate the location of the facility, thus increasing their own utility. Therefore, the government's goal is to design a strategy-proof mechanism where no agent can benefit from misreporting, to determine the location of the facility. Typically, we also consider optimizing some social objective function, such as maximizing the social utility (i.e., the sum of utilities of all agents). However, when money transfer is not allowed, it is not always possible to design a mechanism that is strategy-proof and achieves the maximum social utility. Hence, Procaccia and Tennenholtz [30] studied the facility location game from the perspective of approximation algorithms, measuring the performance of strategy-proof mechanisms using approximation ratios, and proposing approximate mechanism design without money for facility location games. Subsequently, with the introduction of many variants, this field has flourished.

One noteworthy research direction is the extension of the single facility location game to the two facility location game. By altering agent preferences and facility types, both homogeneous facility location and heterogeneous facility location models have been introduced. The homogeneous facility location game captures the scenario where the government plans to build two identical facilities (e.g., bus stops) on a road, and each resident wishes to be as close as possible to their nearest facility. In comparison to homogeneous facility location, the heterogeneous facility location game is more versatile. We consider scenarios where the government plans to build two different facilities (e.g., a hospital and a plaza) on a road, and different residents may have different preferences for the facility: some may prefer to be as close to the facility as possible, some may prefer to be as far away from the facility as possible, and others may be indifferent to the location of the facility. In this case, the utility of the residents depends not only on the distances to the facilities but also on their preferences for the facilities. It is evident that preferences play a crucial role in facility location problems. Besides the above preference types, Fong et al. [19] introduces fractional preferences, where the preferences of residents indicate how often they use two facilities.

However, in real-life situations, considering factors like budget constraint and resource limitation, it may not be feasible to build both facilities simultaneously, requiring a choice between the two. Elkind et al. [11] and Deligkas et al. [9], consider a scenario in which certain facilities are chosen and located from a set of candidate facilities. However, the preferences of residents for facilities are often not straightforward supportive or indifferent. For example, the government plans to build a sports facility on a street with two candidate options, a basketball court and a badminton court. Each resident may have different preferences based on their degree of fondness for the two sports. If a resident is deeply passionate about basketball but has little interest in badminton, they may obtain 100% happiness from the building of a basketball court. If a resident enjoys both basketball and badminton but has different levels of preference for each, say 70% affinity for basketball and a 30% interest in badminton, choosing to build the badminton court would only yield 30% happiness for her since she will not be able to play basketball.

To capture the scenarios described above, we consider a model where one facility is chosen from two candidate facilities and located on the line segment $[0, 1]$. The agents with private positions and fractional preferences, indicated as support weights for facilities, are distributed along this line segment. The utilities of agents depend not only on the distance to the facility but also on their preferences for the chosen facility. According to the private information held by the agent, we consider the following three scenarios:

- Known-preferences setting: The preference is public information, and the position is private information. Each agent can only misreport her own position.
- Known-positions setting: The position is public information, and the preference is private information. Each agent can only misreport her own preference.
- General setting: Both the position and preference are private information. Each agent can simultaneously misreport her position and preference.

Our objective is to design a mechanism that chooses and locates one facility out of two facilities while motivating agents to truthfully report their positions and preferences, aiming to approximately maximize the social utility, where the social utility is defined as the sum of utilities of all agents.

1.1 Related Work

There are numerous works in the literature on facility location games, and we briefly introduce those most relevant to our study. Facility location games can be traced back to the work of Moulin [28], which characterized strategy-proof, Pareto-efficient, and anonymous mechanisms as a class of generalized median mechanisms when the preferences of agents are single-peaked on a line. Procaccia and Tennenholtz [30] studied the facility location game from the perspective of approximation algorithms, measuring the performance of strategy-proof mechanisms using approximation ratios. Subsequently, the model was extended to different metric spaces [1, 10, 15, 16, 27, 34], various utility functions [17, 20], diverse social objectives [4, 13, 23, 26], other incentives [21, 29, 33, 35], and several other models [2, 3, 14, 18, 25, 37].

Procaccia and Tennenholtz [30] first investigated the case of two homogeneous facilities with two types of social objectives, where the costs of agents are the distances from their positions to the nearest facility. Subsequently, Lu et al. [24] improved the lower bound for deterministic strategy-proof mechanisms under the social cost to an asymptotically tight bound, and Fotakis and Tzamos [20] ultimately raised to asymptotically bound of $n - 2$. Filos-Ratsikas et al. [17] extended single-peaked preference to double-peaked preference, where each agent has two most preferred positions. Cheng et al. studied the obnoxious facility location model [7] and extended it to trees and circles [8], where the preference of each agent is to be as far away from facilities as possible. Zou and Li [38] and Feigenbaum and Sethuraman [12] introduced the dual preference model, combining the standard facility location model with the obnoxious facility location model, where the preference of each agent is either to be as far away from facilities as possible or as close as possible. Serafino and Ventre [31, 32] were the first to study two heterogeneous facility location model where the cost of an agent is the sum of distances to both facilities. Subsequently, Chen et al. [6] proposed the optional preference model for the facility location game with two heterogeneous facilities on a line, where the preferences of agents for facilities are either supportive or indifferent. Later, Li et al. [22] generalized it to more general metric spaces and improved the upper bound for the minimum distance cost function compared to [6]. Anastasiadis and Deligkas [2] combined dual preference and optional preference, considering the triple-preference model where the preferences of agents for facilities are either supportive, obnoxious, or indifferent. Xu et al. [36] studied the facility location game with minimum distance requirement, where two facilities need to be located with a constraint that the distance between the facility positions must be at least a specified value. Fong et al. [19] proposed the fractional preference model for the facility location game with two facilities that serve the similar purpose on a line, where the preferences of agents indicate how often they use to facilities. For a more detailed overview of these models, please refer to a survey by Chan et al. [5].

The model closest to ours may be [9, 11]. Elkind et al. [11] considered choosing k facilities out of m facilities and locating them at candidate positions. They introduced a multi-winner facility location model, which combines facility location and multi-winner voting with approval ballots, where the preferences of agents for facilities depend on the positions of facilities. Deligkas et al. [9] considered the selection of one facility from two facilities, where agents have approval preferences for the facility, and extended it to a more general model of choosing k out of m facilities. In our model, we further investigate the case where agents have fractional preferences for facilities, which can be regarded as an extension of the preference domain in [9]. However, due to the existence of the preference $(1, 1)$ in the approval preference model, our model cannot encompass all instances in the approval preference model. This also implies that the lower bound of the problem in our model may not necessarily adhere to the lower bounds stated in [9].

1.2 Our Contribution

We consider the utility version of agents and first study the model of choosing one facility from two facilities when agents have fractional preferences for facilities. For three different settings, we investigate

Table 1: A summary of our results. UB and LB stand for upper bound and lower bound, respectively. SP and GSP stand for strategy-proof and group strategy-proof, respectively. The lower bound of 1.4 (marked with ⋆) can be improved to 1.554, as detailed in Appendix.

Setting	Deterministic	Randomized
Known-preferences	UB: 2 GSP (Mech.3.1) LB: 2 SP (Them.3.2)	UB: $\frac{4}{3}$ GSP (Mech.3.2) LB:1.043 SP (Them.3.6)
Known-positions	UB: 6 GSP (Mech.4.1) LB:1.4⋆ SP (Them.4.2)	UB: 4 SP (Mech.4.2) LB: 1.2 SP (Them.4.5)
General	UB: 6 GSP (Mech.4.1) LB: 2 SP (Them.3.2)	UB: 4 GSP (Mech.5.2) LB: 1.2 SP (Them.4.5)

both deterministic and randomized strategy-proof mechanisms. These results are summarized in Table 1.

In the known-preferences setting, we first provide a deterministic group strategy-proof mechanism with 2-approximation which is also the best deterministic strategy-proof mechanism. By assigning a probability distribution to the two possible outputs of the above mechanism, we obtain a class of randomized mechanisms. By appropriately setting the probability distribution, we achieve a randomized group strategy-proof mechanism with approximation ratio of $\frac{4}{3}$. Finally, we provide a lower bound of 1.043 for any randomized strategy-proof mechanism.

In the known-positions setting, a deterministic group strategy-proof mechanism with 6-approximation can be obtained from the results in the general setting. Subsequently, we provide a lower bound of 1.554 for any deterministic strategy-proof mechanism. Then, we present a randomized strategy-proof mechanism with 4-approximation. Finally, we provide a lower bound of 1.2 for any randomized strategy-proof mechanism.

In the general setting, we first introduce a deterministic group strategy-proof mechanism with 6-approximation. For any deterministic strategy-proof mechanism, the lower bound of 2 for the setting of known-preferences also applies to this setting. Subsequently, we introduce a randomized group strategy-proof mechanism with 4-approximation. Similarly, for any randomized strategy-proof mechanism, the lower bound of 1.2 for the setting of known-positions also applies to this setting.

Finally, we extend the model to a more general scenario, choosing k facilities out of m distinct facilities and locating them. For the known-preferences setting, we extend the deterministic group strategy-proof mechanism under the model of choosing one facility out of two facilities, obtaining a deterministic group strategy-proof mechanism with 2-approximation and prove that this approximation ratio is the best achievable among all deterministic strategy-proof mechanisms. For the known-positions setting, we provide a lower bound of $2 - \frac{1}{k}$ for any deterministic strategy-proof mechanism when $k \geq 2$, and a lower bound of 1.554 when $k = 1$. Similarly, for the general setting, the lower bound of 2 for the setting of known-preferences also applies.

1.2.1 Paper Organization. In Section 2, we formalize our model. In Section 3, Section 4, and Section 5, we consider deterministic and randomized strategy-proof mechanisms in the settings of known-preferences, known-positions, and general case, respectively. In Section 6, we study an extended model where we choose k facilities from m facilities. In Section 7, we summarize our work and discuss some open questions.

2 PRELIMINARIES

Given a set of agents, $N = \{1, \dots, n\}$ where each agent $i \in N$ has a position $x_i \in [0, 1]$ and a fractional preference $p_i = (p_{i,1}, p_{i,2})$ over two facilities F_1 and F_2, where $0 \leq p_{i,1}, p_{i,2} \leq 1$ and $p_{i,1} + p_{i,2} = 1$, we aim to design a mechanism to choose one facility from a set of facilities $\{F_1, F_2\}$ and locate it at $[0, 1]$. To simplify our analysis, we assume $x_1 \leq x_2 \leq \dots \leq x_n$ without loss of generality. We use $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ to denote the *position profile* and *preference profile* of the n agents, respectively. The distance between any two points $x, y \in [0, 1]$ is $d(x, y) = |x - y|$. Denote an instance by $I(\mathbf{x}, \mathbf{p})$ or simply by I.

A (deterministic) mechanism is a function f which maps an instance $I(\mathbf{x}, \mathbf{p})$ consisting of the position and preference profile of the n agents to an output (F_j, y_j) consisting of a facility $F_j \in \{F_1, F_2\}$ that is to be located at $y_j \in [0, 1]$. For a given output (F_j, y_j), the utility of each agent $i \in N$ is defined as $u_i((F_j, y_j), (x_i, p_i)) = p_{i,j} \cdot (1 - d(x_i, y_j))$, which depends both on the distance of the agent from the facility position and on her preference for the chosen facility.

A randomized mechanism maps an instance I to an output that places facility $F_j \in \{F_1, F_2\}$ at $y_j \in [0, 1]$ with some cumulative distribution function $q_{F_j}(\cdot)$ such that $\sum_{j \in \{1,2\}} \int_0^1 dq_{F_j}(y_j) = 1$. We denote the probability distribution that the mechanism outputs for the two facilities as $\mathbf{q} = (q_{F_1}, q_{F_2})$, and the (expected) utility of each agent $i \in N$ is

$$u_i(\mathbf{q}, (x_i, p_i)) = \sum_{j \in \{1,2\}} p_{i,j} \cdot \int_0^1 (1 - |x_i - y_j|) dq_{F_j}(y_j).$$

Given a mechanism f, every agent can deliberately misreport her private information, encompassing personal position and preference, with the aim of manipulating the output of the mechanism and achieving a higher utility. A mechanism f is *strategy-proof* (SP) if no agent can benefit from misreporting, regardless of the reports of the others, that is, for every $i \in N$,

$$u_i\left(f(\mathbf{x}, \mathbf{p}), (x_i, p_i)\right) \geq u_i\left(f(\mathbf{x}', \mathbf{p}'), (x_i, p_i)\right),$$

for any $x_i, x_i' \in [0, 1]$, any $p_i, p_i' \in [0, 1]^2$, any $\mathbf{x}_{-i} \in [0, 1]^{n-1}$ and any $\mathbf{p}_{-i} \in [0, 1]^{2n-2}$, where $(\mathbf{x}', \mathbf{p}') = ((x_i', \mathbf{x}_{-i}), (p_i', \mathbf{p}_{-i}))$, $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $\mathbf{p}_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$. Similarly, a randomized mechanism is strategy-proof in expectation if no agent can increase her expected utility by misreporting. A mechanism f is *group strategy-proof* (GSP), if no coalition of agents can increase the utility of each member by misreporting, regardless of the reports of other agents.

Given an instance $I(\mathbf{x}, \mathbf{p})$, our goal is to design strategy-proof mechanisms to choose a single facility F_j from two facilities F_1, F_2 and locate it at y_j to maximize the social utility, where social utility is defined as the sum of the utilities of all agents, that is, $SU((F_j, y_j), I) = \sum_{i \in N} u_i((F_j, y_j), (x_i, p_i))$. For instance I, let $SU(f(I), I) = \sum_{i \in N} u_i((f(I), (x_i, p_i))$ be the social utility obtained by mechanism f and $OPT(I) = max_{(F_j, y_j)} SU((F_j, y_j), I)$ be the maximum social utility. A strategy-proof mechanism f achieves an approximation ratio of $\alpha \geq 1$ if for every instance I,

$$OPT(I) \leq \alpha \cdot SU(f(I), I).$$

Based on the private information held by each agent, we proceed to explore the following three cases:

1. In the known-preferences setting, the preference of each agent is assumed to be public and the position is private. The agents can misreport only their positions.
2. In the known-positions setting, the position of each agent is assumed to be public and the preference is private. The agents can misreport only their preferences.
3. In the general setting, both the position and the preference of each agent are private information. The agents can simultaneously misreport their positions and preferences.

Note that the positive results in the general setting, which encompasses the (group) strategy-proof mechanisms with proven approximation guarantees, are also applicable as positive results for the setting with known-preferences and known-positions. Similarly, the negative results in the restrictive setting, representing the lower bounds on the approximation ratio of the (group) strategy-proof mechanisms, can be considered as the negative results for the general setting.

3 KNOWN-PREFERENCES SETTING

In this section, we explore the scenario where preferences are discernible. We assume that the preferences of the agents are public information and the positions are private information. Therefore, agents can only misreport their positions.

3.1 Deterministic mechanism

We begin by focusing on deterministic mechanisms. Our first result is that combining the ideas of weighting and median presents a deterministic GSP mechanism with approximation ratio of 2 in the known-preferences setting. The high-level idea is that we first assign weights to two facilities based on the preferences of agents, and choose the facility with higher total weight, then the facility is located at the "median" (according to the chosen facility's weights) agent's position. We further show that the approximation ratio of this mechanism is the best of all deterministic SP mechanisms.

Mechanism 3.1. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p})$, let $n_j = \sum_{i=1}^{n} p_{i,j}$ and define $m_j = \arg\min_k \left\{\sum_{i=1}^{k} p_{i,j} \geq \frac{1}{2} n_j\right\}$ for $j \in \{1, 2\}$. Locate F_1 at the position x_{m_1} of agent m_1 if $n_1 \geq n_2$, and otherwise locate F_2 at the position x_{m_2} of agent m_2.*

In the following we give a key lemma.

Lemma 3.1. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p})$, for $j \in \{1, 2\}$, let $SU((F_j, x_{m_j}), I)$ be the social utility when F_j is chosen and located at the position x_{m_j} of agent m_j. We have*

$$\frac{1}{2} \cdot n_j \leq SU((F_j, x_{m_j}), I) \leq n_j.$$

Due to length limitation, most of the proofs are included in Appendix.

Theorem 3.1. *In the known-preferences setting, Mechanism 3.1 is a deterministic GSP mechanism with approximation ratio of 2.*

Proof. We first show that the mechanism is GSP, that is, for any instance I and any group G of agents with any misreport, there exists at least one member who cannot benefit from it. Assume that the output of the mechanism is (F_j, x_{m_j}) when all members in G truthfully report their positions. According to the mechanism, misreports on position do not affect which facility to be chosen. Therefore, for any misreport by G, the mechanism still chooses F_j. Note that if the misreport by G cannot change the location of F_j, then none of the agents in G can benefit from misreporting. If the misreport by G changes the location of F_j to $y' > x_{m_j}$, then G must contain some agent i with position $x_i \leq x_m$ and preference $p_{i,j} > 0$, but misreports a position $x'_i > x_m$. Clearly, agent i cannot benefit from the misreport. The analysis is similar in the case of $y' < x_{m_j}$.

We now analyze the approximation ratio of the mechanism. Let F_k be the facility chosen by the mechanism and F_o be the facility chosen by optimal solution for instance I. By Lemma 3.1, we have

$$\frac{1}{2} \cdot n_k \leq SU((F_k, x_{m_k}), I) \leq n_k.$$

Considering F_o, since the maximum possible utility of each agent i is $p_{i,o}$, we have

$$OPT(I) \leq \sum_{i \in N} p_{i,o} = n_o.$$

Therefore,

$$SU((F_k, x_{m_k}), I) \geq \frac{1}{2} \cdot n_k \geq \frac{1}{2} \cdot n_o \geq \frac{1}{2} \cdot OPT(I),$$

where the second inequality holds because F_k is the facility with the highest total weight, that is, $n_k \geq n_{3-k}$. □

[9] provides a lower bound of 2 for any deterministic SP mechanism in the approval preference model under the known-preference setting. We observe that the instance utilized in their proof are applicable to our model as well.

Theorem 3.2. *In the known-preferences setting, there is no deterministic SP mechanism with approximation ratio better than 2.*

3.2 Randomized mechanism

In this subsection, we consider randomized mechanisms. We present a class of randomized mechanisms obtained by assigning probability distributions to the two possible outputs of Mechanism 3.1. By setting an appropriate probability distribution, a randomized SP mechanism with approximation ratio of $\frac{4}{3}$ can be achieved, which is also the best in such class of mechanisms. Finally, we provide a lower bound of 1.043 on the approximation ratio of any randomized SP mechanism in the known-preferences setting.

Mechanism 3.2. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p})$, locate F_1 at x_{m_1} with probability $\gamma(\boldsymbol{p})$ and F_2 at x_{m_2} with probability $1 - \gamma(\boldsymbol{p})$, where $\gamma(\cdot) \in [0, 1]$ is a function of $\boldsymbol{p}$.*

Lemma 3.2. *In the known-preferences setting, Mechanism 3.2 is GSP.*

Proof. Since Mechanism 3.2 is a probability distribution over two deterministic GSP mechanisms, wherein the selection of probabilities depends solely on the known-preferences of agents, any group G has no incentive to misreport their positions. This implies that Mechanism 3.2 is GSP. □

It is natural to choose a facility according to its support weight, which is described as follows.

Theorem 3.3. *In the known-preferences setting, setting* $\gamma = \frac{n_1}{n_1+n_2}$*, Mechanism 3.2 is a GSP mechanism with approximation ratio of* $\frac{1+\sqrt{3}}{2} \approx 1.366$.

Next, by appropriately adjusting $\gamma(\mathbf{p})$, we can obtain a SP mechanism with approximation ratio of $\frac{4}{3}$.

Theorem 3.4. *In the known-preferences setting, setting* $\gamma = \begin{cases} \frac{3n_1-2n_2}{4n_1-2n_2}, & n_1 \geq n_2 \\ \frac{n_2}{4n_2-2n_1}, & n_1 < n_2 \end{cases}$*, Mechanism 3.2 is a GSP mechanism with approximation ratio of* $\frac{4}{3}$.

Theorem 3.5. *In the known-preferences setting, the approximation ratio of SP mechanisms obtained by any probability distribution determined by* γ *in Mechanism 3.2 is at least* $\frac{4}{3}$.

Theorem 3.6. *In the known-preferences setting, there is no randomized SP mechanism with approximation ratio better than* $\frac{24}{23} \approx 1.043$.

4 KNOWN-POSITIONS SETTING

In this section, we discuss the scenario where the positions of the agents are known. We assume that the positions of the agents are public information, while the preferences are considered private information. Therefore, the agents can only misreport their preferences.

4.1 Deterministic mechanism

In the known-position setting, we first present a deterministic GSP mechanism with approximation ratio of 6. Furthermore, we provide a lower bound of 1.4 on the approximation ratio for any deterministic SP mechanism, which can be improved to 1.554 by appropriately increasing the number of agents and adjusting preferences, as detailed in Appendix.

Mechanism 4.1. *Given an instance* $I(\boldsymbol{x}, \boldsymbol{p})$*, Let* n_{F_j} *be the number of agents* i *with* $p_{i,j} > p_{i,3-j}$ *for* $j \in \{1, 2\}$*. Locate* F_1 *at* $\frac{1}{2}$ *if* $n_{F_1} \geq n_{F_2}$ *and otherwise locate* F_2 *at* $\frac{1}{2}$.

Theorem 4.1. *In the known-positions setting, Mechanism 4.1 is a deterministic GSP mechanism with approximation ratio of 6.*

Proof. In the general setting, Mechanism 4.1 is also GSP. To avoid verbosity, the proof is omitted here. Please refer to Theorem 5.1 for the specific proof of the approximation ratio and strategyproofness of Mechanism 4.1. □

Before giving the proof of the lower bound, we first give a property of the SP mechanism, which was proposed by Lu et al. [24]. We provide its proof under our model.

Lemma 4.1. *[24] In an SP mechanism, a coalition of agents with the same position and preference cannot benefit even if they misreport simultaneously.*

Theorem 4.2. *In the known-positions setting, there is no deterministic SP mechanism with approximation ratio better than* $\frac{7}{5}$.

4.2 Randomized mechanism

In this subsection, our main result is to present a randomized mechanism that is both SP and 4-approximate under the known-positions setting. Additionally, we provide a lower bound of 1.2 on the approximation ratio for any randomized SP mechanism in the known-positions setting.

Mechanism 4.2. *Given an instance* $I(\boldsymbol{x}, \boldsymbol{p})$*, let* l_j *be the leftmost agent with* $p_{i,j} > 0$ *and* r_j *be the rightmost agent with* $p_{i,j} > 0$ *for* $j \in \{1, 2\}$*. Locate* F_j *at* $\frac{x_{l_j}+x_{r_j}}{2}$ *with probability* $\frac{1}{2}$.

We first demonstrate that Mechanism 4.2 is not GSP by a simple instance.

Theorem 4.3. *Mechanism 4.2 is not GSP.*

Theorem 4.4. *In the known-positions setting, Mechanism 4.2 is an SP mechanism with approximation ratio of* 4.

Now, we present an example to demonstrate the tightness of the approximation ratio of Mechanism 4.2. We consider an instance I, where there are $n-1$ agents with the same preference $(\varepsilon, 1-\varepsilon)$ positioned at 0, and one agent with the preference $(\frac{1}{2}, \frac{1}{2})$ positioned at 1, where ε is a sufficiently small positive value. For I, the optimal solution is to locate F_2 at 0, resulting in an optimal social utility of $(n-1)\cdot(1-\varepsilon)$. The expected social utility of the mechanism is

$$\begin{aligned} SU(f(I), I) &= \frac{1}{2}\cdot(\frac{1}{2}\cdot\frac{1}{2} + (n-1)\cdot\frac{1}{2}\cdot\varepsilon) \\ &+ \frac{1}{2}\cdot(\frac{1}{2}\cdot\frac{1}{2} + (n-1)\cdot\frac{1}{2}(1-\varepsilon)) \\ &= \frac{1}{4} + \frac{1}{4}\cdot(n-1). \end{aligned}$$

Therefore, the approximation ratio is at least

$$\frac{(n-1)\cdot(1-\varepsilon)}{\frac{1}{4}+\frac{1}{4}\cdot(n-1)} = 4\cdot(1-\varepsilon) - \frac{4}{n}(1-\varepsilon) \to 4 - \frac{4}{n},$$

when ε tends to 0.

Theorem 4.5. *In the known-positions setting, there is no randomized SP mechanism with approximation ratio better than* $\frac{6}{5}$.

5 GENERAL SETTING

In this section, we assume that the positions and preferences of the agents are private information, and agents can simultaneously misreport both their positions and preferences.

5.1 Deterministic mechanism

For the scenario where agents can simultaneously misreport both their locations and preferences, our first main result is a deterministic GSP mechanism with approximation ratio of 6. Additionally, based on the result from Theorem 3.2, we derive a lower bound of 2 on the approximation ratio for any deterministic SP mechanism in the general setting. It remains an open question to either improve the lower bound on the approximation ratio or design a mechanism with approximation ratio better than 6 to narrow the gap.

We first provide an example to demonstrate that Mechanism 3.1 is no longer SP in this setting. Consider an instance I with three agents, who are positioned at 0 with preferences $(\frac{3}{8}, \frac{5}{8})$, $(\frac{3}{8}, \frac{5}{8})$ and $(\frac{5}{8}, \frac{3}{8})$ respectively. For this instance, Mechanism 3.1 locates F_2 at 0 and the utility of the agent with the preference $(\frac{5}{8}, \frac{3}{8})$ is $\frac{3}{8}$. However, if she misreports her preference as $(1, 0)$, Mechanism 3.1 will locate F_1 at 0 and her utility i will increase to $\frac{5}{8}$.

Note that Mechanism 3.1 is not SP primarily because it chooses the facility with the highest total weight, which always motivates agents to report a higher weight for the facility they prefer. To address this issue, we adopt a voting-based approach instead of the weighted approach.

Specifically, we categorize all the agents as three types:

1. F_1 agent: If $p_{i,1} > \frac{1}{2} > p_{i,2}$, indicate that agent i votes for F_1.
2. F_2 agent: If $p_{i,2} > \frac{1}{2} > p_{i,1}$, indicate that agent i votes for F_2.
3. $F_{1,2}$ agent: If $p_{i,1} = p_{i,2} = \frac{1}{2}$, indicate that agent i is indifferent between the two facilities.

It is worth noting that when choosing which facility to locate at $\frac{1}{2}$, the preferences of $F_{1,2}$ agents can be disregarded. Hence, an intuitive idea would be to locate F_1 at $\frac{1}{2}$ if the number of F_1 agents is no less than the number of F_2 agents, and locate F_2 at $\frac{1}{2}$ otherwise. See Mechanism 4.1 for details.

Theorem 5.1. *Mechanism 4.1 is a deterministic GSP mechanism with approximation ratio of* 6.

Proof. We first show that the mechanism is GSP, i.e., for any instance I and any group G of agents with any misreport, there exists at least one member who cannot benefit from it. The mechanism only has two possible outputs, either locating F_1 at $\frac{1}{2}$ or locating F_2 at $\frac{1}{2}$. Without loss of generality, we assume the mechanism outputs F_1 at $\frac{1}{2}$ when all members in G truthfully report their positions and preferences. Note that for F_1 agents and $F_{1,2}$ agents, they already achieve the highest utility among all possible output results of the mechanism. Thus, if G contains F_1 agents or $F_{1,2}$ agents, they cannot benefit from misreporting. If G only contains F_2 agents, any misreport of G will not alter the fact that $n_{F_1} \geq n_{F_2}$, and the output of mechanism remains unchanged. Therefore, none of the agents in G can benefit from misreporting.

We now analyze the approximation ratio of the mechanism. Let N_{F_1}, N_{F_2}, and $N_{F_{1,2}}$ be the set of F_1 agents, F_2 agents, and $F_{1,2}$ agents, respectively. Let n_{F_1}, n_{F_2}, and $n_{F_{1,2}}$ be the number of F_1 agents, F_2 agents, and $F_{1,2}$ agents, respectively. Thus, we have $n_{F_1}+n_{F_2}+n_{F_{1,2}} = n$. Without loss of generality, we assume that the mechanism locates F_1 at $\frac{1}{2}$. Hence, $n_{F_1} \geq n_{F_2}$. We now proceed to analyze different cases for the optimal facility.

Case 1: If the optimal facility is F_1, located at y_1, then

$$\frac{OPT(I)}{SU((F_1, \frac{1}{2}), I)} = \frac{\sum_{i\in N} p_{i,1}(1-d(x_i, y_1))}{\sum_{i\in N} p_{i,1}(1-d(x_i, \frac{1}{2}))} \leq \frac{\sum_{i\in N} p_{i,1}\cdot 1}{\sum_{i\in N} p_{i,1}\cdot \frac{1}{2}} = 2,$$

where the inequality holds because the distance from each agent to the optimal facility is greater than or equal to 0 and the distance to $\frac{1}{2}$ is less than or equal to $\frac{1}{2}$.

Case 2: If the optimal facility is F_2, located at y_2, then

$$\begin{aligned}\frac{OPT(I)}{SU((F_1, \frac{1}{2}), I)} &= \frac{\sum_{i\in N} p_{i,2}(1-d(x_i, y_2))}{\sum_{i\in N} p_{i,1}(1-d(x_i, \frac{1}{2}))} \leq \frac{\sum_{i\in N} p_{i,2}\cdot 1}{\sum_{i\in N} p_{i,1}\cdot \frac{1}{2}} \\ &= \frac{\sum_{i\in N_{F_1}} p_{i,2} + \sum_{i\in N_{F_{1,2}}} \frac{1}{2} + \sum_{i\in N_{F_2}} p_{i,2}}{\frac{1}{2}\sum_{i\in N_{F_1}} p_{i,1} + \sum_{i\in N_{F_{1,2}}} \frac{1}{4} + \frac{1}{2}\sum_{i\in N_{F_2}} p_{i,1}} \\ &\leq \frac{\sum_{i\in N_{F_1}} \frac{1}{2} + \sum_{i\in N_{F_{1,2}}} \frac{1}{2} + \sum_{i\in N_{F_2}} 1}{\frac{1}{2}\sum_{i\in N_{F_1}} \frac{1}{2} + \sum_{i\in N_{F_{1,2}}} \frac{1}{4}} \\ &= \frac{\frac{1}{2}n_{F_1} + \frac{1}{2}n_{F_{1,2}} + n_{F_2}}{\frac{1}{4}n_{F_1} + \frac{1}{4}n_{F_{1,2}}} \\ &= \frac{\frac{1}{2}(n+n_{F_2})}{\frac{1}{4}(n-n_{F_2})} \\ &\leq 6,\end{aligned}$$

where the first inequality holds because the distance from each agent to the optimal facility is greater than or equal to 0 and the distance to $\frac{1}{2}$ is less than or equal to $\frac{1}{2}$, the second inequality holds due to the definition of F_1 agent, which satisfies $p_{i,1} > \frac{1}{2} > p_{i,2}$, and the definition of the preference of agent i, which satisfies $0 \leq p_{i,1}, p_{i,2} \leq 1$, and the third inequality holds because $n_{F_1} \geq n_{F_2}$, implying $n_{F_2} \leq \frac{n}{2}$. □

Now, we present an example to demonstrate the tightness of the approximation ratio analysis of Theorem 5.1 discussed above. Consider an instance I, where there are $\frac{n}{2}$ agents with the same preference $(\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon)$ positioned at 0, and $\frac{n}{2}$ agents with the same preference $(0, 1)$ positioned at 0. Here, ε is a sufficiently small positive value. For I, the optimal solution is to locate F_2 at 0 and the optimal social utility is

$$OPT(I) = \frac{n}{2} + \frac{n}{2}\cdot(\frac{1}{2}-\varepsilon) = \frac{(3-2\varepsilon)n}{4}.$$

Because $n_{F_1} \geq n_{F_2}$, the mechanism locates F_1 at $\frac{1}{2}$. Therefore, the social utility of the mechanism is

$$SU((F_1, \frac{1}{2}), I) = \frac{n}{2}\cdot\frac{1}{2}\cdot(\frac{1}{2}+\varepsilon) = \frac{(1+2\varepsilon)n}{8}.$$

Therefore, the approximation ratio is at least

$$\frac{OPT(I)}{SU((F_1, \frac{1}{2}), I)} = \frac{(3-2\varepsilon)n/4}{(1+2\varepsilon)n/8} = \frac{6-4\varepsilon}{1+2\varepsilon} \to 6,$$

when ε tends to 0.

5.2 Randomized mechanism

In this subsection, we consider randomized mechanisms in the general setting. We first provide an example to demonstrate that Mechanism 4.2 is not SP when allowing agents to misreport both their position and preference simultaneously.

Consider an instance I with two agents: agent 1 with the preference $(\frac{1}{2}, \frac{1}{2})$ is positioned at $\frac{1}{2}$, while agent 2 with $(\frac{1}{2}, \frac{1}{2})$ is positioned at 1. For this instance, Mechanism 4.2 locates F_j, $j \in \{1, 2\}$ at $\frac{3}{4}$ with probability of $\frac{1}{2}$. Consequently, the expected utility of agent 1 is $\frac{3}{8}$. Now, consider another instance I' with two agents,where agent 1 with $(\frac{1}{2}, \frac{1}{2})$ is positioned at 0 and agent 2 with $(\frac{1}{2}, \frac{1}{2})$ is positioned at 1. For I', Mechanism 4.2 locates F_j, $j \in \{1,2\}$ at $\frac{1}{2}$ with probability of $\frac{1}{2}$. Therefore, agent 1 in I has an incentive to misreport her position as 0 to switch to I', causing the mechanism to locate the facility at her true position, thus increasing her expected utility to 1. Therefore, Mechanism 4.2 is not SP.

Combining the median mechanism with Mechanism 4.2, we can obtain an SP but not GSP mechanism.

Mechanism 5.1. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p})$, let $S_j = \{i \in N : p_{i,j} > 0\}$ and m_{S_j} be the median agent in S_j for $j \in \{1,2\}$. Locate F_j at $x_{m_{S_j}}$ with probability $\frac{1}{2}$ for $j \in \{1,2\}$. If the number of agents in S_j is even, we choose the median agent on the left.*

We first give a simple instance to prove that Mechanism 5.1 is not GSP.

Theorem 5.2. *Mechanism 5.1 is not GSP.*

Proof. Consider an instance I with five agents: agent 1 with the preference $(0,1)$ is positioned at 0, agent 2 with $(0,1)$ is positioned at $\frac{1}{4}$, agent 3 with $(1-\varepsilon, \varepsilon)$ is positioned at $\frac{1}{2}$, and agent 4 and agent 5 with the same preferences $(1,0)$ are positioned at $\frac{3}{4}$ and 1 respectively. Here, $0 < \varepsilon < \frac{1}{2}$. For this instance, Mechanism 5.1 locates F_1 at $\frac{3}{4}$ and F_2 at $\frac{1}{4}$ with probability $\frac{1}{2}$, respectively. Consequently, the expected utility of agent 1 is $\frac{3}{8}$ and the expected utility of agent 3 is also $\frac{3}{8}$. However, if agent 1 and agent 3 misreport their preferences as $(\frac{1}{2}, \frac{1}{2})$ and $(1,0)$ respectively, Mechanism 5.1 will locates F_1 at $\frac{1}{2}$ and F_2 at 0 with probability $\frac{1}{2}$, and the expected utility of agent 1 and agent 3 will become $\frac{1}{2} > \frac{3}{8}$ and $\frac{1}{2} - \frac{\varepsilon}{4} > \frac{3}{8}$ respectively. Therefore, Mechanism 5.1 is not GSP. □

Theorem 5.3. *Mechanism 5.1 is an SP mechanism with approximation ratio of* $2n$.

In fact, the approximation ratio of Mechanism 5.1 cannot be better than $n-1$. Consider an instance I, where there are $\frac{n-1}{2}$ agents with the same preference $(\varepsilon, 1-\varepsilon)$ positioned at 0, $\frac{n-1}{2}$ agents with the same preference $(1-\varepsilon, \varepsilon)$ positioned at 1, and one agent with $(0,1)$ positioned at 1. Here, $\varepsilon > 0$ can be sufficiently small. For I, the optimal solution is to locate F_1 at 1 or locate F_2 at 0 and the resulting optimal social utility is $OPT(I) = \frac{n-1}{2} \cdot (1-\varepsilon)$. The mechanism locates F_1 at 0 with probability $\frac{1}{2}$ and locates F_2 at 1 with probability $\frac{1}{2}$. The expected social utility is

$$SU(f(I), I) = \frac{n-1}{2} \cdot \frac{\varepsilon}{2} + \frac{1}{2} + \frac{n-1}{2} \cdot \frac{\varepsilon}{2} = \frac{1+(n-1)\varepsilon}{2}.$$

Therefore, the approximation ratio is at least

$$\frac{OPT(I)}{SU(f(I), I)} = \frac{(1-\varepsilon)\cdot(n-1)/2}{(1+(n-1)\varepsilon)/2} = \frac{(n-1)(1-\varepsilon)}{1+(n-1)\varepsilon} \to n-1,$$

when ε tends to 0.

Through the above examples, we observe that placing the facility at the position of the median agent approving it, affected by the fractional preferences, i.e., when the agents farther from the median agent exhibit significant preferences for the facility, the mechanism will result in a poor approximation ratio. Consequently, in the following we randomly choose a facility and locate it in the middle position.

Mechanism 5.2. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p})$, locate F_j, $j \in \{1,2\}$ at $\frac{1}{2}$ with probability $\frac{1}{2}$.*

Theorem 5.4. *Mechanism 5.2 is a GSP mechanism with approximation ratio of* 4.

Proof. Clearly, the mechanism is GSP, since any misreport about positions and preferences by any group G would not affect the output of the mechanism. We use f to represent Mechanism 5.2. Given any instance I, let F_o be the optimal facility. For each facility F_j, $j \in \{1,2\}$, let $SU((F_j, \frac{1}{2}), I)$ be the social utility when F_j is located at $\frac{1}{2}$. Note that when a facility is located at $\frac{1}{2}$, the maximum distance from any agent i to the facility is at most $\frac{1}{2}$, i.e., $d(x_i, \frac{1}{2}) \le \frac{1}{2}$. Therefore, the expected social utility of the mechanism is

$$\begin{aligned}
SU(f(I), I) &= \frac{1}{2} \cdot SU((F_1, \frac{1}{2}), I) + \frac{1}{2} \cdot SU((F_2, \frac{1}{2}), I) \\
&= \frac{1}{2} \cdot \sum_{i \in N} p_{i,1}(1 - d(x_i, \frac{1}{2})) + \frac{1}{2} \cdot \sum_{i \in N} p_{i,2}(1 - d(x_i, \frac{1}{2})) \\
&\ge \frac{1}{2} \cdot \sum_{i \in N} p_{i,1}(1 - \frac{1}{2}) + \frac{1}{2} \cdot \sum_{i \in N} p_{i,2}(1 - \frac{1}{2}) \\
&= \frac{1}{4}(\sum_{i \in N} p_{i,1} + \sum_{i \in N} p_{i,2}) \\
&= \frac{1}{4} n.
\end{aligned}$$

Since for F_o, the maximum possible utility for each agent i is 1, we have

$$OPT(I) \le n.$$

Therefore, the approximation ratio of the mechanism is at most 4. □

Now, we present an example to demonstrate the tightness of the approximation ratio analysis of Theorem 5.4 discussed above. Consider an instance I, where all n agents with the same preference $(1,0)$ are positioned at 0 with. For I, the optimal solution is to locate F_1 at 0 and the optimal social utility is n. Since the expected social utility of Mechanism 5.2 is $\frac{n}{4}$, its approximation ratio is at least 4.

Furthermore, based on Theorem 4.5, we derive a lower bound of 1.2 on the approximation ratio for any randomized SP mechanism in the general setting.

Theorem 5.5. *In the general setting, there is no randomized SP mechanism with approximation ratio better than* 1.2.

6 CHOOSING K OUT OF M FACILITIES

The model discussed above involves the choice and location of one facility from two facilities. We now extend this model to a more general scenario. Consider the case where there are $m \ge 2$ distinct facilities, and we need to choose $k(< m)$ facilities and locate them. We continue to use $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ to denote the *position profile* and *preference profile* of the n agents, respectively. Here, $p_i = (p_{i,1}, \ldots, p_{i,m})$ signifies the preference vector of agent i for the m facilities $\{F_1, \ldots, F_m\}$, subject to the constraint that $0 \le p_{i,j} \le 1$ for any $1 \le j \le m$ and $\sum_{1 \le j \le m} p_{i,j} = 1$. Let an instance be denoted as $I(\mathbf{x}, \mathbf{p}, m, k)$ or simply I. A mechanism is a function that maps an instance $I(\mathbf{x}, \mathbf{p}, m, k)$ to an output $(S, \mathbf{y})$, where S represents a subset of facilities chosen by the mechanism with cardinality k, and $\mathbf{y} = (y_j)_{j \in S}$ corresponds to the location of the k chosen facilities. Given the output $(S, \mathbf{y})$ of a mechanism, the utility of each agent $i \in N$ is defined as the sum of their utilities for the k chosen facilities, i.e.,

$$u_i((S, \mathbf{y}), (x_i.p_i)) = \sum_{F_j \in S} u_i((F_j, y_j), (x_i, p_i)) = \sum_{F_j \in S} p_{i,j} \cdot (1 - d(x_i, y_j)).$$

Our goal is to maximize the social utility by choosing k facilities from the m distinct facilities and locating them, where the social utility is defined as the sum of n agents' utilities, i.e., $SU((S, \mathbf{y}), I) = \sum_{i \in N} u_i((S, \mathbf{y}), (x_i, p_i))$.

For this model, our first result is a generalization of Mechanism 3.1 to obtain a 2-approximate GSP mechanism in the known-preferences setting. Furthermore, We show that this mechanism has the best approximation ratio among all deterministic SP mechanisms. Finally, we provide a lower bound on the approximation ratio for any deterministic SP mechanism in the known-positions setting.

Mechanism 6.1. *Given an instance $I(\boldsymbol{x}, \boldsymbol{p}, m, k)$, let $n_j = \sum_{i=1}^n p_{i,j}$ and $m_j = \arg\min_k \left\{\sum_{i=1}^k p_{i,j} \geq \frac{1}{2} n_j\right\}$ for $j \in \{1, \ldots, m\}$. Choose the facility set S that includes the k facilities corresponding to the top k largest values of n_j, and locate F_j at position x_{m_j} of agent m_j for any $F_j \in S$, breaking ties in any deterministic way.*

Theorem 6.1. *In the known-preferences setting, Mechanism 6.1 is a deterministic GSP mechanism with approximation ratio of 2.*

Proof. The proof of group strategy-proofness of Mechanism 6.1 is similar to that of Theorem 3.2, which is omitted here.

We now turn to analyze the approximation ratio of the mechanism. Given any instance I, let $(S, \mathbf{y})$ be the output of Mechanism 6.1. Denote $S = \{F_{s_1}, \ldots, F_{s_k}\}$ and let $O = \{F_{o_1}, \ldots, F_{o_k}\}$ be the set of optimal facilities for I. For any facility $F_{s_j} \in S$, let $SU((F_{s_j}, x_{m_{s_j}}), I)$ be the social utility when only F_{s_j} is chosen and located at $x_{m_{s_j}}$. By Lemma 3.1, we have

$$\frac{1}{2} n_{s_j} \leq SU((F_{s_j}, x_{m_{s_j}}), I) \leq n_{s_j}.$$

Since for any $F_{o_j} \in O$, the maximum possible utility of each agent i is p_{i,o_j}, we have

$$OPT(I) \leq \sum_{F_{o_j} \in O} \sum_{i \in N} p_{i,o_j} = \sum_{F_{o_j} \in O} n_{o_j}.$$

Therefore,

$$\begin{aligned} SU((S, \mathbf{y}), I) &= \sum_{F_{s_j} \in S} SU((F_{s_j}, x_{m_{s_j}}), I) \geq \frac{1}{2} \cdot \sum_{F_{s_j} \in S} n_{s_j} \\ &\geq \frac{1}{2} \cdot \sum_{F_{o_j} \in O} n_{o_j} \geq \frac{1}{2} \cdot OPT(I), \end{aligned}$$

where the second inequality holds due to the fact that the k facilities in set S correspond to the top-k highest support weights. □

Theorem 6.2. *In the known-preferences setting, when $m \geq k(1 + k)$, there is no deterministic SP mechanism with approximation ratio better than 2.*

Theorem 6.3. *In the known-positions setting, there is no deterministic SP mechanism with approximation ratio better than 1.554 when $k = 1$, and no deterministic SP mechanism with approximation ratio better than $2 - \frac{1}{k}$ when $k \geq 2$ and $m \geq k(1 + k)$.*

7 CONCLUSIONS AND OPEN PROBLEMS

In this paper, we investigated the facility location game with fractional preferences under resource constraints. In this model, we need to choose k facilities from a set of m facilities and locate them. The preferences of agents indicate their weights of support for the facilities. We focus on the scenario of choosing one facility from two facilities. Based on private information held by the agents, we considered three settings, each providing upper and lower bounds on the approximation ratio for deterministic and randomized (group) strategy-proof mechanisms. The results are summarized in Table 1.

For the known-preferences setting, we provided a 2-approximate deterministic group strategy-proof mechanism, which is also the best deterministic strategy-proof mechanism. Subsequently, by randomizing it, we obtained a $\frac{4}{3}$-approximate group strategy-proof mechanism and a lower bound of 1.043 for any randomized strategy-proof mechanism. For the known-positions setting, we derived an upper bound of 6 and a lower bound of 1.554 for any deterministic group strategy-proof mechanism. Whether it is possible to fully utilize the position information may be crucial for narrowing the gap. We also provided a 4-approximate randomized SP mechanism and a lower bound of 1.2 for any randomized SP mechanism. For the general setting, we presented a 6-approximate deterministic group strategy-proof mechanism, a lower bound of 2 for any deterministic SP mechanism and a 4-approximate randomized GSP mechanism.

We also considered a more general scenario, choosing $k(< m)$ facilities from $m \geq 2$ different facilities and locating them. For the known-preferences setting, we obtained a deterministic group strategy-proof mechanism with 2-approximation which is the best deterministic strategy-proof mechanism. For the known-positions setting, we provided a lower bound of 2 for any deterministic strategy-proof mechanism. For the general setting, the lower bound of 2 in the known-preferences setting also applies. However, for the general setting, even for known-positions setting, the deterministic strategy-proof mechanism under the scenario of choosing one facility out of two facilities is no longer strategy-proof (Appendix B), and we cannot provide any strategy-proof mechanism with bounded approximation ratio.

For future work, a natural direction is to narrow the gap between the upper and lower bounds of deterministic and randomized mechanisms in different settings. We can also consider various variants of preferences. For example, the agents may have dual preferences or triple preferences instead of fractional preferences. In addition, it is interesting to consider the obnoxious facility model, where we need to choose and locate facilities that the agents want to stay away from. We can also consider scenarios with candidate positions, meaning that we need to locate facilities at predefined positions.

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